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A NEW GENERAL INTEGRAL TRANSFORM DECOMPOSITION METHOD FOR SOLVING FRACTIONAL DIFFERNTIAL EQUATIONS

Agwunobi Chioma Joy, Manjak Nibron Haggai, Okai Jeremiah and Domven Lohcwat.
Department of mathematical sciences,
Abubakar Tafawa Balewa University, Bauchi, Nigeria.

ABSTRACT

In this work, we introduce the general integral transform decomposition method to provide analytical solution for linear and non linear fractional differential equations. The technique is a combination of a new integral transform proposed by Hossein Jafari called the general integral transform and Adomian decomposition method. The new general integral transform is a generalization of all Laplace-type integral transforms. The proposed method does not require discretiztion or restrictive assumptions; it is straight forward, easy to implement and gives a series solution that converges rapidly. Several illustrative examples are considered and the results obtained are compared with the results obtained from other methods to show the feasibility, efficiency of the method.

Keywords: *Fractional Calculus, General integral transform, Adomian polynomials*

1.0 INTRODUCTION

Fractional differential equations are differential equations involving fractional order derivatives of the unknown function. It is a branch of fractional calculus which generalizes the operation of differentiation and integration to non integer order. In recent times, fractional derivatives have become an important tool in modeling various problems that arise in physics, engineering, biology, economics and other sciences. This is as a result of the fact that many physical phenomena depends not only on their instantaneous state but also on previous time history and fractional differential equations provides an excellent instrument for the description of such memory and hereditary properties which is neglected by integer models, hence making it more suitable for modeling of systems whose evolution depends on their previous and current states. Podlunby (1999) presented a survey of the application of fractional differential equations in various fields of sciences and engineering, some of which include viscosity, electrical circuits, control theory, diffusion. The fractional order models were seen to be more adequate than their integer order counterpart. Rihan (2012) used fractional differential equations to model cancer-immune system interaction. The fractional differential equations were found to be naturally related to biological systems with memory or after effects; such effects are usually neglected in integer order differential equations. Despite the noticeable progress in fractional calculus, there is no universally agreed method for solving fractional differential equations. Several methods have been proposed for solving fractional differential equations. Some of the methods include Homotopy perturbation method, Adomian decomposition method, variational iteration method. Hossein (2006) applied Adomian decomposition method to solving linear and nonlinear fractional diffusion wave equation that describes diffusion in special types of porous media and are also used to model anomalous diffusion in plasma transport. Zaid (2008) proposed a generalization of the differential transform method and successfully applied it to fractional differential equations. The proposed method was based on a generalization of the Taylor's formula which is less computational than classical Taylor's formula. Zaid (2006) applied homotopy perturbation method in a modified form, to solve linear and non linear quadratic Riccatti differential equations of fractional order. The results obtained from the proposed method which eliminates the small parameter assumption encountered in the basic homotopy perturbation method showed that the method is effective and reliable. Integral transforms is also one of the methods that have also been widely used to solve fractional differential equations due to the simplicity they bring when dealing with differential equations. Some of the well known integral transforms such as Laplace transform, Sumudu transform, Natural transform, have been applied severally in finding solutions of fractional differential equations. It must be stated that integral transform alone cannot handle non linear equations due to the nonlinearity present. In recent times, researchers have considered the possibility of coupling integral transforms and decomposition methods for solving non linear fractional differential equations. Sambath and Balachandran (2016) used Laplace Adomian decomposition method (LADM) to obtain series solution of non linear system of fractional differential equation arising from a fish farm model. The method which is a combination of the Laplace transform and Adomian decomposition method was shown to be effective when applied to several examples. Elzaki transform was also successfully combined with Adomian decomposition method by Nehad et al (2020). The method was seen to be straight forward and an effective method for solving linear and non linear fractional differential equations and was used to evaluate fractional order telegraph equations. Shehu and Ibrahim (2016) in their paper combined the natural transform and Homotopy perturbation method to propose the natural homotopy perturbation method (NHPM). They used their method to solve linear and non linear partial fractional differential equations. Hossien (2020) introduced a new integral transform which he called "new general integral transform". By comparing his integral transform and some existing transforms in the Laplace family such as Sumudu transform, Elzaki transform, Natural transform, Aboodh transform, the new general integral transform was shown to be a generalization of most type of transform in the class of Laplace transform. The transform was also applied to solve higher order ordinary differential equation with constant and variable coefficient, integral equations and fractional integral equations. In this research, we propose a combination of the new general integral transform and Adomian decomposition method which we shall call the general integral transform decomposition method (GITDM) to solve linear and non linear fractional differential equations. The algorithm derived from the proposed method is easy to implement and produces rapidly convergent approximate series solution.

2.0 DEFINITION OF TERMS

Definition 1

Gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{t}dt \qquad \operatorname{Re}(z)>0 \qquad \dots(1)$$

The gamma function plays an important role in fractional calculus and has the following important properties

$$\Gamma(z+1)=z!$$
$$\Gamma(z+1)=z\Gamma(z) \qquad \dots(2)$$
$$\Gamma(1)=1$$

Definition 2

The Caputo fractional derivative of $f(t)$ is defined as

$$D^{\alpha} f(t)=D^{-(n-\alpha)}\left(D^{n} f(t)\right)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha+1} f(\tau)d\tau \qquad \dots(3)$$

For $n-1<\alpha\le n, n\in N, t>0$

Some useful properties of Caputo fractional derivative are

- $D^{\alpha}t^{n}=\dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}$
- $D^{\alpha}C=0 \qquad \dots(4)$

3.0 MATERIALS AND METHOD

3.1 The general integral transform

The general integral transform $F(s)$ of $f(t)$is defined by the formula

$$T\{f(t);s\}=F(s)=p(s)\int_0^\infty f(t)e^{-q(s)t}dt \qquad \dots(5)$$

Where $f(t)$ is an integrable function defined for $t\ge 0$, $p(s)\neq 0$ and $q(s)$ are positive real functions and the integral exists.

It is worth mentioning that when

$p(s)=1$ and $q(s)=s$ gives the Laplace transform

$p(s)=s$ and $q(s)=s$ gives sumudu transform

$p(s)=1/s$ and $q(s)=s$ gives Elzaki transform

$p(s)=u$ and $q(s)=s/u$ gives the natural transform

Other integral transforms in the Laplace family are also special cases of the general integral transforms for appropriate values of $p(s)$ and $q(s)$.

3.1.1Properties of general integral transform

The general integral transform has the following properties

- $T\{f^{n}(t);s\}=q^{n}(s)F(s)-p(s)\sum_{k=0}^{n-1}q^{n-k-1}(s)f^{k}(0)$ $\quad\ldots(6)$
- $T\{tf^{n}(t)\}=-\frac{p(s)}{q(s)}\frac{d}{ds}\left[\frac{1}{p(s)}T\{f^{n}(t)\}\right]\ldots(7)$
- If $F_1(s)$ and $F_2(s)$ are the integral transform of $f_1(t)$ and $f_2(t)$ respectively, then the general integral transform of the convolution of f_1 and f_2 is

$$f_1(t)*f_2(t)=\int_0^{\infty}f_1(t)f_2(t-\tau)d\tau=\frac{1}{p(s)}F_1(s)F_2(s)\quad\ldots(8)$$

3.1.2 General Integral Transform of Some Functions

The table below gives the general integral transform of some functions

Function	General Integral Transform
a ($a=$ constant)	$\frac{ap(s)}{q(s)}$
t	$\frac{p(s)}{q(s)^2}$
t^n	$\frac{\Gamma(\alpha+1)p(s)}{q^{\alpha+1}(s)},\alpha>0$
$\sin(at)$	$\frac{ap(s)}{a^2+q^2(s)}$
$\cos(at)$	$\frac{q(s)p(s)}{a^2+q^2(s)}$
e^t	$\frac{p(s)}{q^2(s)-1}$

3.1.3 Inverse General Integral Transform

Given a continuous function $f(t)$, if $F(s)=T\{f(t)\}$ then $f(t)$ is the inverse general integral transform of $F(s)$ and denoted as

$$f(t)=T^{-1}\{F(s)\}\quad\ldots(9)$$

3.1.4 General Integral Transform of Caputo Fractional Derivative

Let D^{α} be the Caputo fractional derivative of order α, the general integral transform of Caputo fractional derivative of $D^{\alpha}f(t)$ is

$$T\{D^{\alpha}f(t)\}=q^{\alpha}(s)F(s)-p\left(s\sum_{k=0}^{n-1}q^{\alpha-k-1}(s)f^{k}(0)\right)\quad\ldots(10)$$

__**Proof**__

$D^{\alpha}f(t)=D^{-(n-\alpha)}(D^{n}f(t))$

Taking transform of both sides

$$T\{D^{\alpha}f(t)\}=T\{D^{-(n-\alpha)}D^{n}f(t)\}$$
$$=q^{-(n-\alpha)}(s)T\{D^{n}f(t)\}$$
$$=q^{-(n-\alpha)}(s)\left[q^{n}F(s)-p(s)\left(\sum_{k=0}^{n-1}q^{n-k-1}(s)f^{k}(0)\right)\right]$$
$$=q^{-(n-\alpha)}(s)q^{n}F(s)-p(s)\left(\sum_{k=0}^{n-1}q^{-(n-\alpha)}(s)q^{n-k-1}(s)f^{k}(0)\right)$$
$$=q^{\alpha}(s)F(s)-p\left(s\sum_{k=0}^{n-1}q^{\alpha-k-1}(s)f^{k}(0)\right)$$

3.2 Adomian Decomposition Method

The basic principle of the Adomian decomposition method is as follows

Consider a non linear differential equation of the form

$$Lu(t)+Ru(t)+Nu(t)=f(t)\quad\ldots(11)$$

Where L is an invertible linear operator (of the highest order), R is the remaining linear operators, N represents nonlinear operators and f is the source term.

If L is a linear operator of order $n+1$, we denote L^{-1} as the inverse operator of L and

$$L^{-1}Lu(t)=u(t)-\sum_{i=0}^{n}\tfrac{1}{i!}t^{i}u^{i}(0)\quad\ldots(12)$$

Solving for $u(t)$ by applying L^{-1} to (11)

$$L^{-1}Lu(t)+L^{-1}Ru(t)+L^{-1}Nu(t)=L^{-1}f(t)\quad\ldots(13)$$

$$u(t)=L^{-1}f(t)+\sum_{i=0}^{n}\tfrac{1}{i!}t^{i}u^{i}(0)-L^{-1}Ru(t)-L^{-1}Nu(t)$$

The Adomian decomposition method suggest that the solution $u(t)$ be decomposed into an infinite series

$$u(t)=\sum_{i=0}^{\infty}u_i(t) \qquad \ldots(14)$$

And the non linear terms by infinite series of the Adomian polynomial given by

$$Nu(t)=\sum_{i=0}^{\infty}A_i(t) \qquad \ldots(15)$$

where $A_i(t)=\frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i(t)\right)\right]_{\lambda=0}, i=0,1,2,\ldots$

$$\ldots(16)$$

the first few terms are given as follows

$$A_0=f(u_0)$$
$$A_1=f'(u_0)u_1$$
$$A_2=f'(u_0)u_2+\tfrac{1}{2!}f''(u_0)u_1^2 \qquad \ldots(17)$$
$$A_3=f'(u_0)u_3+\tfrac{2}{2!}f''(u_0)u_1u_2+\tfrac{1}{3!}f'''(u_0)u_1^3$$

From (14), the solution of the equation (11) can be written as

$$u(t)=\sum_{i=0}^{\infty}u_i(t)$$
$$=L^{-1}f(t)+\sum_{i=0}^{n}\tfrac{1}{i!}t^i u^i(0)-L^{-1}\sum_{i=0}^{\infty}Ru_i(t)-L^{-1}\sum_{i=0}^{\infty}A_i(t)$$

$$\ldots(18)$$

The solution component of $u(t)$ are then determined recursively as

$$u_0(t)=L^{-1}f(t)+\sum_{i=0}^{n}\tfrac{1}{i!}t^i u^i(0)$$
$$u_1(t)=L^{-1}Ru_0(t)-L^{-1}A_0(t)$$
$$u_2(t)=L^{-1}Ru_1(t)-L^{-1}A_1(t)$$
$$\vdots$$
$$u_{i+1}(t)=L^{-1}Ru_i(t)-L^{-1}A_i(t)$$

3.3 THE GENERAL INTEGRAL TRANSFORM DECOMOSITION METHOD (GITDM)

In this section, we discuss the general integral transform decomposition method (GITDM) for solving linear and non linear fractional differential equations

Consider the fractional differential equation of the form

$$D^{\alpha}y(t)+Ry(t)-Ny(t)=f(t), \quad n-1<\alpha\le n, n\in N$$

$$\ldots(20)$$

With initial conditions

$$y^k(0)=c_k, \; k=1,2,\cdots n-1 \qquad \ldots(21)$$

Where D^{α} is the Caporto fractional derivative of order α, R and N are linear and non linear operators respectively and $f(t)$ is the source term.

The general integral transform decomposition method requires applying the general integral transform to both sides of equation (20)

$$T\{D^{\alpha}y(t)\}+T\{Ry(t)\}+T\{Ny(t)\}=T\{f(t)\} \qquad \ldots(22)$$

Using the differentiability property (6) and initial conditions (21)

$$q^{\alpha}(s)T\{y(t)\}-p(s)\sum_{0}^{n-1}q^{\alpha-k-1}(s)c_k=T\{f(t)\}-T\{Ry(t)\}$$
$$-T\{Ny(t)\}$$

$$\ldots(23)$$

Operating the inverse of general integral transform on both sides of (23)

$$y(t)=T^{-1}\left[q^{-\alpha}(s)T\{f(t)\}+q^{-\alpha}(s)p(s)\sum_{0}^{n-1}q^{\alpha-k-1}(s)c_k\right]$$
$$-T^{-1}\left[q^{-\alpha}(s)T\{Ry(t)+Ny(t)\}\right]$$

$$\ldots(24)$$

The GITDM describes the solution $y(t)$ by the infinite series

$$y(t)=\sum_{0}^{\infty}y_i(t) \qquad \ldots(25)$$

And the non linear terms $Ny(t)$ is decomposed into the Adomian polynomials

$$Ny(t)=\sum_{0}^{\infty}A_i(t) \qquad \ldots(26)$$

Applying (25) and (26) into (24)

$$\sum_{0}^{\infty}y(t)=T^{-1}\left[q^{-\alpha}(s)T\{f(t)\}+q^{-\alpha}(s)p(s)\sum_{0}^{m-1}q^{\alpha-k-1}(s)c_k\right]$$

$$-T^{-1}\left[q^{-\alpha}(s)T\left\{R\sum_{0}^{\infty}y_i(t)+\sum_{0}^{\infty}A_i\right\}\right] \qquad (27)$$

From (27) we deduce the following recursive formula

$$y_0(t)=T^{-1}\left[q^{-\alpha}(s)T\{f(t)\}+q^{-\alpha}(s)p(s)\sum_{0}^{m-1}q^{\alpha-k-1}(s)c_k\right]$$

$$y_{i+1}=-T^{-1}\left[q^{-\alpha}(s)T\left\{R\sum_{0}^{\infty}y_i(t)+\sum_{0}^{\infty}A_i\right\}\right] \qquad \ldots(28)$$

From (28) we can obtain approximate solution of (20) as

$$y(t) \approx \sum_{0}^{k} y_i(t) \text{, where } \lim_{k\to\infty} \sum_{0}^{k} y_i(t) = y(t) \qquad \ldots(29)$$

4.0 NUMERICAL RESULTS

EXAMPLE 1: Consider the linear fractional differential equation

$$y''(t) + D^{\alpha} y(t) + y(t) = 8 \;,\;\; t > 0, 1 < \alpha \le 2 \qquad \ldots(30)$$

Subject to the initial condition

$$y(0) = y'(0) = 0 \qquad \ldots(31)$$

The exact solution of $(30)$when $\alpha = 1$ is given as

$$y(t) = 8\left(1 - e^{-\sqrt{3}/2}\left(\cos\tfrac{\sqrt{3}}{2}t + \tfrac{1}{\sqrt{3}}\sin\tfrac{\sqrt{3}}{2}t\right)\right) \qquad \ldots(32)$$

Applying the GITDM to $(30)$and utilizing the initial condition(31), we obtain the following iterations.

$$y_0 = \frac{8t^2}{\Gamma(3)}$$

$$y_1 = -\frac{8t^4}{\Gamma(5)} - \frac{8t^{4-\alpha}}{\Gamma(5-\alpha)}$$

$$y_2 = \frac{8t^6}{\Gamma(7)} + \frac{16t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{8t^{6-2\alpha}}{\Gamma(7-2\alpha)}$$

$$y_3 = \frac{8t^8}{\Gamma(9)} - \frac{24t^{8-\alpha}}{\Gamma(9-\alpha)} - \frac{24t^{8-2\alpha}}{\Gamma(9-2\alpha)} - \frac{8t^{8-3\alpha}}{\Gamma(9-3\alpha)} \qquad \ldots(33)$$

Table 1 shows the exact solution and approximate solution of (30) using GIADM. The results are compared with the result obtained using Adomian decomposition method (ADM) and variational iteration method (VIM).

Table 1: numerical solution for example1

t	ADM	VIM	GITDM	EXACT
0.0	0.000000	0.000000	0.000000	0.000000
0.1	0.039874	0.039874	0.039750	0.039750
0.2	0.158512	0.158512	0.157036	0.157036
0.3	0.353625	0.353625	0.347370	0.347370
0.4	0.622083	0.622083	0.604695	0.604695
0.5	0.960047	0.960047	0.921766	0.921768
0.6	1.363093	1.363093	1.290448	1.290457
0.7	1.826257	1.826257	1.701978	1.702008
0.8	2.344224	2.344224	2.147195	2.147287
0.9	2.911278	2.911278	2.616753	2.617001
1.0	3.521462	3.521462	3.101303	3.101906

From the numerical result, it is clear that with few iterations, the approximate solution obtained using GIADM are in high agreement with the exact solution. The efficiency can be further enhanced by computing more terms.

EXAMPLE 2: Consider the non linear fractional differential equation

$$D^{\alpha} y(t) + y^2(t) = 1,\; 0 < \alpha \le 1 \qquad \ldots(34)$$

Subject to the initial condition

$$y(0) = 0 \qquad \ldots(35)$$

The exact solution of (34) when $\alpha = 1$is

$$y(t) = \frac{1 - e^{2t}}{1 + e^{2t}} \qquad \ldots(36)$$

Applying the GITDM to (34) and utilizing the initial conditions given in(35), we obtain the following iterations

$$y_0 = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$y_1 = -\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}$$

$$y_2 = \frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}$$

$$y_3 = -\frac{4\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)} + \frac{\Gamma(2\alpha+1)^2\Gamma(6\alpha+1)t^{7\alpha}}{\Gamma(\alpha+1)^4\Gamma(3\alpha+1)^2\Gamma(7\alpha+1)} \qquad \ldots(37)$$

Table 2 shows the exact solution and approximate solution of (30) using GIADM. The results are compared with the result obtained using homotppy perturbation method (HPM) and variational iteration method (VIM).

Table2: **numerical result for example2**

t	VIM	MHPM	GITDM	EXACT
0	0	0	0	0
0.1	0.099667	0.099668	0.099668	0.099668
0.2	0.197375	0.197375	0.197375	0.197375
0.3	0.291312	0.291312	0.291312	0.291313
0.4	0.379946	0.379944	0.379944	0.379949
0.5	0.462103	0.462078	0.462078	0.462117
0.6	0.536983	0.536857	0.536857	0.53705
0.7	0.604124	0.603631	0.603631	0.604368
0.8	0.663300	0.661706	0.661706	0.664037
0.9	0.710023	0.709919	0.709919	0.716298
1	0.757166	0.746032	0.746032	0.761594

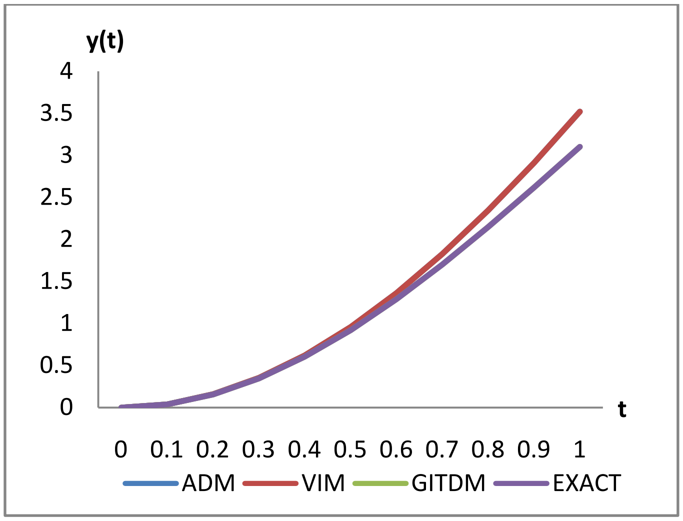

Figure1: plot showing approximate solution of (30) in compararism with the exact solution and results obtained from other methods.

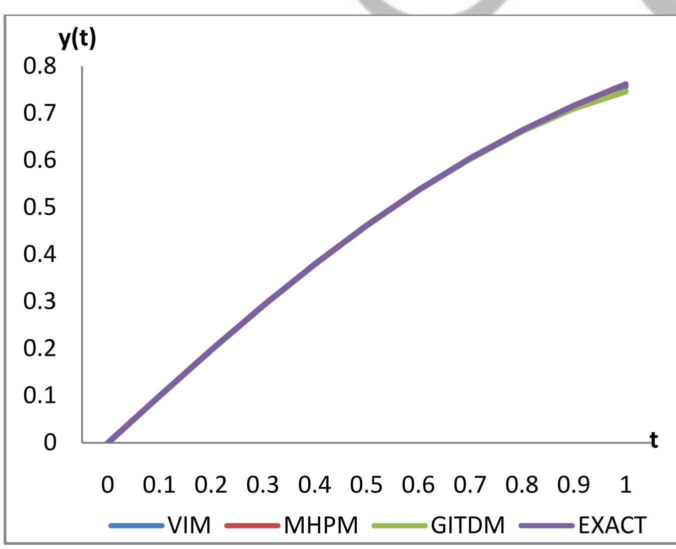

Figure 2: plot showing the approximate solution of (34) in comparism with the exact solution and results obtaining

5.0 CONCLUSION

The general integral transform decomposition method (GITDM) has been successfully implemented to obtain approximate solution of linear and nonlinear fractional differential equation. The method is seen to be efficient, easy to implement, and has shown remarkable performance by producing result that is in high agreement with exact solution and solutions obtained from other methods.

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