GSJ: Volume 12, Issue 4, April 2024, Online: ISSN 2320-9186

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Cantor's Ternary Set is also Countable

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Abstract:. The Cantor set is a famous set first introduced by German mathematician Georg Cantor in 1883. It is simply a subset of the interval [0, 1] . This set is considered to be uncountable. This paper explains that cantor's Ternary Set can be written as Countable union of Countable intersection of Closed Nested Intervals and with the help of Cantor's Nested intervals theorem , Cantor's Ternary Set is Countable.

Keywords: Rational numbers, Countable, union, intersection, nested intervals, Subset and limit.

Lemmas

1. **Distributive Law:** If A, B, C are three Sets then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
2. **If A and B are two sets Such that $A \subset B$ then $A \cap B = A$**
3. If A, B, C, D are two sets then $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$
4. $k^n \to 0$ for $-1 < k < 1$ as $n \to \infty$
5. Set of Rational numbers is countable
6.

6. **Cantor's Nested Intervals Theorem:** Let $\{I_n\}$ be a sequence of non-empty closed intervals

$I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \dots\dots \supseteq I_n \supseteq \dots\dots\dots$ Then
(a) $\bigcap_{n=1}^{\infty} I_n$ is a non empty closed interval and

(b) if $\lim_{n\to\infty}(b_n - a_n) = 0$, then

$\bigcap_{n=1}^{\infty} I_n$ contains exactly one point.

Proof: Let $\{I_n\}$ be a sequence of non empty closed intervals $I_n = [a_n, b_n]$ that are Nested; i.e.

$I_1 \supseteq I_2 \supseteq \dots\dots\dots \supseteq I_n \supseteq \dots\dots\dots\dots$, consider the sequences of $\{a_n\}$ and $\{b_n\}$ of left and right end points of the intervals I_n . Since the intervals are Nested, we have $\forall\, n \epsilon N$,

$$a_1 \le a_2 \le \cdots \le a_n \le a_{n+1} \le b_{n+1} \le b_n \le \cdots \le b_2 \le b_1.$$

Thus, the sequence $\{a_n\}$ is monotonic increasing and bounded above by b_1 .Likewise, the sequence $\{b_n\}$ is monotonic decreasing and bounded below by a_1 . By Monotonic Convergence theorem,

$\exists\ a = \lim_{n\to\infty} a_n$= sup{ a_n ; $n\epsilon$N}, and $\exists\ b = \lim_{n\to\infty} a_n$= inf{ b_n ; $n\epsilon$N}, .

Now, $\forall\ n\epsilon$N, $a_n \le b_n$.Thus $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$ i.e. $a \le b$

We shall prove that $[a,b] = \bigcap_{n=1}^{\infty} I_n$

Let $x\epsilon\ [a,b]$ then $\forall\ n\epsilon$N$a_n \le a \le x \le b \le b_n$, so $x \in I_n$. thus $\forall\ n \in \mathbb{N}, [a,b] \subseteq I_n$. Therefore,

$[a,b] \subseteq \bigcap_{n=1}^{\infty} I_n$ ... ... (0.1)

We now show that the set containment goes the other way also. Suppose $y \in \bigcap_{n=1}^{\infty} I_n$. then $\forall\ n \in \mathbb{N}, a_n \le y \le b_n$. thus(by limit preserve inequalities) $\lim_{n\to+\infty} a_n \le y \le \lim_{n\to\infty} b_n$.

That is $a \le y \le b$. Therefore $\bigcap_{n=1}^{\infty} I_n \subseteq [a,b]$ ... ...(0.2)

From (0.1) & (0.2), we have $\bigcap_{n=1}^{\infty} I_n = [a,b]$ ....(0.3)

Since $a \le b$, This interval is non empty.

(b)Suppose further, that $\lim_{n\to\infty}(b_n - a_n) = 0$. Then by the algebra of limits, $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$ but ,

$\lim_{n\to\infty} b_n = b$ *and* $\lim_{n\to\infty} a_n = a$ Thus b-a =0, Therefore by (0.3) $I_n = \{a\}$

INTRODUCTION

Cantor's Ternary Set is obtained by dividing [0, 1] into three equal parts and deleting the middle most open interval ,then further dividing the remaining two closed intervals into three equal parts and deleting their middle most open intervals respectively, Continuing the same way. Cantor's set is the set C left after this procedure of deleting the open middle third subinterval is performed infinitely many times.

$\mathbf{C_0} = [0, 1]$

$\mathbf{C_1} = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$

$\mathbf{C_2} = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$

$\mathbf{C_3} = \left[0,\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{1}{3}\right] \cup \left[\frac{18}{27},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{21}{27}\right] \cup \left[\frac{24}{27},\frac{25}{27}\right] \cup \left[\frac{26}{27},1\right]$

$\mathbf{C_4} = \left[0,\frac{1}{81}\right] \cup \left[\frac{2}{81},\frac{1}{27}\right] \cup \left[\frac{2}{27},\frac{7}{81}\right] \cup \left[\frac{8}{81},\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{19}{81}\right] \cup \left[\frac{20}{81},\frac{7}{27}\right] \cup \left[\frac{8}{27},\frac{25}{81}\right] \cup \left[\frac{26}{81},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{55}{81}\right] \cup \left[\frac{56}{81},\frac{19}{27}\right] \cup \left[\frac{20}{27},\frac{61}{81}\right] \cup \left[\frac{62}{81},\frac{7}{9}\right] \cup \left[\frac{8}{9},\frac{73}{81}\right] \cup \left[\frac{74}{81},\frac{25}{27}\right] \cup \left[\frac{26}{27},\frac{79}{81}\right] \cup \left[\frac{80}{81}, 1\right]$

. ... ………………..

$\mathbf{C_0}$ has $2^0 = 1$ closed interval , $\mathbf{C_1}$ has $2^1 = 2$ closed intervals, $\mathbf{C_3}$has $2^2 = 4$ closed intervals

$\mathbf{C_4}$has $2^3 = 8$ closed intervals, **… … ..** $\mathbf{C_n}$has 2^n closed intervals**..**

Here $\mathbf{C_0} \supset \mathbf{C_1} \supset \mathbf{C_2} \supset \mathbf{C_3} \supset \mathbf{C_4} \supset \ldots \ldots \ldots .. \supset \mathbf{C_n} \supset \ldots\ldots ..$

Now $\mathbf{C_1} \cap \mathbf{C_2} \cap \ldots .. \cap \mathbf{C_n} = \mathbf{C_n}$

Consider the Sets Containing the end points of intervals in$\boldsymbol{C_i}$ $\boldsymbol{s}$ are denoted by $\mathbf{K}_i s$

$$\boldsymbol{K_0} = \{\mathbf{0}, \mathbf{1}\}$$

$$\boldsymbol{K_1} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{3}}, \frac{\mathbf{2}}{\mathbf{3}}, \mathbf{1}\right\}$$

$$\boldsymbol{K_2} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{9}}, \frac{\mathbf{2}}{\mathbf{9}}, \frac{\mathbf{3}}{\mathbf{9}}, \frac{\mathbf{6}}{\mathbf{9}}, \frac{\mathbf{7}}{\mathbf{9}}, \frac{\mathbf{8}}{\mathbf{9}}, \mathbf{1}\right\}$$

$$\mathbf{K_3} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{27}}, \frac{\mathbf{2}}{\mathbf{27}}, \frac{\mathbf{3}}{\mathbf{27}}, \frac{\mathbf{6}}{\mathbf{27}}, \frac{\mathbf{7}}{\mathbf{27}}, \frac{\mathbf{8}}{\mathbf{27}}, \frac{\mathbf{9}}{\mathbf{27}}, \frac{\mathbf{18}}{\mathbf{27}}, \frac{\mathbf{19}}{\mathbf{27}}, \frac{\mathbf{20}}{\mathbf{27}}, \frac{\mathbf{21}}{\mathbf{27}}, \frac{\mathbf{24}}{\mathbf{27}}, \frac{\mathbf{25}}{\mathbf{27}}, \frac{\mathbf{26}}{\mathbf{27}}, \mathbf{1}\right\}$$

$$\boldsymbol{K_4} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{81}}, \frac{\mathbf{2}}{\mathbf{81}}, \frac{\mathbf{3}}{\mathbf{81}}, \frac{\mathbf{6}}{\mathbf{81}}, \frac{\mathbf{7}}{\mathbf{81}}, \frac{\mathbf{8}}{\mathbf{81}}, \frac{\mathbf{9}}{\mathbf{81}}, \frac{\mathbf{18}}{\mathbf{81}}, \frac{\mathbf{19}}{\mathbf{81}}, \frac{\mathbf{20}}{\mathbf{81}}, \frac{\mathbf{21}}{\mathbf{81}}, \frac{\mathbf{24}}{\mathbf{81}}, \frac{\mathbf{25}}{\mathbf{81}}, \frac{\mathbf{26}}{\mathbf{81}}, \frac{\mathbf{27}}{\mathbf{81}}, \frac{\mathbf{54}}{\mathbf{81}}, \frac{\mathbf{55}}{\mathbf{81}}, \frac{\mathbf{56}}{\mathbf{81}}, \frac{\mathbf{57}}{\mathbf{81}}, \frac{\mathbf{60}}{\mathbf{81}}, \frac{\mathbf{61}}{\mathbf{81}}, \frac{\mathbf{62}}{\mathbf{81}}, \frac{\mathbf{63}}{\mathbf{81}}, \frac{\mathbf{66}}{\mathbf{81}}, \frac{\mathbf{67}}{\mathbf{81}}, \frac{\mathbf{68}}{\mathbf{81}}, \frac{\mathbf{69}}{\mathbf{81}}, \frac{\mathbf{72}}{\mathbf{81}}, \frac{\mathbf{73}}{\mathbf{81}}, \frac{\mathbf{74}}{\mathbf{81}}, \frac{\mathbf{75}}{\mathbf{81}}, \frac{\mathbf{78}}{\mathbf{81}}, \frac{\mathbf{79}}{\mathbf{81}}, \frac{\mathbf{80}}{\mathbf{81}}, \mathbf{1}\right\}$$

. .. … … ..

$$\mathbf{K_n} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{3^n}}, \frac{\mathbf{2}}{\mathbf{3^n}}, \frac{\mathbf{3}}{\mathbf{3^n}}, \frac{\mathbf{6}}{\mathbf{3^n}}, \frac{\mathbf{7}}{\mathbf{3^n}}, \frac{\mathbf{8}}{\mathbf{3^n}}, \frac{\mathbf{3^2}}{\mathbf{3^n}}, \frac{\mathbf{2.3^2}}{\mathbf{3^n}}, \frac{\mathbf{2.3^2+1}}{\mathbf{3^n}}, \frac{\mathbf{2.3^2+2}}{\mathbf{3^n}}, \frac{\mathbf{2.3^2+3}}{\mathbf{3^n}}, \frac{\mathbf{24}}{\mathbf{3^n}}, \ldots\ldots\ldots.., \frac{\mathbf{3^{n-i}}}{\mathbf{3^n}}, \frac{\mathbf{2(3^{n-i})}}{\mathbf{3^n}}, \ldots.. \frac{\mathbf{3^n-1}}{\mathbf{3^n}}, \mathbf{1}\right\}$$

Clearly, $\boldsymbol{K_0} \subset \boldsymbol{K_1} \subset \boldsymbol{K_2} \subset \boldsymbol{K_3} \subset \boldsymbol{K_4} \subset \cdots \subset \boldsymbol{K_n} \subset \cdots$

Thus $\mathbf{K_n}$ Contains all the end points of all the intervals of $C_0, C_1, C_2, \ldots . C_n$

Now $\mathbf{C_n} = \left[\mathbf{0}, \frac{\mathbf{1}}{\mathbf{3^n}}\right] \cup \left[\frac{\mathbf{2}}{\mathbf{3^n}}, \frac{\mathbf{1}}{\mathbf{3^{n-1}}}\right] \cup \left[\frac{\mathbf{2}}{\mathbf{3^{n-1}}}, \frac{\mathbf{7}}{\mathbf{3^n}}\right] \cup \left[\frac{\mathbf{8}}{\mathbf{3^n}}, \frac{\mathbf{1}}{\mathbf{3^{n-2}}}\right] \cup \left[\frac{\mathbf{2}}{\mathbf{3^{n-2}}}, \frac{\mathbf{19}}{\mathbf{3^n}}\right] \cup \left[\frac{\mathbf{20}}{\mathbf{3^n}}, \frac{\mathbf{7}}{\mathbf{3^{n-1}}}\right] \cup \left[\frac{\mathbf{8}}{\mathbf{3^{n-1}}}, \frac{\mathbf{25}}{\mathbf{3^n}}\right] \cup \left[\frac{\mathbf{26}}{\mathbf{3^n}}, \frac{\mathbf{1}}{\mathbf{3^{n-3}}}\right] \cup \left[\frac{\mathbf{2}}{\mathbf{3^{n-3}}}, \frac{\mathbf{55}}{\mathbf{3^n}}\right] \ldots \ldots . \cup \left[\frac{\mathbf{3^n-1}}{\mathbf{3^n}}, \mathbf{1}\right]$

Now every Interval of $\mathbf{C_n}$ is an intersection of 2^n nested intervals.

$$\left[0, \frac{1}{3}\right] \supset \left[0, \frac{1}{9}\right] \supset \ldots . . . \ldots\ldots\ldots\ldots \supset \left[0, \frac{1}{3^n}\right] \quad \ldots \ldots (1)$$

$$\left[0, \frac{1}{3}\right] \supset \left[\frac{2}{9}, \frac{1}{3}\right] \supset \left[\frac{8}{27}, \frac{1}{3}\right] \supset \left[\frac{26}{81}, \frac{1}{3}\right] \supset \ldots \ldots \ldots \ldots \supset \left[\frac{3^{n-1}-1}{3^n}, \frac{1}{3}\right] \quad \ldots \ldots (2)$$

$$\left[0, \frac{1}{3}\right] \supset \left[0, \frac{1}{9}\right] \supset \left[\frac{2}{27}, \frac{1}{9}\right] \supset \left[\frac{8}{81}, \frac{1}{9}\right] \supset \left[\frac{26}{243}, \frac{1}{9}\right] \supset \ldots \ldots \ldots \ldots \supset \left[\frac{3^{n-2}-1}{3^n}, \frac{1}{9}\right] \quad \ldots . . (3)$$

$$\left[0, \frac{1}{3}\right] \supset \left[\frac{2}{9}, \frac{1}{3}\right] \supset \left[\frac{2}{9}, \frac{7}{27}\right] \supset \left[\frac{2}{9}, \frac{19}{81}\right] \supset \left[\frac{2}{9}, \frac{55}{243}\right] \supset \ldots \ldots \ldots \ldots \supset \left[\frac{2}{9}, \frac{1+2.3^n}{3^n}\right] \ldots \ldots . . (4)$$

$$\left[\frac{2}{3}, 1\right] \supset \left[\frac{2}{3}, \frac{7}{9}\right] \supset \left[\frac{2}{3}, \frac{19}{27}\right] \supset \left[\frac{2}{3}, \frac{55}{81}\right] \supset \ldots . . \ldots \ldots \ldots \ldots \supset \left[\frac{2}{3}, \frac{1+2.3^{n-1}}{3^n}\right] \ldots \ldots (5)$$

. . .   .

GSJ: Volume 12, Issue 4, April 2024
ISSN 2320-9186

… … …

… … …

$\left[\frac{2}{3},\frac{7}{9}\right]\supset\left[\frac{20}{27},\frac{7}{9}\right]\supset\left[\frac{62}{81},\frac{7}{9}\right]\supset\left[\frac{188}{243},\frac{7}{9}\right]\supset\ldots\ldots\ldots\ldots\supset\left[\frac{7.3^{n-2}-1}{3^n},\frac{7}{9}\right]\ldots\ldots(j)$

. . . … …

$\left[\frac{2}{3},1\right]\supset\left[\frac{8}{9},1\right]\supset\left[\frac{26}{27},1\right]\supset\left[\frac{80}{81},1\right]\supset\ldots\ldots\ldots\ldots\supset\left[\frac{3^n-1}{3^n},1\right]\ldots\ldots(2^n)$

from (1) $\bigcap_{k=1}^{n}\left[0,\frac{1}{3^k}\right]=\left[0,\frac{1}{3^n}\right]$ here $\left|\frac{1}{3^n}-0\right|\to 0$ as $n\to\infty$

from (2) $\bigcap_{k=1}^{n}\left[\frac{3^{k-1}-1}{3^k},\frac{1}{3}\right]=\left[\frac{3^{n-1}-1}{3^n},\frac{1}{3}\right]$ here $\left|\frac{1}{3}-\frac{3^{n-1}-1}{3^n}\right|=\frac{1}{3^n}\to 0$ as $n\to\infty$

… … .. …. . …

from (2^n) $\bigcap_{k=1}^{n}\left[\frac{3^k-1}{3^k},1\right]=\left[\frac{3^n-1}{3^n},1\right]$ here $\left|1-\frac{3^n-1}{3^n}\right|=\frac{1}{3^n}\to 0$ as $n\to\infty$

So we can write C_n **as the union of Countable intersection of of nested intervals**.

$$C_n=\left(\bigcap_{\{K=1\}}^{n}\left[0,\frac{1}{3^k}\right]\right)\bigcup\left(\bigcap_{k=1}^{n}\left[\frac{3^{k-1}-1}{3^k},\frac{1}{3}\right]\right)\bigcup\left(\bigcap_{k=2}^{n}\left[\frac{3^{k-2}-1}{3^k},\frac{1}{9}\right]\right)\bigcup\left(\bigcap_{k=2}^{n}\left[\frac{2}{9},\frac{1+2.3^{k-2}}{3^k}\right]\right)\ldots.\bigcup\left(\bigcap_{k=2}^{n}\left[\frac{7.3^{k-2}-1}{3^k},\frac{7}{9}\right]\right)\ldots\bigcup\left(\bigcap_{k=1}^{n}\left[\frac{3^k-1}{3^k},1\right]\right).$$

We know that Cantor's Ternary Set is defined as

$$C=\bigcap_{n=0}^{\infty}C_n=\lim_{n\to\infty}\bigcap_{k=1}^{n}C_k=\lim_{n\to\infty}C_n$$

C_n is union of countable intersection of Nested Intervals. Thus C is countable union of countable intersection of Nested Intervals.

$$C=\lim_{n\to\infty}C_n$$
$$=\left(\lim_{n\to\infty}\bigcap_{\{K=1\}}^{n}\left[0,\frac{1}{3^k}\right]\right)\bigcup\left(\lim_{n\to\infty}\bigcap_{k=1}^{n}\left[\frac{3^{k-1}-1}{3^k},\frac{1}{3}\right]\right)\bigcup\left(\lim_{n\to\infty}\bigcap_{k=2}^{n}\left[\frac{3^{k-2}-1}{3^k},\frac{1}{9}\right]\right)\bigcup\left(\lim_{n\to\infty}\bigcap_{k=2}^{n}\left[\frac{2}{9},\frac{1+2.3^{k-2}}{3^k}\right]\right)\ldots$$

GSJ: Volume 12, Issue 4, April 2024
ISSN 2320-9186

$$\ldots\ldots\bigcup\left(\lim_{n\to\infty}\bigcap_{k=2}^{n}\left[\frac{7.3^{k-2}-1}{3^k},\frac{7}{9}\right]\right)\ldots\bigcup\left(\lim_{n\to\infty}\bigcap_{k=1}^{n}\left[\frac{3^k-1}{3^k},1\right]\right)$$

$$\lim_{n\to\infty}\bigcap_{\{K=1\}}^{n}\left[0,\frac{1}{3^k}\right]=\bigcap_{\{n=1\}}^{\infty}\left[0,\frac{1}{3^n}\right]$$

Which is an infinite intersection of Nested Intervals, and length of

$\left[\mathbf{0},\frac{\mathbf{1}}{\mathbf{3^n}}\right]$ **Converging to 0 as n $\to\infty$**

Then by **Cantor's Nested Intervals Theorem**

$$\bigcap_{\{n=1\}}^{\infty}\left[0,\frac{1}{3^n}\right]$$

Is an infinite intersection which is Non empty and contains exactly one point. As 0 belongs to $\left[0,\frac{1}{3^n}\right]\ \forall\ n$

Thus

$$\bigcap_{n=1}^{\infty}\left[0,\frac{1}{3^n}\right]=\{0\},$$

Likewise $\bigcap_{n=1}^{\infty}\left[\frac{3^{n-1}-1}{3^n},\frac{1}{3}\right]=\left\{\frac{1}{3}\right\},\bigcap_{n=2}^{\infty}\left[\frac{3^{n-2}-1}{3^n},\frac{1}{9}\right]=\left\{\frac{1}{9}\right\},$

$$\bigcap_{n=2}^{\infty}\left[\frac{2}{9},\frac{1+2.3^{n-2}}{3^n}\right]=\left\{\frac{2}{9}\right\},\ldots\ldots\ldots.\bigcap_{n=2}^{\infty}\left[\frac{7.3^{n-2}-1}{3^n},\frac{7}{9}\right]=\left\{\frac{7}{9}\right\}$$

$$\bigcap_{n=2}^{\infty}\left[\frac{3^{n-1}-1}{3^n},1\right]=\{1\}$$

$$\boldsymbol{C}=\left\{0,\ldots\frac{1}{81},\frac{2}{81}\ldots,\frac{1}{9},\frac{2}{9},\ldots.\frac{1}{3},\frac{2}{3}\ \ldots\ldots 1\right\}=\bigcup_{n=0}^{\infty}\boldsymbol{K_n}=\lim_{n\to\infty}\boldsymbol{K_n}$$

Conclusion

Thus ***C***, the Cantor's Ternary Set can be written as countable Union of countable intersection of Closed nested intervals Cantor's Ternary Set is obtained by deleting Uncountable set from an uncountable set, Countably. This process is unable to delete the end points of the closed intervals. Thus as n approaches to infinity, we are left with only end points of $\mathbf{C_0}$, $\mathbf{C_1}, \mathbf{C_2}$,.........$\mathbf{C_n}$, … … ..each end point of the intervals is a rational number. Set of rational numbers is countable We know that every subset of a countable set is also countable. $\boldsymbol{C}\subset\mathbb{Q}$ Which implies ***C*** is a Countable Set.

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