Global Scientific JOURNALS

GSJ: Volume 9, Issue 4, April 2021, Online: ISSN 2320-9186
www.globalscientificjournal.com

Certain Preserver Conditions for Automorphisms on Unital C*-Algebras

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Abstract

In this paper, we characterize certain preserver conditions for automorphisms on unital C*-algebras. This is done by giving an indepth characterization and establishing sufficient conditions for which a surjective map between unital C*-algebras preserves norm.

1. Introduction

Preserver problems for linear transformations have been studied over a long period of time by several mathematicians including Li, Tsing, Mazur, Ulam among others [1]. Preserver problems that have been considered are spectral, norm and numerical range preserver problems in matrix algebras [2]. However, norm preserver conditions for automorphisms on unital C^*-algebras have not be considered.In this thesis, we consider norm preserver conditions for automorphisms on unital C^*-algebras. This is done by first establishing sufficient conditions for which a surjective map between unital C^*-algebras is an algebra automorphism. When this is done, we shall also identify norm preserving conditions for automorphisms on unital C^*-algebras [3]. This is been done by use of technical approach of extreme points techniques and utilization of the famous Monlar theorem .Results show that for an algebra A which is a unital C^*-algebra that is commutative then it is isomorphic to the space $C(P)$ of all continuous functions on a compact set P [4]. Since uniform algebra is a sub algebra of the space $C(P)$, it follows that for uniform algebras $X \subset C(P)$ and $Y \subset C(S)$ having Choquet boundary δX and δY, and a surjective transformation $T : X \longrightarrow Y$ that preserves the norm of the sums of the moduli of algebra elements, then T induces a automorphism ϕ between the Choquet boundaries of X and Y such that $|Tm| = |m \circ \phi|$ on the Choquet boundary of Y [5]. If, on top of the earlier property, T either preserves both elements i and 1 or the peripheral spectra of C-peaking function and also preserves the norms of all linear combinations of algebra elements, then T is a composition operator which makes it an algebra automorphism [6]. We also show that if a surjection T either preserving both i and 1 or preserving the peripheral spectra of the C-peaking functions that preserves the norms of the sum of algebra elements as well as , then T is a composition operator and thus an algebra automorphism [7]. The results obtained are useful to matrix theory and applications in quantum computing. Having a spectral condition, it also required the mapping $T : U \longrightarrow Y$ to be a linear operator [8]. There are several other results that require preservation of all or part of the spectra of the elements of the algebra or a subset of the elements of the algebra but do not require the mapping T to be linear. The first of such results was Kowalski and Slodkowski [9] which demanded that the spectrum of the difference between algebra elements be preserved in order to have the mapping preserve the algebraic structure as well as the distance between algebra elements. The spectral condition in the results implied that $\|Tm - Tn\| = \|m - n\|, \forall m, n \in G$, that is, T preserves distances be- tween the algebra elements. This spectral condition brought the isometry conclusion which was not a surprise. We also see that the Mazur-Ulam Theorem [10] implies that T is an R−linear mapping, so the additivity requirement for an isomorphism is met. Unital operators are mappings that preserve the unit element that is the mapping $T : V \longrightarrow W$ between unital algebras has the property $T(1_V) = 1_W$. Spectral preserver problems started taking a multiplicative direction where the unit element was to be preserved. One such result was from [11] which was extended by Rao and Roy to surjective self-maps from any uniform algebra to itself and for an arbitrary compact Hausdorff set P. The results in [2] were significantly improved, one year later, by [3]. This was done by allowing T to be an operator between any two uniform algebras instead of requiring it to be a self -map and by only requiring the preservation of a subset of the spectra (the peripheral spectra) of products of algebra elements. For algebra elements m and n, if $\sigma(m) = \sigma(n)$ then $\sigma_\pi(m) = \sigma_\pi(n)$ but not vice versa. Later [6] extended this theorem to standard operator algebras. Luttman and Tonev were joined with lambert to show that instead of the preservation of the peripheral spectra of products of algebra elements, T need only preserve at least one element of the peripheral spectra of products. The requirement that T be unital was removed and added the requirement that T preserve the peripheral spectra of all algebra elements. However, this requirement is not more than the previous results because the theorem requires that T be unital, in which case $\sigma_\pi(Tm) = \sigma_\pi(TmT1) = \sigma_\pi(m \cdot 1) = \sigma_\pi(m)$, so a map that satisfies the hypothesis of theorem does in fact preserve the peripheral spectra of algebra elements. The proofs of these theorems largely depend on variations of the classical result by [2] which was refined by [5]. A stronger version of the lemma is found in [9]. In [8] the authors took the additive direction by showing that a surjection that preserves the peripheral spectra of sums of algebra elements as well as the sup-norms of the sums of the moduli of algebra elements will preserve the distances between algebra elements as well as the structure of the algebra.

2. Justification of the problem

Preserver problems in the general sense appear in many parts of mathematics and hence the study of them certainly deserves attention. Preserver problems play a crucial role in Quantum Mechanics. In the Hilbert space formulation of Quantum mechanics, several mathematical objects appear which have sets equipped with certain scalar valued functions and/or algebraic operations and/or binary relations which have important physical con- tent. The corresponding automorphisms, that is, the bijective maps on those sets which preserve the relevant structures represent different kinds of quantum mechanical symme- tries. To describe those symmetries and study the relations among them are important problems which were considered by a number mathematicians and theoretical physicist. We consider C^*-algebra primarily for their use in quantum mechanics to model algebras of physical variables that can be measured.

3. Research methodology

As we have seen, the set of linear multiplicative functionals on a commutative Banach algebra and the set of maximal ideals for that algebra are in bijective correspondence, so we can make the following definition.

Definition 2.1. Let D be a commutative Banach algebra with unit. The set M_D of all nonzero linear multiplicative functionals of D is called the maximal ideal space of D. Though the space M_D does not posses a natural algebraic structure, we can equip it with the weak$-^*$topology it inherits as a subset of D^*, the collection of all bounded linear functionals on D. When applied to the maximal ideal space, we call this toplolgy the Gelfand topology. We recall that under this topology, a net of elements φ_α in M_D tends to $\varphi \in M_D$ if and only if $\varphi_\alpha(m) \to \varphi(m)$, $\forall m \in D$. Thus, under the Gelfand topology, convergence of functionals in M_D is point wise convergence. A weak$-^*$ limit of linear multiplicative functionals is itself a non-zero linear multiplicative functional because $(\lim \varphi_\alpha)(1) = \lim \varphi_\alpha(1) = 1$. We also note that the space M_D is compact in the weak$-^*$topology by the Banach-Alaoglu theorem.

Definition 2.2. Let m be an element in a commutative Banach algebra D. The Gelfand transform of m is the function $\hat{m}$ on M_D defined by $\hat{m}(\varphi) = \varphi(m)$, $\forall\varphi\varphi \in M_D$. The Gelfand transform of m is clearly continuous on M_D with respect to the Gelfand topology since if $\varphi_\alpha \to \varphi$, then $\varphi_\alpha(m) \to \varphi(m)$, which implies that $\hat{m}(\varphi_\alpha) \to \hat{m}(\varphi)$.

4. Results and discussion

In this section, we present our results. We use the fact that if $X \subset C(P)$ and $Y \subset C(S)$ will be uniform algebras on compact sets P and S respectively and Since X and Y are sub algebras of the unital C^*-

algebra A and also C(S) being isomorphic to the unital C*-algebra A, the results discussed in this Chapter concerning uniform algebras can be extended to unital C*-algebra and isomorphism changed to automorphism since the transformation will be in the same space. The following proposition gives sufficient conditions under which surjective maps, in unital C*-algebra, are automorphisms.

Proposition 4.1 *Any peripherally-additive and norm-additive in modulus surjection $T : X \subset A \longrightarrow Y \subset A$ is an isometric algebra isomorphism.*

Proof. We note that automorphisms are multiplicative. As from the statement of Proposition 4.1, if T satisfies the equation

$$\|Tm + \alpha Tn\| = \|m + \alpha n\|, \forall m, n \in X \subset A \text{ and each } \alpha \text{ with } |\alpha| = 1,$$

then T is norm-additive and norm additive in modulus

Lemma 4.2. *Any surjection $T : X \subset A \longrightarrow Y \subset A$ that satisfies the equation*

$$\|Tm + \alpha Tn\| = \|m + \alpha n\|, \forall m, n \in X \subset A \text{ and all } \alpha \in \mathbb{T},$$

is a $\phi-$composition operator in modulus on $\delta Y \subset A$. If in addition

i $T(1) = 1$ and $T(i) = i$ or

ii T preserves the peripheral spectra of all C−peaking functions of $X \subset A$, Then T is an isometric unital algebra isomorphism.

Proof. According to Proposition 4.1, every norm-linear operator is norm-additive and norm- additive in modulus, so Theorem 4.2.7 yields: (Norm-linear Operators). Any norm-linear surjection $T : X \subset A \longrightarrow Y \subset A$ between uniform algebras is a $\psi-$composition operator in modulus on $\delta Y \subset A$. If, in addition ,

i $T(1) = 1$ and $T(i) = 1$ or

ii T preserves the peripheral spectra of all C−peaking functions of X ,

then T is an isometric unital algebra isomorphism. We note that the operator T in Theorem 4.2.11 is not assumed a priori to be linear or continuous. Both the norm-linearity and either condition [1] or [ii] are necessary conditions for T to be an isomorphism in Theorem 4.2.11. For example , the operator $Tm = -m$ is norm-linear since $\|\lambda Tm + \mu Tn\| = \|\lambda(-m) + \mu(-n)\| = \|\lambda m + \mu n\|$ but does not preserve the peripheral spectra of all C−peaking functions of $X \subset A$ (for example, $\sigma_\pi(1) = 1$ but $\sigma_\pi(T(1)) = \sigma_\pi(-1) = -1$), nor does it satisfy condition [i] because $T(1) = -1$ and $T(i) = -i$. This operator is not an algebra isomorphism because it is not multiplicative : $T(mn) = -mn$ but $TmTn = (-m)(-n) = mn$. On the other hand, while the operator

$$Tm = \frac{m|m|}{\|m\|}, m = 0, \text{ on } C(P)$$

clearly preserves the peripheral spectra of all algebra elements, so it preserves the periph- eral spectra of C−peaking functions in particular, and it satisfies $T(1) = 1$ and $T(i) = i$, it is also not norm-linear. For example , on $C[0, 1]$, if $m(a) = \frac{1}{2}a + 1$ and $n(a) = -a + 1$, then we have $\|m + n\| = max_{a\in[0,1]} \cdot -\frac{1}{2} a + 2\cdot = 2$ so T is not norm-additive and thus not norm-linear. This operator is not an algebra isomorphism because it, too, is not multiplicative: for example, if $m(a) = a$ and $n(a) = -a + 1$ on $C[0, 1]$, then $\|mn\| = \frac{1}{4}$, so

$$T(mn) = \frac{a(-a+1)|a(-a+1)|}{\frac{1}{4}} = 4a(-a + 1)\, |a(-a + 1)| = 4a\, |a|\, (-a + 1)\, |-a + 1|$$

and

$$TmTn = a\,|a|\,(-a + 1)\,|-a + 1|,$$

which are assuredly not equal.

The following corollary states that multiples of norm-linear operators are also algebra isomorphisms.

Theorem 4.3 *A mapping $T : X \subset A \longrightarrow Y \subset A$ that satisfies*

$$\|\lambda Tm + \mu Tn\| = C\,\|\lambda m + \mu n\| \text{ for some real number } C > 0,\ \forall \lambda, \mu \in \mathbb{C}$$
and

$$\forall m, n \in X \subset A$$

is a $\psi-$composition operator in modulus on $\delta Y \subset A$. If in addition, beginitemize i $T(1) = C$ and $T(i) = Ci$ or

ii $\sigma_\pi(Th) = \sigma_\pi(Ch)$, $\forall h \in \mathbb{C} \cdot \mathrm{P}(X)$,

Definition 4.6. *An operator $T : X \subset A \longrightarrow Y \subset A$ between uniform algebras for which*

$$\sigma_\pi(Tm + Tn) \cap \sigma_\pi(m + n)\ f = \emptyset$$

is called a weakly peripherally-additive operator, and an operator T for which

$$\sigma_\pi(\lambda Tm + \mu Tn) \cap \sigma_\pi(\lambda m + \mu n)\ f = \emptyset$$

is called a weakly peripherally-linear operator.

It is clear that weakly-peripherally additive operators are norm-additive and that weakly peripherally-linear operators are norm-linear, so Theorem 4.2.7 and 4.2.11 alsi imply the following improvements of the the major result of [24], which are in the spirit of Lambert, Luttman and Tonev's improvement in [12] to the results in [14].

Proposition 4.7. *Any Norm-additive in modulus, additive bijection $T : X \subset A \longrightarrow Y \subset A$ is a $\psi-$composition operator in modulus on $\delta Y \subset A$. If, in addition, either*

i $T(1) = 1$ *and* $T(i) = i$ *or*

ii T *preserves the peripheral spectra of all* C*−peaking functions of* $X \subset A$,

then T is a $\psi-$composition operator on $\delta Y \subset A$. Hence, the operator $T^{\dagger} : X|_{\delta X \subset A} \longrightarrow Y|_{\delta Y \subset A}$ that T induces is an algebra isomorphism and the restriction algebras $X|_{\delta X \subset A}$ and $Y|_{\delta Y \subset A}$ are algebraically isomorphic.

Proof. Since by Lemma 4.2 every surjective, norm-additive operator is injective and addi- tive, Proposition 4.1.10 and Proposition 4.2.6 give the following characterization of norm- additive operators that are also norm-additive in modulus. (Norm-Additive operators). Any norm-additive and norm-additive in modulus surjection $T : X \subset A \longrightarrow Y \subset A$ between uniform algebras is a $\psi-$composition operator in modulus on $\delta Y \subset A$. If , in addition, either

i $T(1) = 1$ and $T(i) = i$ or

ii T preserves the peripheral spectra of all C−peaking functions of X ,

then T is an isometric unital algebra isomorphism. We note that the operator T in Theorem 4.2.7 is not assumed to be linear or continuous. The Mazur-Ulam theorem (Theorem 1.1.1) implies that any surjective operator that preserves the distance between algebra elements is R−linear, so $\|Tm - Tn\| = \|m - n\|$ implies that $\|Tm + Tn\| = \|Tm - T(-n)\| = \|m - (-n)\| = \|m + n\|$.

Thus, Theorem 4.3 also holds for surjective norm-additive in modulus isometries T (for which $\|Tm - Tn\| = \|m_n\|$) with $T(0) = 0$, so it extends the Banach-Stone result (Theorem 1.2.3) mentioned in Chapter 1 to the case of uniform algebras. Theorem 4.3 implies the following relationship between norm-linear and norm-additive operators.

5. Conclusion

In summary, we have determined sufficient conditions for which a surjective map between unital C^*-algebras is an algebra automorphism, we have shown that If A is a unital C^*-algebra which is commutative then it is isomorphic to the space $C(P)$ of all continuous functions on a compact set P and uniform algebra is a sub algebra of the space $C(P)$. Therefore if $X \subset C(P)$ and $Y \subset C(S)$ are uniform algebras with Choquet boundary δX and δY , it is shown that if $T : X \longrightarrow Y$ is a surjection that preserves the norm of the sums of the moduli of algebra elements, then T induces a homoemorphism ϕ between the Choquet boundaries of X and Y such that $|Tm| = |m \circ \phi|$ on the Choquet boundary of Y . If, in addition, T preserves the norms of all linear combi- nations of algebra elements and either preserves both i and 1 or the peripheral spectra of C-peaking function, then T is a composition operator and thus an algebra automorphism.

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