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# A ONE STEP METHOD FOR THE SOLUTION OF OSCILLATORY AND EXPONENTIAL ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT:

This work presents a One Step Numerical Method for the direct solution of general first order Ordinary Differential Equations. The formulawas developed using the interpolation techniques as basis function and augmented by adopting the improved Euler – Cauchy approach to solve some Initial Value Problemsof Oscillatory and Exponential Ordinary Differential Equations. Accuracy of the method was tested with numerical examples and the results showed a good performance better than others. The method results were compared with Cauchy – Euler result and error analysis was computed.

KEYWORDS: Numerical Formulae, Interpolation, Oscillatory, Exponential and Basis Function

## 1.0 INTRODUTION

In this paper, we present a simpler method for computing numerical solution to the initial value problems

$$y' = f(x,y), y(x_o) = y_o \tag{1}$$

and estimating the errors of the computed solutions using a basis function.

Theorem 1. [2],[4]

Let $f(x)$ be a continuous function of $x$ and $y$. A function $y(x)$ that is continuous on an open interval $I$ which contains $x_o$ is a solution of the Initial value problem (1) if and only if $y(x)$ is a solution of the integral equation

$$y(x) = x_o + \int_{x_o}^{x} f\big(x, y(x)\big)dx \tag{2}$$

Proof:

Suppose that $y(x)$ satisfies the initial value problem (1) on an interval $I$ containing $x_o$, since $y(x)$ satisfies the differential equation $y(x) = f\big(x, y(x)\big)$ on $I$, $y(x)$ is a continuous function of $x$ on $I$. $f\big(x, y(x)\big)$ is also a continuous function of $x$ on $I$, since $f(x,y)$ is a continuous function of $x$ and $y$. Consequently, $f\big(x, y(x)\big)$ is integrable on $I$. Integrating the differential equation $y(x) = f\big(x, y(x)\big)$, we obtain

$$y(x) - y(x_o) = \int_{x_o}^{x} y(x)dx = \int_{x_o}^{x} f\big(x, y(x)\big)dx \quad (3)$$

For $x \in I$. Imposing the initial condition $y(x_o) = x_o$, hence, $y(x)$ is the solution of the initial value problem (1) for $x \in I$, then $y(x)$ satisfies the integral equation (2) for $x \in I$. Now suppose that $y(x)$ is a continuous function of $x$ on some interval $I$ containing $x_o$ and that $y(x)$ satisfies the integral equation (2). Substituting $x = x_o$ into the integral equation (2), we see that $y(x)$satisfies the initial condition $y(x_o) = x_o$. Differentiating the integral equation (2), we find that$y(x)$ satisfies the differential equation (1) for $x \in I$. So if $y(x)$ satisfies the integral equation (2) for $x \in I$, $y(x)$ satisfies the initial value problem (1) for $x \in I$.

Applying this theorem, it is established that the solution of initial value problems (1) on the interval $(x_n, x_{n+1})$ is (3). Many scholars [1], [3] have solved the initial value problems using various methods with the aim of finding the value of the approximate solution $y_{n+1}$which depends on the solution $y(x)$ on the interval $(x_n, x_{n+1})$ and the function $f$ in the set

$$S = \{(x, y)| x_n \le x \le x_{n+1},\ y(x)| x_n \le x \le x_{n+1} \quad (4)$$

## 2.0 THE DERIVATIVE METHOD

### 2.1 Derivation of the Numerical Method

Consider the basis function is of the type

$$F(x) = Ae^{\lambda x} + Bx^2 + Cx + D \quad (5)$$

Which is oscillatory and exponential function, where $A, B, and\ C$ are determined coefficients and $D$ is a constant. $\lambda$ is varies to obtained the robustness of the derived Numerical Method.

Let $F^{'}(x) = f_n$, $F^{''(x)} = f_n^1$, $F^{'''(x)} = f_n^2$,

Then, the three consecutives derivatives give

$$f_n = A\lambda e^{\lambda x} + 2Bx + C \quad (6)$$

$$f_n^1 = A\lambda^2 e^{\lambda x} + 2B \quad (7)$$

$$f_n^2 = A\lambda^3 e^{\lambda x} \quad (8)$$

Through expansion from (8), (7) and (6)

$$A = \frac{f_n^2}{\lambda^3 e^{\lambda x}},\ B = \frac{f_n^1}{2} - \frac{f_n^2}{2\lambda},\ C = f_n - \frac{f_n^2}{\lambda^2} - x\left[f_n^1 - \frac{f_n^2}{\lambda}\right] \quad (9)$$

Basically, it is a known fact that

$$F(x_{n+1}) - F(x_n) = y_{n+1} - y_n$$

Hence,

$$y_{n+1} - y_n = \left(Ae^{\lambda x_{n+1}} + Bx_{n+1}^2 + Cx_{n+1} + D\right) - \left(Ae^{\lambda x_n} + Bx_n^2 + Cx_n + D\right)$$

$$= A\left(e^{\lambda x_{n+1}} - e^{\lambda x_n}\right) + B(x_{n+1}^2 - x_n^2) + C(x_{n+1} - x_n) \quad (10)$$

By simplification,

$\left(e^{\lambda x_{n+1}} - e^{\lambda x_n}\right) = e^{\lambda x_n}(e^{\lambda h} - 1)$ , $(x_{n+1}^2 - x_n^2) = (2n+1)h^2$ and $(x_{n+1} - x_n) = h$

Therefore, from $x_n = a + nh$,

$$y_{n+1} - y_n = \frac{f_n^2}{\lambda^3}\left(e^{\lambda h} - 1\right) + \left(\frac{f_n^1}{2} - \frac{f_n^2}{2\lambda}\right)(2n+1)h^2 + \left(f_n - \frac{f_n^2}{\lambda^2} - x\left[f_n^1 - \frac{f_n^2}{\lambda}\right]\right)h$$

and we now have the numerical scheme

$$y_{n+1} = y_n + hf_n + \tfrac{1}{2}h^2 f_n^1 - \tfrac{1}{\lambda^3} f_n^2\left(1 + \tfrac{1}{2}h^2\lambda^2 + h\lambda - e^{\lambda h}\right) \quad (11)$$

The existence and uniqueness of the solution of the initial value problem (1) is guaranteed by the following theorem:

Theorem 2 [3]

We assume that $f(x, y)$ satisfies the following conditions:

(i) $f(x, y)$ is a real function

(ii) $f(x, y)$ is defined and continuous in the strip $x \in [a, b]$ , $y \in (-\infty, \infty)$

(iii) there exists a constant L such that for any $x \in [a, b]$ and any $v_1$ and $v_2$, then

$|f(x, v_1) - f(x, v_2)| \le L|v_1 - v_2|$ where L is called the Lipschitz constant. Then for any $y_o$, the initial value problem (1) has a unique solution $y(x)$ for $x \in [a, b]$.

The proof to this theorem is all over the textbooks on Numerical Methods. [1, 2]

However, the properties of the Numerical Scheme (11) to establish the effectiveness, and robustness of the method is done and will be in the next paper.

## 3.0 IMPLEMENTATION OF THE METHODS

### 3.1 The Implementation of the Numerical Scheme (11) on First Order ODE

Numerical Scheme (11) was tested on some problems and the results are shown below:

Problem 1. $y' = y,\ y(0) = 1, 0 \le x \le 1, \lambda = 1$, which is autonomous

Hence, we have the result

Table 1: Results of problem 1, for $h = 0.1$

| $X_n$ | Method (11) | Exact | Errors |
|---|---|---|---|
| 0.00 | 1.000000000000000 | 1.000000000000000 | 0.000000000000000000 |

| | | | |
|---|---|---|---|
| 0.10 | 1.105170918075648 | 1.105170918075648 | 0.0000000000000000 |
| 0.20 | 1.221402758160170 | 1.221402758160170 | 2.220446049250313e-16 |
| 0.30 | 1.349858807576003 | 1.349858807576003 | 2.220446049250313e-16 |
| 0.40 | 1.491824697641271 | 1.491824697641270 | 4.440892098500626e-16 |
| 0.50 | 1.648721270700129 | 1.648721270700128 | 6.661338147750939e-16 |
| 0.60 | 1.822118800390510 | 1.822118800390509 | 8.881784197001252e-16 |
| 0.70 | 2.013752707470478 | 2.013752707470477 | 1.332267629550188e-15 |
| 0.80 | 2.225540928492469 | 2.225540928492468 | 1.332267629550188e-15 |
| 0.90 | 2.459603111156951 | 2.459603111156950 | 1.332267629550188e-15 |
| 1.00 | 2.718281828459047 | 2.718281828459046 | 1.332267629550188e-15 |

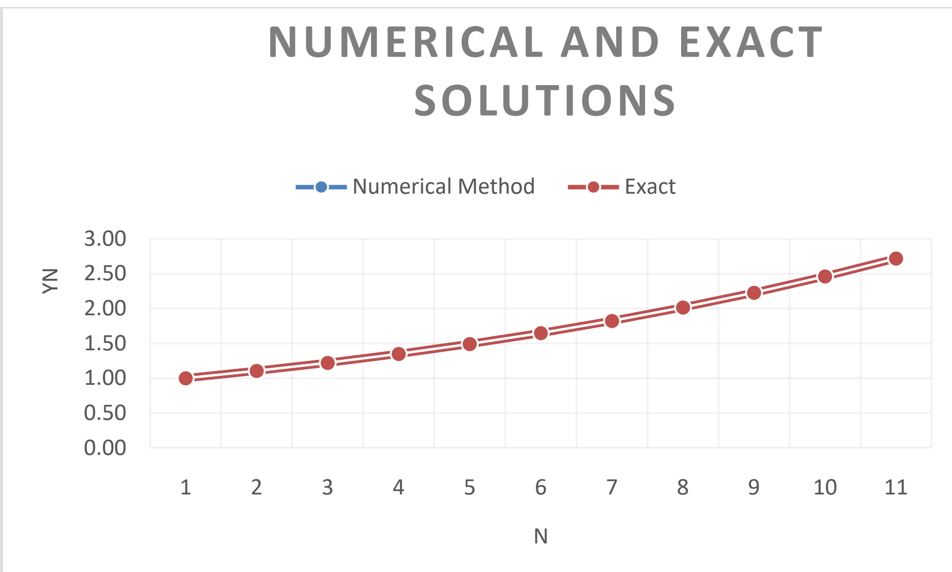


Figure 1: Graphical representation of problem 1 results on table 1 showing the convergence of the method.

Problem 2. $y^{'} = x + y,\ y(0) = 1$, $0.0 \leq x \leq 1.0$, $\lambda = 1$, $h = 0.1$, which is non - autonomous

Hence, we have the result

Table 2: Results of problem 2, for $h = 0.1$

| $X_n$ | Method (11) | Exact | Errors |
|---|---|---|---|
| 0.00 | 1.000000000000000 | 1.000000000000000 | 0.0000000000000000000 |
| 0.10 | 1.110341836151296 | 1.110341836151295 | 2.220446049250313e-16 |
| 0.20 | 1.242805516320340 | 1.242805516320340 | 8.881784197001252e-16 |
| 0.30 | 1.399717615152007 | 1.399717615152007 | 6.661338147750939e-16 |
| 0.40 | 1.583649395282542 | 1.583649395282541 | 1.332267629550188e-15 |
| 0.50 | 1.797442541400258 | 1.797442541400256 | 1.554312234475219e-15 |
| 0.60 | 2.044237600781020 | 2.044237600781018 | 2.220446049250313e-15 |
| 0.70 | 2.327505414940955 | 2.327505414940953 | 2.220446049250313e-15 |
| 0.80 | 2.651081856984938 | 2.651081856984936 | 2.220446049250313e-15 |
| 0.90 | 3.019206222313903 | 3.019206222313899 | 3.552713678800501e-15 |
| 1.00 | 3.436563656918095 | 3.436563656918091 | 3.552713678800501e-15 |

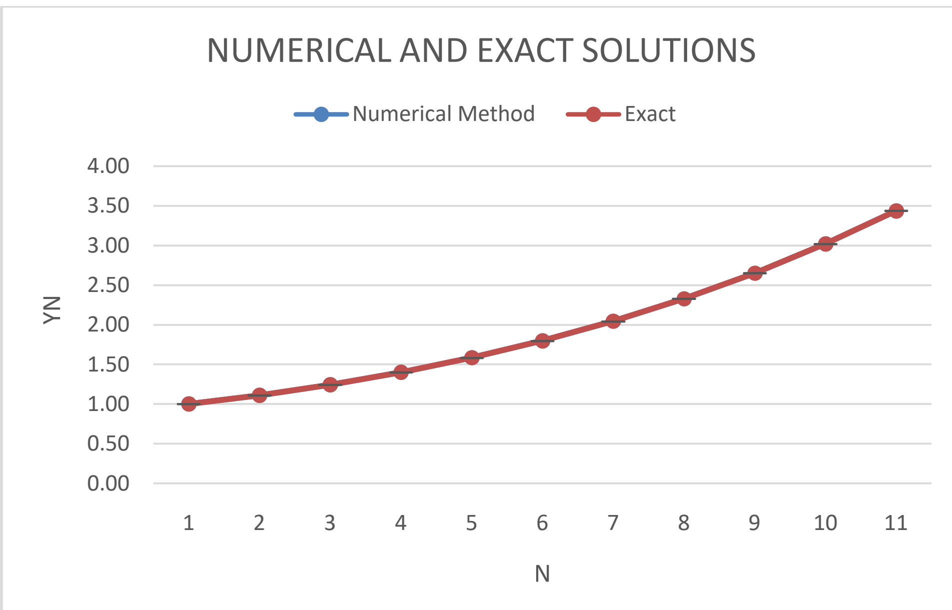


Figure 3: Graphical representation of problem 2 results on table 2 showing the convergence of the method.

### 3.1.1Comparison of Results

The result of problem$y' = x + y,\ y(0) = 1$, $0.0 \leq x \leq 1.0$, $\lambda = 1$, $h = 0.1$, of Method(11) is compared with Euler – Cauchy method, which clearly revealed that method (11) is better

Table 3: Comparison of results between Derived Method and Euler – Cauchy Method

| $X_n$ | Method (11) | Euler- Cauchy | Exact |
|---|---|---|---|
| 0.00 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.10 | 1.1103418 | 1.1100000 | 1.1103418 |
| 0.20 | 1.2428055 | 1.2425750 | 1.2428055 |
| 0.30 | 1.3997176 | 1.3996268 | 1.3997176 |
| 0.40 | 1.5836493 | 1.5837303 | 1.5836493 |
| 0.50 | 1.7974425 | 1.7977322 | 1.7974425 |
| 0.60 | 2.0442376 | 2.0447791 | 2.0442376 |
| 0.70 | 2.3275054 | 2.3283485 | 2.3275054 |
| 0.80 | 2.6510818 | 2.6522840 | 2.6510818 |
| 0.90 | 3.0192062 | 3.0208337 | 3.0192062 |
| 1.00 | 3.4365636 | 3.4386926 | 3.4365636 |

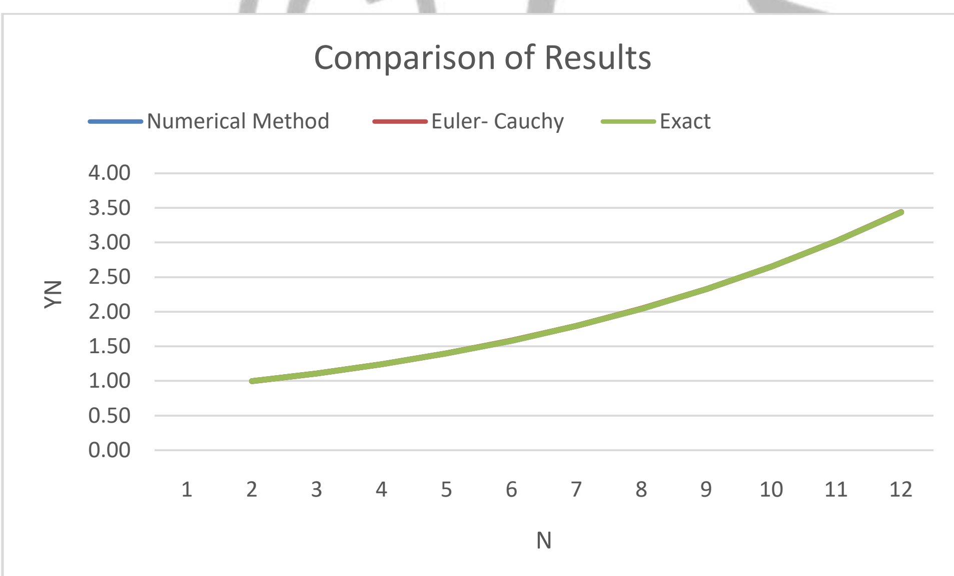


Figure 3: The graphical representation of problem 2 results on table 3 showing the comparison of results of convergence of the method.

3.1.2 Error Analysis of the New Method and Euler – Cauchy Method

Table 4: Error analysis of results of problem 2 in comparing derived method (11) with Euler – Cauchy method

| Xn | Error Method (11) | Error Euler |
|---|---|---|
| 0.00 | 0.0000000000E+00 | 0.0000000 |
| 0.10 | 2.2204460493E-16 | 0.0003418 |
| 0.20 | 8.8817841970E-16 | 0.0002305 |
| 0.30 | 6.6613381478E-16 | 0.0000908 |
| 0.40 | 1.3322676296E-15 | 0.0000809 |
| 0.50 | 1.5543122345E-15 | 0.0002897 |
| 0.60 | 2.2204460493E-15 | 0.0005415 |
| 0.70 | 2.2204460493E-15 | 0.0008431 |
| 0.80 | 2.2204460493E-15 | 0.0012021 |
| 0.90 | 3.5527136788E-15 | 0.0016275 |
| 1.00 | 3.5527136788E-15 | 0.0021289 |

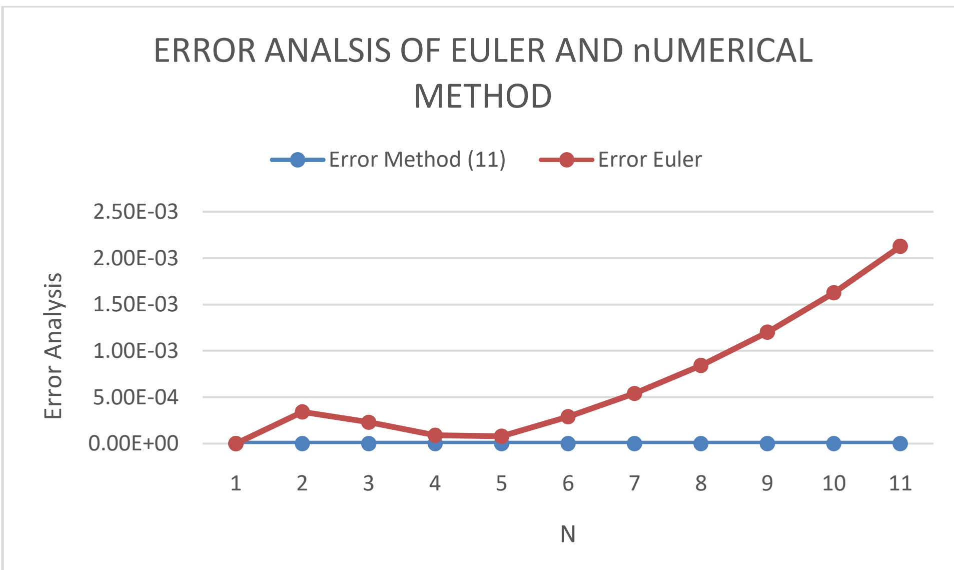


Figure 4: Graphical representation of problem 1 results on table 4 showing the error analysis of the comparison in table 4 of convergence of the method.

From this, with other examples, it shows the method converges faster which indicates that the method (11) is effective and robust with the properties.

## 4.0 CONCLUSION AND RECOMMENDATION

We have presented a One Step Numerical Formula for the direct solution of general first order Ordinary Differential Equations. The formula wasdeveloped using the interpolation techniques as basis function and augmented by adopting the improved Euler – Cauchy method approach for first order ODE problems. The accuracy of the method was tested with numerical examples and the results showed a good performance.

## 4.0 References

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