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A Trigonometrically Fitted Predictor-Corrector Method for solving Oscillatory Second Order Ordinary Differential Equations

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Abstract

A predictor-corrector class of Continuous Trigonometrically-Fitted Method for Solving Oscillatory (CTMSO) Second Order Ordinary Differential Equations in this research paper is developed . The method coefficients is proportional to the approximate solution frequency and step size. The CTMSO generates a discrete trigonometrically-fitted second order ordinary differential equation as a by-product. The main predictors needed for the evaluation of the implicit methods are obtained to be of the same order with the method at whatever point of collocation. The method stability qualities are described, and the method usefulness and efficiency are demonstrated by solving linear and nonlinear initial value oscillatory problems.

Keywords: Linear multistep,interpolation techniques,Trigonometric-fitting, predictor-corrector.

1 Introduction

The numerical solution of the second order initial value problem is examined.

$$y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0, x\epsilon[a, b] \quad (1)$$

Equation (1) is the result of a variety of physical processes, across a wide range of applications notably in engineering, such as the movement of a vehicle either a rocket or a satellite, electric circuit, fluid dynamic as well as other areas of application, it is well-thought out that this type of equation can be solved directly or indirectly or by converting the problem to a set of first-order differential equations before attempting to address the problem using any of the available techniques Chan et al. (2014), Gholamtabar Lambert (1973), Kayode and Adegboro (2018). Other methods based on exponential fitting techniques have been developed (see Simos (1998a, 2002), Van de Vyver (2005a), Van de Vyver (2006b), Monovasilis et al. (2013), and Nguyen et al. 2007). The exponentially-fitted approaches are motivated by the idea that if the frequency, or a good estimate of it, is known ahead of time, these methods will be more advantageous than polynomial-based methods. Ngwane and Jator (2014) created a continuous trigonometrically-fitted second derivative approach whose coefficients are dependent on frequency and stepsize, and the method is built with trigonometric basis functions. Numerical experiments show that the method for numerically solving ordinary differential equations with oscillatory solutions is effective. The CTMSO presented in this study, on the other hand, avoids the computation of higher order derivatives, which can increase computing cost, particularly when applied to nonlinear systems. We propose a CTMSO of order 4 in this study, and its application is extended to oscillatory issues. The following is a breakdown of how this article is structured. Derivation of the CTMSO for solving the problem in Section 2. Section 3 delves into the CTMSO's analysis and execution. Section 4 provides numerical examples to demonstrate the CTMSO's accuracy and efficiency. Finally, Section 5 contains the paper conclusion.

2 Derivation of the Method

CTMSO is obtained by approximating the exact solution $y(x)$ by searching the solution $y(x,u)$, which provides a discrete method as a by-product. The method has the form

$$y(x) = \sum_{j=0}^{k} a_j x^j + a_{k+1} sin(wx) + a_{k+2} cos(wx) \tag{2}$$

will be used as a basis function to approximate the solution of the second order initial value problems of the form
The second derivative of (2) is given as:

$$y'' = \sum_{n=j}^{k} j(j-1) a_j x^{i-2} - w^2 a_{k+1} sin(wx) - w^2 a_{k+2} cos(wx) \tag{3}$$

Through interpolation of (2) at $x_{n+j}, j = 0, k-1$, collocation of (3) at $x_{n+j}, j = 0(3)k$ to obtain $k+3$ system of equation

$$y(x_{n+j}, u) = y_{n+j}, j = 0(2) \tag{4}$$

$$\frac{d^2}{dx^2}(y(x_{n+j}, u)) = f_{n+j}, j = 0(2)k \tag{5}$$

Equations (2) and (3) lead to a system of 3k system equations which is solved by Cramer's rule to obtain $a_j's$. Our continuous CTMSO is constructed by substituting the values of $a_j's$ into equation (2). After some algebraic manipulation, the CTMSO is expressed in the form

$$y(x) = \alpha_n(x,w) + \alpha_{n+2}(x,) + h^2(\beta_n(x,w) f_n + \beta_{n+1}(x,w) f_{n+1} + \beta_{n+2}(x,w) f_{n+2} + \beta_{n+3}(x,w) f_{n+2}) \tag{6}$$

where, w is the frequency, $\alpha_n(w,x), \alpha_{n+2}(w,x), \beta_n(w,x), \beta_{n+1}(w,x), \beta_{n+2}(w,x), \beta_{n+3}(w,x)$ are continuous coefficients. The continuous coefficients in Equation (6) is used to generate the method of the form in Equation (2). Thus, evaluating (6) at $x = x_{n+2}$ and letting $u = wh$, we obtain the coefficients of (2) as follows:

$$\begin{aligned}
\alpha_0 &= -\frac{1}{2} \\
\alpha_2 &= \frac{3}{2} \\
\beta_0 &= -\frac{1}{4}\left(\frac{\sin(u)\,u^2 + 4\,\cos(3\,u)\sin(u)\cos(u) - 4\,\sin(3\,u)(\cos(u))^2 + 2\,\cos(u)\sin(u) + 2\,\sin(3\,u)}{w^2\sin(u)(-1+\cos(u))}\right) \\
\beta_1 &= \frac{1}{2}\left(\frac{u^2}{w^2}\right) \\
\beta_2 &= \frac{1}{4}\left(\frac{4\,u^2\cos(u) - u^2 + 6\,\cos(u) - 6}{w^2(-1+\cos(u))}\right) \\
\beta_3 &= -\frac{1}{2}\left(\frac{2\,\cos(u) + u^2 - 2}{w^2(-1+\cos(u))}\right)
\end{aligned} \tag{7}$$

The frequency flow of (6) can be shown below

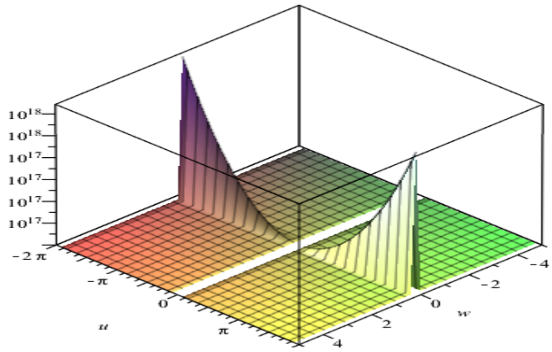

Figure 1: 3D plot Frequency for the method

3 Error Analysis and Stability

3.1 Local Truncation Error

The Taylor series is used for small values of u (see Simos (1998). Thus the coefficients in equation (7) can be expressed as

$$\begin{aligned}
\beta_0 &= \frac{1}{24}u^2w^2 + \frac{1}{480}\frac{u^4}{w^2} + \frac{1}{12096}\frac{u^6}{w^2} + \frac{1}{345600}\frac{u^8}{w^2} + \frac{1}{10644480}\frac{u^{10}}{w^2} + \frac{691}{237758976000}\frac{u^{12}}{w^2} \\
\beta_1 &= \frac{1}{2}(\frac{u^2}{w^2}) \\
\beta_2 &= \frac{7}{8}\frac{u^2}{w^2} - \frac{1}{160}\frac{u^4}{w^2} - \frac{1}{4032}\frac{u^6}{w^2} - \frac{1}{115200}\frac{u^8}{w^2} - \frac{1}{3548160}\frac{u^{10}}{w^2} - \frac{691}{79252992000}\frac{u^{12}}{w^2} \\
\beta_3 &= \frac{1}{12}\frac{u^2}{w^2} + \frac{1}{240}\frac{u^4}{w^2} + \frac{1}{6048}\frac{u^6}{w^2} + \frac{1}{172800}\frac{u^8}{w^2} + \frac{1}{5322240}\frac{u^{10}}{w^2} + \frac{691}{118879488000}\frac{u^{12}}{w^2}
\end{aligned} \quad (8)$$

For practical computations when u is small, it is advisable to use the series expansion (8) . Thus the Local Truncation Error for method (7) subject to equation (8) is obtained as

Local Truncation Error for CTMSO

$$\frac{h^6}{12096}(w^2y^{(4)}(x_n) + y^6(x_n)) + 0^8$$
$$\frac{-h^6}{4032}(w^2y^{(4)}(x_n) + y^6(x_n)) + 0^8$$
$$\frac{h^6}{6048}(w^2y^{(4)}(x_n) + y^6(x_n)) + 0^8$$

The local truncation error are $(\frac{1}{12096}, \frac{1}{2}, \frac{-1}{4032}, \frac{1}{6048})$ and it has at lease order of order 4

Remark 1. The CTMSO (12) is consistent as it has order $p > 1$ and zero-stable, hence it is convergent since zero stability + consistency = convergence

3.2 Stability

Proposition 1. The trigonometrically-fitted second derivative method (7) is applied to a test equation $y'' = -\lambda^2 y$, where λ is a real constant (see Jator et al. (2013)), it yields

$$y_{n+2} = M(\gamma^2; u)y_{n+1}, \gamma = h\lambda; u = kh \tag{9}$$

with

$$M(\gamma^2; u) = \frac{A_0 + \gamma^2\beta_0}{A_1 - \gamma^2\beta_1} \tag{10}$$

where the matrix $M(\gamma^2; u)$ is the amplification matrix which determines the stability of the method.

Proof. We begin by applying (7) to the test equation $y'' = \lambda^2 y$ respectively, by letting $\gamma = h\lambda, u = kh$, we obtain a linear equation which is used to solve for y_{n+2} with (10) as consequence.

Definition 1. A region of stability is a region in the $\gamma - u$ plane, in which the rational function $|M(\gamma; u)| \leq 1$

Definition 2. The method (7) is zero stable provided the root of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple (Lambert (1973)).

Definition 3. Method (7) is consistent if it has order $p > 1$ (Nwagne and Jator (2017)) The trigonometrically-fitted second derivative method is consistent as it has order $p > 1$ and zero stable, hence convergent. Since Convergence =Zero stability + consistency

The method is zero stable provided the roots $R_j, j = 1; 2; 3$ of the first characteristic polynomial $\rho(R)$ specified by $\rho(R) = 2r^2 - 3r + 1 = 0$ for $r = \frac{-1}{2}, 1, 1$ and the multiplicity does not exceed 1 (see Obarhua and Adegboro (2021)).

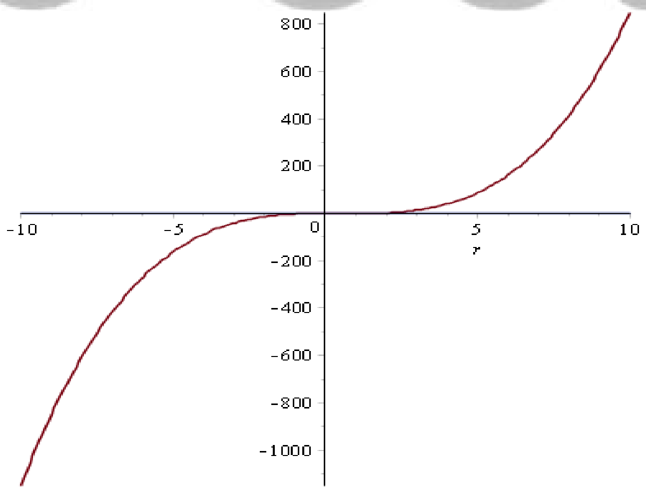

Figure 2: 2D plot for Zero Stability CTMSO

Linear Stability and Region of Absolute Stability of the Method

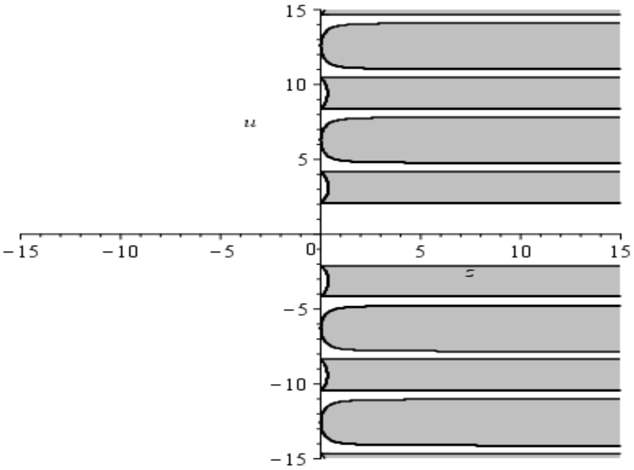

Figure 3: Region of Absolute Stability of the method

4 Numerical Examples

The performance and accuracy of the newly developed CTMSO are reviewed in this part for a number of well-known oscillatory IVPs, both linear and nonlinear situations. For the computation, the fitting frequency of each problem is utilized as the default frequency. The approximation solutions' absolute errors or maximum errors are estimated and compared to results from existing approaches in the literature. $r(-t)$ represents an error of the kind $r * 10^{-t}$. All calculations were completed using written Maple 2016.1 code and the Windows 8.1 operating system.

Example 1

$$y'' = -100y + 99sin(x)$$

$$y(0) = 1, y'(0) = 11, x\epsilon[0, 1000], h = \frac{1}{3200}, w = 5000$$

where the analytical solution is given by

$$y(x) = cos10x + sin(10x) + sinx$$

This requires only $3N + 1$ function evaluations in N steps compared to $3N + 1$ and 4N function evaluation in N,For instance , if we let $n = 0$ in the continuous scheme, then y_1 is obtained on the sub interval $[x_0; x_1]$, as y_0 is obtained from the IVP, in a similar way, if we let $n = 1, y_2$ is obtained on the subinterval $[x_1; x_2]$, as y_1 is known from the previous computation and so on until we reach the final subinterval $[x_{N-1}; x_N]$. Hence this methods performs better.

Example 2

Consider the Scalar test equation

$$y'' = -w^2y, y(0) = 1, y'(0) = 0, w = 10, h = \frac{\pi}{200}$$

Exact $y(x) = coswx$

Example 3

$$y'' = -w^2y, y(0) = 1, y'(0) = -2, w = 10, h = \frac{\pi}{200}$$

Exact $y(x) = sinwx$

Example 4

Linear Kramarz problem.

$$y'' = \begin{pmatrix} 2498 & 4998 \\ -2499 & -4999 \end{pmatrix} y(t), y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0 \le t \le 100$$

Exact solution: $(2cos(t), -sin(t))^t$

Example 5

Almost Periodic Problem) Van de Vyver

$$y_1'' = y_1 + \frac{1}{1000}cos(x), y_1(0) = 1, y_1'(0) = 0$$

$$y_2'' = y_2 + \frac{1}{1000}sin(x), y_2(0) = 0, y_2'(0) = 0.9995, x_{end} = 10$$

with the theoretical solution: $y_1(x) = cos(x) + 0.0005xsin(x), y_2(x) = sin(x) - 0.0005cos(x)$

Example 6

$$t_1'' = -2 + 2osx, 0 \le x \le 1$$

Exact solution: $t(x) = cos(x) + xsin(x)$

Example 7

Resonance Vibration of a Machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger which moves vertically up and down by a fly wheel makes the impact force on the metal sheet and therefore the supporting base, intermittent and cyclic. The bearing base on which the metal sheet is situated has a mass, $M = 2000kg$. The force acting on the base follows a function: $f(t) = 2000sin(10t)$,in which t=time in seconds. The base is supported by an elastic pad with an equivalent spring constant $k = 2 * 10^5 N/M$. Determine the differential equation for the instantaneous position of the base $y(t)$ if the base is initially depressed down by an amount 0.1m.

Solution: The mass- spring system above is modeled as differential equation:

The Bearing base mass = 2000kg

Spring constant $k = 2 * 10^5 N/m$

Force (ma) on the metal sheet= $m\frac{d^2y}{dt^2} = my''$

i.e. $ma = my'' = 2000sin(10t)$; where $a = y''$

Initial conditions on the system are

$y(t_0) = y_0; \frac{dy}{dt}|t = 0 = y'(t_0) = y'(0); t_0 = 0, y_0' = 0.1$

Therefore, the governing equation for the instantaneous position of the base y(t) is given by

$My'' + ky = F(t); y(t_0) = y_0, y'(t_0) = y'0$

Theoretical solution: $y(t) = \frac{1}{10}cos10t + \frac{1}{200}sin10t - \frac{t}{20}cos10t$

Table 1: Comparison of the new error with Simon (1998), Ngwane and Jator (2017)

N	Simon (1998)	Ngwane and Jator (2017)	new method CTMSO
1000	$1.4e-1$	$2.14e-1$	$2.1e-12$
2000	$3.4-2$	$5.98e-5$	$1.6e-12$
4000	$1.1e-3$	$2.06e-5$	$9.8e-13$
8000	$8.4e-5$	$1.26e-6$	$6.5e-13$
16000	$5.5e-6$	$7.79e-8$	$3.7e-13$
32000	-	$4.67e-9$	$5.3e-14$

Table 1 shows the comparison for computed error for CTMSO, error signifies the efficiency of the new method solved with problem 1.

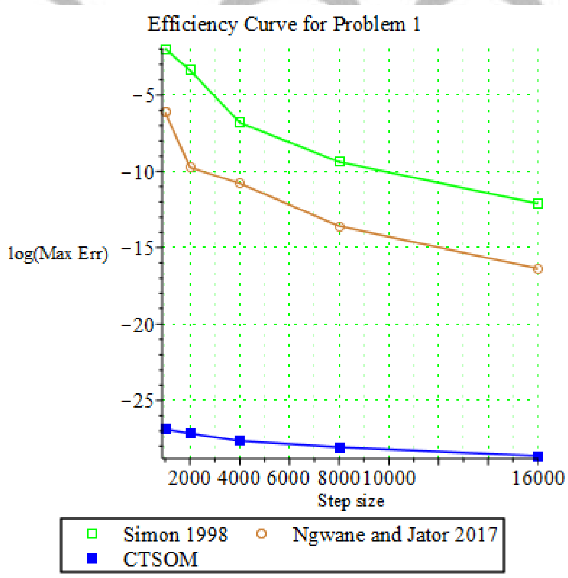

Figure 4: Efficiency curve for example 1

Table 2: Comparison of the new error with Ali Shorki (2014) for problem 2

x	Ali Shorki (2014), $p = 5$	CTMSO $p = 4$
5π	$2.3659e-04$	$9.19640760e-07$
10π	$5.1547e-04$	$9.55693810e-07$
15π	$6.2689e-04$	$4.62577449e-07$
20π	$8.3654e-04$	$5.60483146e-07$

Table 2 Error comparison for the new method CTMSO which signifies the accuracy of the new method solved with problem 2 over existing method.

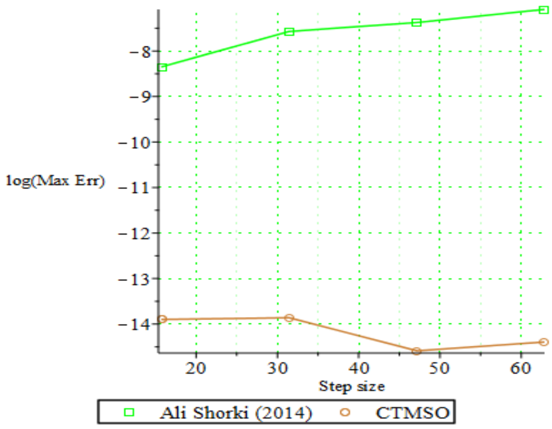

Figure 5: Efficiency curve for example 2

Table 3: Result of problem 3 for CTMSO , $h = \frac{\pi}{200}$ of $p = 4$

x	y-exact	y-computed	Errors
5π	0.999876521928723	0.999877444367191	$9.22438468e-07$
10π	0.999506118208559	0.999507076711142	$9.58502583e-07$
15π	0.998888880312983	0.998893520728652	$4.64041567e-06$
20π	0.998024960672684	0.998030590827594	$5.63015491e-06$
25π	0.996914572637924	0.996920306632039	$5.73399411e-07$
30π	0.995557990425848	0.995567466109030	$947568318e-06$

Table 3 shows the computed result for CTMSO, error signifies the accuracy of the new method solved with problem 3.

Table 4: Comparison of the new error with Nguyen et al. (2012), Nwagne and Jator (2017), Adeniran and Longe (2017).

x	Nguyen. et al(2007)	x	Ngwane and Jator (2014)	Adeniran and Longe (2017)	CTMSO
73		10	$1.3e-15$	$5.9e-15$	$2.20e-17$
143	$9.0-12$	43	$8.4e-15$	$1.1e-15$	$1.20e-17$
170	$3.7e-12$	80	$7.1e-15$	0	$8.10e-18$

Table 4 shows the comparison table, error for the new method CTMSO signifies the accuracy of the new method solved with problem 4 over existing method.

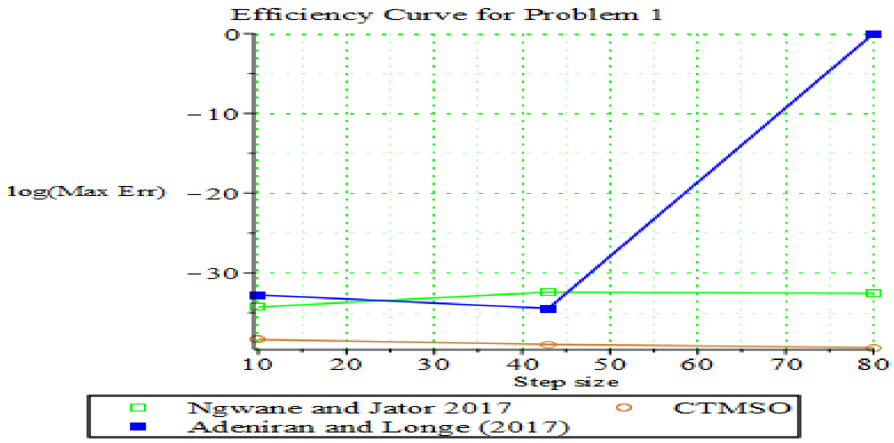

Figure 6: Efficiency Curve for example 4

Table 5: Comparison of the new error with Denba e tal.(2015),Senu (2009),Van de Vyver (2006), Anastassi and Kosti (2016)

N	METHOD	FCN	MAXE
10	CTMSO	40	$1.00041315e-6$
22	Demba (2017)	88	$1.894082e-1$
146	Senu (2009)	584	$5.907771e-4$
215	Anastassi and Kosti (2016)	1290	$7.375736e-1$
578	Van de Vyver (2006)	2315	$2.175738e-1$
20	CTMSO	80	$8.10309863e-7$
49	Demba (2017)	196	$1.246609e-3$
363	Senu (2009)	1452	$1.215852e-5$
825	Anastassi and Kosti (2016)	4955	$5.685758e-2$
2901	Van de Vyver (2006)	11610	$2.175738e-1$
60	CTMSO	240	$1.26020827e-7$
230	Demba (2017)	920	$1.303536e-6$
1821	Senu (2009)	7284	$1.826680e-8$
3180	Anastassi and Kosti (2016)	19090	$3.817832e-3$
29131	Van de Vyver (2006)	116536	$8.569318e-5$
80	CTMSO	320	$8.9999990e-8$
574	Demba (2017)	2296	$1.303536e-6$
4574	Senu (2009)	18296	$1.826680e-8$
24548	Anastassi and Kosti (2016)	147308	$3.817832e-3$
292676	Van de Vyver (2006)	1170722	$8.569318e-5$

Table 5 above shows the comparison table, error for the new method CTMSO signifies the efficiency of the new method solved with problem 4 over existing method.

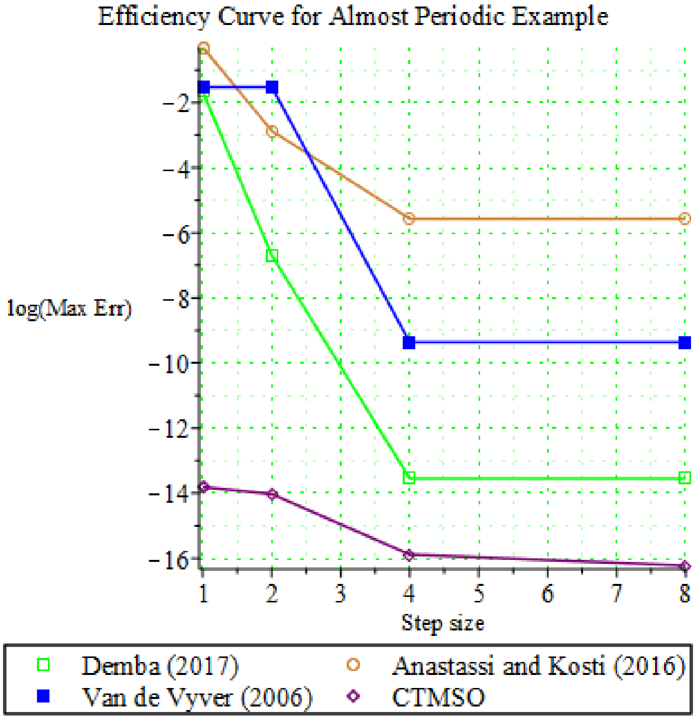

Figure 7: Efficiency Curve for example 5

Table 6: Comparison of the new error with Ismail (2021), Adeyeye and Omar (2017)Kuboye and Omar (2015)

N	METHOD	MAXE
0.01	CTMSO	$e+00$
0.01	Ismail (2021)	$2.212510e-16$
0.01	Adeyeye and Omar (2017)	$8.881784e-15$
0.01	Kuboye and Omar (2015)	$1.428607e-11$

Table 6 above shows the comparison table, error for the new method CTMSO signifies the efficiency of the new method solved with problem 6 over existing method.

Table 7: Table for problem 7, showing the accuracy of the new method

N	y-exact	y-computed	CTMSO $\max\|\|y_i - y(x_i)\|\|$
0.01	0.0999999500016710	0.0999998999887522	$5.001291880e-08$
0.02	0.0999998000134000	0.0999995999588829	$2.000545171e-07$
0.03	0.0999995500453379	0.9999909992110980	$4.501242281e-07$
0.04	0.0999992001077328	0.0999983995987475	$8.005089850e-07$
0.05	0.0999982003653984	0.9999749875707910	$1.251453858e-06$
0.06	0.0999982003653984	0.0999963975445895	$1.802820808e-06$
0.07	0.0999975505816685	0.0999950956725539	$2.454909110e-06$
0.08	0.0999968008703942	0.0999935928127285	$3.208057660e-06$
0.09	0.0999959512423277	0.9999188924400080	$4.061998320e-06$
0.10	0.0999950017083162	0.0999899846564115	$5.017051900e-06$

Table 7 shows the computed result for the new method, error signifies the efficiency of the new method solved with problem 7.

5 CONCLUSION

A non self-stating Continuous Trigonometrically-fitted Discrete technique for solving periodic IVPs with an algebraic 4 order is presented. The methods' convergence and accuracy were established, and the approach was evaluated with several standard oscillatory second order ordinary differential equations problems, where it was shown to be accurate and compare favorably to other ways in literature, as shown in Tables 1-7 above. Example 3 does not have any contest. All computations were carried out using written codes in Maple 2016. and executed on Windows 8.1 operating system.

Competing interests

The authors declare that they have no competing interests.

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