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A STUDY ON WEDDERNBURN – ARTIN THEOREM FOR RINGS

Fahad Suleiman, Sammani Abdullahi

Department of Mathematics & Statistics

School of Science & Technology

Federal Polytechnic Kaura Namoda

Zamfara State – Nigeria

+2348037346623 & +2347068706358

fssalai@gmail.com & dagasammaniabdullahi@gmail.com

Abstract: The study depicts that a Wedderburn Artin – theorem for rings is considered to be a semisimple ring R which is isomorphic to a product of finitely many $m_i by m_i$ matrix rings over division rings D_i, for some integers n_i, both of which are uniquely determined up to permutation of the index *i*.

Keywords: Commutativity, Ideal, matrix, module, ring, Semi simple ring, Wedderburn – Artin theorem.

1. INTRODUCTION

According to Ref. [1], the subject of determining structure of rings and algebras, over which all modules are direct sum of certain cyclic modules has a long a long history. [1] showed that every module over a finite dimensional K – algebra A is a direct sum of simple modules if and only if $A \cong \prod_{i=1}^{m} M_{n_i}(D_i)$ where $m, n_{1,\ldots\ldots,\in N}$ and each D_i is finite dimensional division algebra over k.

It has been observed from the study of [2] that the first wedderburn Artin theorem has two parts one dealing with finite simple algebra. We let R be a ring with identity. The ring R is called left semisimple ring if the left R module is a semisimple module, i.e, if R is a direct sum of minimal left ideals. In this case $R = \bigoplus_{i \in S} I_i$.

Examples of semisimple rings were stated in [4] as:

i. If D is a division ring, then the ring $R = M_n(D)$is left semisimple in the of the definition of Wedderburn Artin theorem. It revealed that R as a left R module is given by $M_n(D) \cong D^n \ldots \oplus D^n$, where each D^n is a simple left $M_n(D)$ module and the j^{th} summand D^n corresponds to the matrices whose only non – zero entries are in the j^{th} column. The left R module $M_n(D)$ has a composition series whose terms are partial sumsof the n summandsD^n.

ii. If $R_1, \ldots R_n$ are left semisimple rings, then the direct product $R \cong \prod_{i=1}^{n} R_i$. Each minimal left ideal of R_i, when included into R, is a minimal left ideal of R. Hence R is the sum of minimal left ideals and is left semisimple.

2. MAIN RESULT

The main result of this paper is given below

Theorem 2.1: If R is any left semisimple ring, then

$$R \cong M_{n_i}(D_1) \times \ldots \times M_{n_r}(D_r)$$

Where each D_i is a division ring and $M_n(D)$ denotes the ring of $n \times n$ matrices over D

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Theorem 2.2: Let R be a finite dimensional algebra with identity over field F. If R is a semisimple ring, then R is a semisimple and hence is isomorphic to $M_n(D)$ for some integer $n \geq 1$ and some finite dimensional algebra D over F. the integer n is uniquely determined by R, and D is unique upto isomorphic.

In order to obtain our main result, we begin prove the above stated theorems.

Proof of theorem 2.1: By the work of [2] we write R as direct sum of minimal left ideals, and then regroup the summand according to their R isomorphic type as $R \cong \oplus_{j=1}^{r} n_j V_j$, where $n_i V_j$ is the direct sum of n_j submodules R isomorphic to V_j and where $V_i \ncong V_j$ for $i \neq j$. The isomorphison is one of unital left R modules. Now put $D_i^o = End_{R(V_i)}$. This is a division ring by Schur's Lemma as it was proved in [5] research.

Using proposition 10.14 of [2], we obtain an isomorphism of rings of rings

$$R^0 \cong End_R R \cong Hom_R\left(\oplus_{i=1}^{r} n_i V_i, \oplus_{j=1}^{r} n_j V_j\right) \qquad 2.1$$

Define a map $p_i : \oplus_{j=1}^{r} n_j V_j \to n_i V_i$ to be the i^{th} projection and $q_i : n_i V_i \to \oplus_{j=1}^{r} n_j V_j$ to be the i^{th} inclusion. Let us see that the right side of 2.1 is isomorphic as a ring to $\prod_i End_{R(n_i V_i)}$ via mapping $f \mapsto (p_1 f q_1, \ldots p_r f q_r)$. What is to be presented here is that $p_j f q_j = 0 for i \neq j$. Now $p_j f q_i$ is a member of $Hom_R\left(n_i V_i, n_j V_j\right)$.

Referring to (2.1) above, we therefore obtain

$$R^0 \cong \prod_{i=1}^{r} Hom_R\left(n_i V_i, n_j V_j\right) = \prod_{i=1}^{r} End_{R(n_i V_i)}$$

$$\cong \prod_{i=1}^{r} M_{n_i}\left(End_{R(V_i)}\right) \qquad \text{by corollary 10.13 of [3]}$$

$$\cong \prod_{i=1}^{r} M_{n_i}(D_i^o) \qquad \text{by definition of } D_i^o$$

Reversing the order of multiplication in R^0 and using transpose map to reverse the order of multiplication in each $M_{n_i}(D_i^o)$, we conclude that $R \cong \prod_{i=1}^{r} M_{n_i}(D_i)$. This proves the existence of the decomposition in the given theorem of the main result. We still need to identify the simple left R module and prove its an appropriate unique statement as contained in [2]. Recalled, in example (i) above, we have decomposition $M_n(D_i) \cong D_i^{n_1} \ldots \oplus D_i^{n_i}$ of left $M_{n_i}(D_i)$ modules, and each term $D_i^{n_i}$ is a simple left $M_{n_i}(D_i)$ module. The decomposition proved allows the researchers to regard each term $D_i^{n_i}$ as simple left R module, $1 \leq i \leq r$. Each of these modules is acted upon by a different coordinate of R, and hence we have projected at least r non – isomorphic simple left R modules as in [3]. The researcher added that any simple R module must be a quotient of R by a maximal left ideal, as we observed in example (ii) hence a composition factor as consequences of the Jordan – Holder theorem in [6]. There are only r non - isomorphic such $V_{j's}$, and we conclude that the number of simple left R modules, up to isomorphism, is exactly r.

For uniqueness, we consider [3] which opined that supposed that $R \cong M_{n_1'}(D_i) \times \ldots \times M_{n_s'}(D_s)$ as rings. Let $V_j' = \left(D_j'\right)^{n_j'}$ be the unique simple left $M_{n_j'}\left(D_j'\right)$ module up to isomorphison, and regard V_j' as a simple left R module. Then we have $R \cong \oplus_{j=1}^{s} n_j' V_j'$ as left R modules. By the Jordan Holder Theorem we must have r=s and, after a suitable numbering, $n_i = n_i'$ and $V_i \cong V_i'$ for $1 \leq i \leq r \leq$. Thus we have ring isomorphison

$$\left(D_j'\right)^0 \cong End_{M_{n_i'}(D_i')} \qquad \text{by lemma 2.1 of [3]}$$

$$\cong End_{R(V_i')}$$

$$\cong End_{R(V_i)} \qquad \text{since } V_i \cong V_i'$$

$$\cong D_i^0$$

Reversing the order of multiplication gives $D_i' \cong D_i$, and hence proved.

Proof of theorem 2.2: from the work of [2] and by finite dimensionality, R has a minimal left ideal V. For *r* in *R*, form the set Vr . This is a left ideal, and we claim that it is minimal or is 0. In fact, the function $v \to vr$ is R linear from V onto Vr . Since V is simple as a left *R* module, *Vr* is simple or 0. The sum $I = \sum_r withr \neq 0\, Vr$ is a two-sided ideal in R, and it is not 0 because $V1 \neq 0$. Since R is simple, $I = R$. Then the left R module R is exhibited as the sum of simple left *R* modules and is therefore semisimple. The isomorphism with $M_n(D)$ and the uniqueness now follow from Theorem 2.1 above.

3. CONCLUSION

The prove yield a very good main result in which it shows that a ring R with unity is semisimple, or left semisimple to be precise, if the free left R-module underlying R is a sum of simple R-module. In his short prove of Wedderburn's theorem, [7] states that when R is simple the Wedderburn – Artin theorem is known as Wedderburn's theorem.

REFERNCES

[1] Behhoodi, M., Daneshvar, A. & Vedadi, M.R. (2016). Two Generalizations of the Wedderburn-Artin Theorem with Applications. *Algebras and Representation Theory*, Vo. 21 No.1, pp 1 – 18.

[2] Knapp, A.W. (2016). Advanced Algebra. New York: East Setauket

[3] Jacobson, Nathan (1989), Basic algebra II (2nd ed.), W. H. Freeman, Lam,

[4] Tsit-Yuen (2001), A First Course in Noncommutative Rings (2nd ed.), Berlin, New York: Springer-Verlag, doi:10.1007/978-1-4419-8616-0,

[5] Kostov, V.P. (1995). A Generalization of the Burnside Theorem and Of Schur's Lemma For Reducible Representations. *Journal of Dynamical and Control Systems,* Vol. 1 No. 4, Pp 551 – 580. DOI: 10.1007/s10468-017-9748-2

[6] Natale, S. (2015). Jordan-Holder Theorem for Finite Dimensional ¨ Hopf Algebras. *Proceedings Of The American Mathematical Society, Vol. 143 No. 12.* Pp 5195 – 5211.

[7] Nicholson, W.K. (1993). A Short Proof of the Wedderburn Artin Theorem. *New Zealand Journal of Mathematics,* Vo. 22 No. 1. Pp 83 – 86.