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Consistency Analysis of Three Dimensional Advection-Diffusion Equation with a Mixed Derivative

George Ochieng Ogongo[1], Oduor Okoya M. E[2], Omolo Ongati[3]

[1,2,3] Department of Pure and Applied mathematics

JOOST, P.O. Box 210-40601, Bondo, Kenya

george.ogongo@yahoo.com, oduorokoya2005@gmail.com, nomoloongati@gmail.com

Abstract

The paper studies consistency analysis for (3+1) Dimensional Advection – Diffusion equation with a mixed derivative. Taylor series expansion is used to generate the finite difference scheme of Alternating Direction Explicit (ADE) scheme and Alternating Direction Implicit (ADI) scheme. The two schemes are found to be consistent with the model equations.

Key words: (3+1) Dimensional Advection Diffusion Equation, Partial Differential Equations (PDE'S), Alternating Direction Explicit (ADE) scheme, Alternating Direction Implicit (ADI) scheme.

Mathematics Subject Classification: Primary 65N30, 65M12, 65M06; Secondary 65D05, 65M22, 65M60

Introduction

Use of Advection – Diffusion equation in various fields of science like transport of heat, sediment, ground water and surface flow pollutants are fully sufficient for researchers to show interest in solving this equation. Many researchers like Bear [1] tried to propose analytical solutions for these type of equations, but in recent years researchers like Beny [2] have shown more interest thereby introducing numerical solutions to these kind of equations. As noted earlier, most of the researchers showed interest to present numerical solutions for Advection – Diffusion Equation instead of analytical solutions.

Brief review of work done by attention to the data was done by Young and et al [66] who developed an algorithm to solve fully conservative, high resolution Advection – Diffusion Equation in irregular geometries. In this algorithm they developed Finite Volume Method to solve this equation. Bobenko [3] in order to numerically integrate the semi – discrete equation arising arising after the spatial discretization of Advection – Reaction – Diffusion Equation applied two variable step linearly implicit Runge – Kutta methods of order 3 amd 4 equations.

Chapra [5] used the Euclerian – Lagrangian localized adjoin method on non – uniform time steps and unstructured meshes to solve the Advection – Diffusion Equation. Doyo [9] tried to develop an algorithm by second and third order accuracy with finite with finite – difference method to solve the convection – diffusion equation. In this algorithm they used to counter error mechanism to reduce numerical dispersion. One of the researchers that tried to solve Advection – Diffusion Equation in implicit condition is Douglas [8]. He solved the equation with Finite Difference Method by using the upwind and Crank – Nicolson schemes.

First,we derive the finite difference forms of ADE and ADI methods for the given model equation and then present an algorithm for each method.

The model equation

The research examines the consistency of the Alternating Direction Explicit (ADE) scheme and Alternating Direction Implicit (ADI) scheme for solving the (3+1) Dimensional Advection-Diffusion equation

$$f_1(x,y,z,t)\frac{\partial^2 C}{\partial x^2}+f_2(x,y,z,t)\frac{\partial^2 C}{\partial y^2}+f_3(x,y,z,t)\frac{\partial^2 C}{\partial z^2}+f_4(x,y,z,t)\frac{\partial^2 C}{\partial x\partial y}+f_5(x,y,z,t)\frac{\partial C}{\partial x}+ f_6(x,y,z,t)\frac{\partial C}{\partial y}=C_t \quad (1)$$

which is used to model physical process of Advection-Diffusion in a (3+1) Dimensional system such as one involving contaminant concentration in aquifer. The coefficients $f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t) f_4(x,y,z,t)$ represent the diffusion parameters (diffusivity) and $f_5(x,y,z,t)$ *and* $f_6(x,y,z,t)$ are the advection parameters (velocity). The equation is parabolic and is derived from the principle of conservation of mass using Fick's law of conservation in fluid flow problems as presented by (Morton 1971).The Alternating Direction Explicit(ADE) scheme developed for the equation is given by:-

$$4qC_{i,j,k}^{n+1}=4C_{i+1,j,k}^{n}-24C_{i,j,k}^{n}+4C_{i-1,j,k}^{n}+4C_{i,j+1,k}^{n}+4C_{i,j-1,k}^{n}+4C_{i,j,k+1}^{n}4C_{i,j,k-1}^{n}+ C_{i+1,j+1,k}^{n}-C_{i+1,j-1,k}^{n}-C_{i-1,j+1,k}^{n}+C_{i-1,j-1,k}^{n}+2qC_{i+1,j,k}^{n}+2qC_{i,j+1,k}^{n} \quad (2)$$

and the Alternating Direction Implicit (ADI) scheme developed for the equation is given by:-

$$4qC_{i,j,k}^{n+1}+4C_{i-1,j,k}^{n+1}-8C_{i,j,k}^{n+1}-4C_{i+1,j,k}^{n+1}=4qC_{i,j,k}^{n}-16C_{i,j,k}^{n}+4C_{i,j+1,k}^{n}+4C_{i,j-1,k}^{n}+ 4C_{i,j,k+1}^{n}+4C_{i,j,k-1}^{n}+C_{i+1,j+1,k}^{n}-C_{i+1,j-1,k}^{n}-C_{i-1,j+1,k}^{n}+C_{i-1,j-1,k}^{n}+2qC_{i+1,j,k}^{n}- 2qC_{i-1,j,k}^{n}+2qC_{i,j+1,k}^{n}-2qC_{i,j-1,k}^{n} \quad (3)$$

Properties of numerical schemes

Many techniques are available for numerical simulation work and in order to quantify how well a particular numerical technique performs in generating a solution to a problem, there are four fundamental criteria that can be applied to compare and contrast different methods. The concepts are accuracy, consistency, stability and convergence. The method of Finite Difference Method is one of the most valuable methods of approximating numerical solution of Partial

Differential Equations (PDEs). Before numerical computations are made, these four important properties of finite difference equations must be considered.

(a) **Accuracy**: Is a measure of how well the discrete solution represents the exact solution of the problem. Two quantities exist to measure this, the local or truncation error, which measures how well the difference equations match the differential equations, and the global error which reacts to the overall error in the solution. This is not possible to find unless the exact solution is known.

(b) **Stability**: A finite difference scheme is stable if the error made at one time step of the calculation do not cause the errors to be magnified as the computations are continued. A neutrally stable scheme is one in which errors remain constant as the computation are carried forward. If the errors decay are eventually damp out, the numerical scheme is said to be stable. If on the contrary, the errors grow with time the numerical scheme is said to be unstable.

(c) **Consistency**: When a truncation error goes to zero, a finite difference equation is said to be consistent or compatible with a partial differential equation. Consistency requires that the original equations can be recovered from the algebraic equations. Obviously this is a minimum requirement for any discretization.

(d) **Convergence**: A solution of a set of algebraic equations is convergent if the approximate solution approaches the exact solution of the Partial Differential Equations (PDEs) for each value of the independent variable. For example, as the mesh sizes approaches zero, the grid spacing and time step also goes to zero.

Lax had proved that under appropriate conditions a consistent scheme is convergent if and only if it is stable. According to $Lax - Richtmyer\ Equivalence\ Theorem$ which states that "given a properly posed linear initial value problem and a finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence"

Consistency of the numerical schemes

The Alternating Directional Explicit scheme generated from the equation assuming that $\Delta x = \Delta y = \Delta z = \Delta t = q$ and for some $f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t), f_4(x,y,z,t) = 1\ and\ f_5(x,y,z,t)\ and\ f_6(x,y,z,t) = \frac{1}{2}$ will be developed.

Consistency of Alternating Direction Explicit (ADE) scheme

We analyse consistency of the Alternating Direction Explicit (ADE) scheme for equation (1). A Taylor expansion of the individual terms in this scheme of equation (2) will yield:-

$$C_{i,j}^{n+1} = (x,y,t+\Delta t) = C(x,y,t) + (\Delta t)C_t + \frac{1}{2}(\Delta t)^2 C_{tt} + \frac{1}{6}(\Delta t)^3 C_{ttt} + \frac{1}{24}(\Delta t)^4 C_{tttt} + \frac{1}{120}(\Delta t)^5 C_{ttttt} + \frac{1}{720}(\Delta t)^6 C_{tttttt} + \cdots 0(\Delta t)^7 \quad (4)$$

$$C_{i+1,j,k}^{n}=(x+\Delta x,y,z,t)=C(x,y,z,t)+(\Delta x)C_x+\frac{1}{2}(\Delta x)^2C_{xx}+\frac{1}{6}(\Delta x)^3C_{xxx}+\frac{1}{24}(\Delta x)^4C_{xxxx}+\frac{1}{120}(\Delta x)^5C_{xxxxx}+\frac{1}{720}(\Delta x)^6C_{xxxxxx}+\cdots 0(\Delta x)^7 \quad (5)$$

$$C_{i-1,j,k}^{n}=(x-\Delta x,y,z,t)=C(x,y,z,t)-(\Delta x)C_x+\frac{1}{2}(\Delta x)^2C_{xx}-\frac{1}{6}(\Delta x)^3C_{xxx}+\frac{1}{24}(\Delta x)^4C_{xxxx}-\frac{1}{120}(\Delta x)^5C_{xxxxx}+\frac{1}{720}(\Delta x)^6C_{xxxxxx}+\cdots 0(\Delta x)^7 \quad (6)$$

$$C_{i,j+1,k}^{n}=(x,y+\Delta y,z,t)=C(x,y,z,t)+(\Delta y)C_y+\frac{1}{2}(\Delta y)^2C_{yy}+\frac{1}{6}(\Delta y)^3C_{yyy}+\frac{1}{24}(\Delta y)^4C_{yyyy}+\frac{1}{120}(\Delta y)^5C_{yyyyy}+\frac{1}{720}(\Delta y)^6C_{yyyyyy}+\cdots 0(\Delta y)^7 \quad (7)$$

$$C_{i,j-1,k}^{n}=(x,y-\Delta y,z,t)=C(x,y,z,t)-(\Delta y)C_y+\frac{1}{2}(\Delta y)^2C_{yy}-\frac{1}{6}(\Delta y)^3C_{yyy}+\frac{1}{24}(\Delta y)^4C_{yyyy}-\frac{1}{120}(\Delta y)^5C_{yyyyy}+\frac{1}{720}(\Delta y)^6C_{yyyyyy}+\cdots 0(\Delta y)^7 \quad (8)$$

$$C_{i,j,k+1}^{n}=(x,y,z+\Delta z,t)=C(x,y,z,t)+(\Delta z)C_z+\frac{1}{2}(\Delta z)^2C_{zz}+\frac{1}{6}(\Delta z)^3C_{zzz}+\frac{1}{24}(\Delta z)^4C_{zzzz}+\frac{1}{120}(\Delta z)^5C_{zzzzz}+\frac{1}{720}(\Delta z)^6C_{zzzzzz}+\cdots 0(\Delta z)^7 \quad (9)$$

$$C_{i,j,k-1}^{n}=(x,y,z-\Delta z,t)=C(x,y,z,t)-(\Delta z)C_z+\frac{1}{2}(\Delta z)^2C_{zz}-\frac{1}{6}(\Delta z)^3C_{zzz}+\frac{1}{24}(\Delta z)^4C_{zzzz}-\frac{1}{120}(\Delta z)^5C_{zzzzz}+\frac{1}{720}(\Delta z)^6C_{zzzzzz}+\cdots 0(\Delta z)^7 \quad (10)$$

$$C_{i+1,j,k}^{n}=C(x,y,z,t)+(\Delta x)C_x+\frac{1}{2}(\Delta x)^2C_{xx}-\frac{1}{6}(\Delta x)^3C_{xxx}+\frac{1}{24}(\Delta x)^4C_{xxxx}-\frac{1}{120}(\Delta x)^5C_{xxxxx}+\frac{1}{720}(\Delta x)^6C_{xxxxxx}+\cdots 0(\Delta x)^7 \quad (11)$$

$$C_{i,j+1,k}^{n}=C(x,y,z,t)+(\Delta y)C_y+\frac{1}{2}(\Delta y)^2C_{yy}-\frac{1}{6}(\Delta y)^3C_{yyy}+\frac{1}{24}(\Delta y)^4C_{yyyy}-\frac{1}{120}(\Delta y)^5C_{yyyyy}+\frac{1}{720}(\Delta y)^6C_{yyyyyy}+\cdots 0(\Delta y)^7 \quad (12)$$

$$C_{i,j,k}^{n}=(x,y,z,t) \quad (13)$$

The scheme in equation (2) can be re-arranged and be written as

$$4qC_{i,j,k}^{n+1}=4qC_{i,j,k}^{n}+4qC_{i-1,j,k}^{n}-24qC_{i,j,k}^{n}+4C_{i,j+1,k}^{n}+4C_{i,j-1,k}^{n}+4C_{i,j,k+1}^{n}+4C_{i,j,k-1}^{n}+C_{i+1,j+1,k}^{n}-C_{i+1,j-1,k}^{n}+C_{i-1,j+1,k}^{n}-C_{i-1,j-1,k}^{n}+2qC_{i+1,j,k}^{n}+2qC_{i,j+1,k}^{n} \quad (14)$$

Adding equation (4) to equation (5) and multiplying by 4 while we let $\Delta x=\Delta y=\Delta z=\Delta t=q$ will now yield

$$4(C_{i+1,j,k}^{n}+C_{i-1,j,k}^{n})=8(x,y,z,t)+4q^2C_{xx}+\frac{1}{3}q^4C_{xxxx}+\cdots \quad (15)$$

Adding equation (6) to equation (7) and multiplying by 4 while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ will now yield

$$4(C_{i,j+1,k}^{n} + C_{i,j-1,k}^{n}) = 8(x,y,z,t) + 4q^2 C_{yy} + \frac{1}{3} q^4 C_{yyyy} + \cdots \quad (16)$$

Adding equation (8) to equation (9) and multiplying by 4 while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ will now yield

$$4(C_{i,j,k+1}^{n} + C_{i,j,k-1}^{n}) = 8(x,y,z,t) + 4q^2 C_{zz} + \frac{1}{3} q^4 C_{zzzz} + \cdots \quad (17)$$

Subtracting equations generated from finite difference equivalents of mixed derivative terms while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ will now yield

$$(C_{i+1,j+1,k}^{n} - C_{i+1,j-1,k}^{n}) = 2qC_y + 2q^2 C_{xy} + q^3 C_{xxy} + \cdots \quad (18)$$

Subtracting equations generated from finite difference equivalents of mixed derivative terms while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ will now yield

$$(C_{i-1,j-1,k}^{n} - C_{i-1,j+1,k}^{n}) = -q^3 C_{xxy} - \frac{1}{3} q^3 C_{xxx} + 2q^2 C_{xy} - 2qC_y - 2qC_x + \cdots \quad (19)$$

Adding equation (4) to equation (5) and multiplying by 2 while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ will now yield,

$$2q(C_{i+1,j,k}^{n} + C_{i,j+1,k}^{n}) = 4qC(x,y,z,t) + 2q^2 C_x + 2q^2 C_y + q^3 C_{xx} + q^3 C_{yy} + \cdots \quad (20)$$

Multiplying equation (12) by -24 will give

$$-24C_{i,j,k}^{n} = -24(x,y,z,t) + \cdots \quad (21)$$

Multiplying equation (3) by $4q$ will give

$$-4qC_{i,j,k}^{n+1} = -4q(x,y,z,t+\Delta t) = -4qC(x,y,z,t) - 4q(\Delta t)C_t - 2q(\Delta t)^2 C_{tt} - q\frac{2}{2}(\Delta t)^3 C_{ttt} - q\frac{1}{6}(\Delta t)^4 C_{tttt} - q\frac{1}{30}(\Delta t)^5 C_{ttttt} + q\frac{1}{180}(\Delta t)^6 C_{tttttt} - \cdots 0(\Delta t)^7 \quad (22)$$

Substituting the RHS of equations (14), (15), (16), (17), (18), (19), (20), and (21), into the ADE scheme in (2) will now yield

$$4q^2 C_{xx} + 4q^2 C_{yy} + 4q^2 C_{zz} + 2q^2 C_{xy} + 2q^2 C_{xy} + 2q^2 C_x + 2q^2 C_y - 4q^2 C_t + 8(x,y,z,t) + \frac{1}{3} q^4 C_{xxxx} + 8(x,y,z,t) + \frac{1}{3} q^4 C_{yyyy} + 8(x,y,z,t) + \frac{1}{3} q^4 C_{zzzz} + 2qC_y + q^3 C_{xxy} - 2qC_y - 2qC_x + 4qC(x,y,t) - q^3 C_{xxy} + q^3 C_{yy} - 24(x,y,z,t) - 4qC(x,y,t) - 2q^3 C_{tt} - 2q^4 C_{ttt} - \frac{1}{6} q^5 C_{tttt} = 0 \quad (23)$$

Dividing equation (23) by $4q^2$ gives

$$C_{xx} + C_{yy} + C_{zz} + C_{xy} + \frac{1}{2} C_x + \frac{1}{2} C_y - C_t + \frac{1}{12} q^2 C_{xxxx} + \frac{1}{12} q^2 C_{yyyy} + \frac{1}{12} q^2 C_{zzzz} + \frac{1}{2q} C_x + \frac{1}{4} C_{yy} - \frac{1}{2} qC_{tt} - \frac{1}{2} q^2 C_{ttt} - \frac{1}{24} q^3 C_{tttt} + 0\big((\Delta x), (\Delta y), (\Delta z), (\Delta t)\big) = 0 \quad (24)$$

Thus the error $E_{i,j,k}^{n}$ for the scheme in equation (24) is

$$\frac{1}{12}q^2C_{xxxx}+\frac{1}{12}q^2C_{yyyy}+\frac{1}{12}q^2C_{zzzz}+\frac{1}{2q}C_x+\frac{1}{4}C_{yy}-\frac{1}{2}qC_{tt}-\frac{1}{2}q^2C_{ttt}-\frac{1}{24}q^3C_{tttt}+0\big((\Delta x),(\Delta y),(\Delta z),(\Delta t)\big)=0 \tag{25}$$

Consistency of ADI Scheme

We analyse consistency of the ADI in the equation (3). A Taylor expansion of the individual terms in this scheme of equation (3) are represented by

$$C_{i+1,j,k}^{n+1}=(x+\Delta x,y,z,t+\Delta x)=C(x,y,z,t)+(\Delta x)C_x+(\Delta t)C_t+\frac{1}{2}(\Delta x)^2C_{xx}+(\Delta x)(\Delta t)C_{xt}+\frac{1}{2}(\Delta t)^2C_u+\frac{1}{6}(\Delta x)^3C_{xxx}\dots 0(\Delta x)^4(\Delta x)^4 \tag{26}$$

$$C_{i,j,k}^{n+1}=(x,y,z,t+\Delta x)=C(x,y,z,t)+(\Delta t)C_t+\frac{1}{2}(\Delta t)^2C_{tt}+\frac{1}{6}(\Delta x)^3C_{ttt}+\frac{1}{24}(\Delta t)^4C_{tttt}+\frac{1}{120}(\Delta t)^5C_{ttttt}+\frac{1}{720}(\Delta x)^6C_{tttttt}\dots 0(\Delta t)^7 \tag{27}$$

$$C_{i-1,j,k}^{n+1}=(x-\Delta x,y,z,t+\Delta t)=C(x,y,z,t)-(\Delta x)C_x+(\Delta t)C_t+\frac{1}{2}(\Delta x)^2C_{xx}-(\Delta x)(\Delta t)C_{xt}+\frac{1}{2}(\Delta t)^2C_{tt}-\frac{1}{6}(\Delta x)^3C_{xxx}\dots 0(\Delta x)^4(\Delta x)^4 \tag{28}$$

$$C_{i,j+1,k}^{n+1}=(x,y+\Delta y,z,t)=C(x,y,z,t)+(\Delta y)C_y+\frac{1}{2}(\Delta y)^2C_{yy}+\frac{1}{6}(\Delta y)^3C_{yyy}+\frac{1}{24}(\Delta y)^4C_{yyyy}+\frac{1}{120}(\Delta y)^5C_{yyyyy}+\frac{1}{720}(\Delta y)^6C_{yyyyyy}+\cdots 0(\Delta y)^7 \tag{29}$$

$$C_{i,j-1,k}^{n}=(x,y-\Delta y,z,t)=C(x,y,z,t)-(\Delta y)C_y+\frac{1}{2}(\Delta y)^2C_{yy}-\frac{1}{6}(\Delta y)^3C_{yyy}+\frac{1}{24}(\Delta y)^4C_{yyyy}-\frac{1}{120}(\Delta y)^5C_{yyyyy}+\frac{1}{720}(\Delta y)^6C_{yyyyyy}-\cdots 0(\Delta y)^7 \tag{30}$$

$$C_{i,j,k+1}^{n}=(x,y,z+\Delta z,t)=C(x,y,z,t)+(\Delta z)C_z+\frac{1}{2}(\Delta z)^2C_{zz}+\frac{1}{6}(\Delta z)^3C_{zzz}+\frac{1}{24}(\Delta z)^4C_{zzzz}+\frac{1}{120}(\Delta z)^5C_{zzzzz}+\frac{1}{720}(\Delta z)^6C_{zzzzzz}+\cdots 0(\Delta z)^7 \tag{31}$$

$$C_{i,j,k-1}^{n}=(x,y,z-\Delta z,t)=C(x,y,z,t)-(\Delta z)C_z+\frac{1}{2}(\Delta z)^2C_{zz}-\frac{1}{6}(\Delta z)^3C_{zzz}+\frac{1}{24}(\Delta z)^4C_{zzzz}-\frac{1}{120}(\Delta z)^5C_{zzzzz}+\frac{1}{720}(\Delta z)^6C_{zzzzzz}+\cdots 0(\Delta z)^7 \tag{32}$$

$$C_{i,j,k}^{n}=(x,y,z,t) \tag{33}$$

$$C_{i+1,j+1,k}^{n}=(x+\Delta x,y+\Delta y,z,t)=C(x,y,z,t)+(\Delta x)C_x+\left(\Delta_y\right)C_y+\frac{1}{2}(\Delta x)^2C_{xx}+(\Delta x)(\Delta y)C_{xy}+\frac{(\Delta y)^2}{2}C_{yy}+\frac{(\Delta y)^3}{6}C_{xxx}+3\frac{(\Delta x)^2\Delta y}{6}C_{xxy}+\cdots \tag{34}$$

$$C_{i+1,j-1,k}^{n}=(x+\Delta x,y-\Delta y,z,t)=C(x,y,z,t)+(\Delta x)C_x-\left(\Delta_y\right)C_y+\frac{1}{2}(\Delta x)^2C_{xx}-(\Delta x)(\Delta y)C_{xy}+\frac{(\Delta y)^2}{2}C_{yy}+\frac{(\Delta y)^3}{6}C_{xxx}-3\frac{(\Delta x)^2\Delta y}{6}C_{xxy}+\cdots \tag{35}$$

$$C_{i-1,j+1,k}^{n} = (x-\Delta x, y+\Delta y, z, t) = C(x,y,z,t) + (\Delta x)C_x - (\Delta_y)C_y + \frac{1}{2}(\Delta x)^2 C_{xx} - (\Delta x)(\Delta y)C_{xy} + \frac{(\Delta y)^2}{2}C_{yy} - \frac{(\Delta y)^3}{6}C_{xxx} - 3\frac{(\Delta x)^2\Delta y}{6}C_{xxy} - \cdots \quad (36)$$

$$C_{i-1,j-1,k}^{n} = (x-\Delta x, y-\Delta y, z, t) = C(x,y,z,t) - (\Delta x)C_x - (\Delta_y)C_y + \frac{1}{2}(\Delta x)^2 C_{xx} - (\Delta x)(\Delta y)C_{xy} + \frac{(\Delta y)^2}{2}C_{yy} - \frac{(\Delta y)^3}{6}C_{xxx} - 3\frac{(\Delta x)^2\Delta y}{6}C_{xxy} - \cdots \quad (37)$$

$$C_{i+1,j,k}^{n} = C(x,y,z,t) + (\Delta x)C_x + \frac{1}{2}(\Delta x)^2 C_{xx} + \frac{1}{6}(\Delta x)^3 C_{xxx} + \frac{1}{24}(\Delta x)^4 C_{xxxx} + \frac{1}{120}(\Delta x)^5 C_{xxxxx} + \frac{1}{720}(\Delta x)^6 C_{xxxxxx} + \cdots 0(\Delta x)^7 \quad (38)$$

$$C_{i-1,j,k}^{n} = C(x,y,z,t) - (\Delta x)C_x + \frac{1}{2}(\Delta x)^2 C_{xx} - \frac{1}{6}(\Delta x)^3 C_{xxx} + \frac{1}{24}(\Delta x)^4 C_{xxxx} - \frac{1}{120}(\Delta x)^5 C_{xxxxx} + \frac{1}{720}(\Delta x)^6 C_{xxxxxx} + \cdots 0(\Delta x)^7 \quad (39)$$

$$C_{i,j+1,k}^{n} = C(x,y,z,t) + (\Delta y)C_y + \frac{1}{2}(\Delta y)^2 C_{yy} + \frac{1}{6}(\Delta y)^3 C_{yyy} + \frac{1}{24}(\Delta y)^4 C_{yyyy} + \frac{1}{120}(\Delta y)^5 C_{yyyyy} + \frac{1}{720}(\Delta y)^6 C_{yyyyyy} + \cdots 0(\Delta y)^7 \quad (40)$$

$$C_{i,j-1,k}^{n} = C(x,y,z,t) - (\Delta y)C_y + \frac{1}{2}(\Delta y)^2 C_{yy} - \frac{1}{6}(\Delta y)^3 C_{yyy} + \frac{1}{24}(\Delta y)^4 C_{yyyy} - \frac{1}{120}(\Delta y)^5 C_{yyyyy} + \frac{1}{720}(\Delta y)^6 C_{yyyyyy} - \cdots 0(\Delta y)^7 \quad (41)$$

$$C_{i,j,k}^{n+1} = (x,y,z,t+\Delta t) = C(x,y,z,t) + (\Delta t)C_t + \frac{1}{2}(\Delta t)^2 C_{tt} + \frac{1}{6}(\Delta t)^3 C_{ttt} + \frac{1}{24}(\Delta t)^4 C_{tttt} + \frac{1}{120}(\Delta t)^5 C_{ttttt} + \frac{1}{720}(\Delta t)^6 C_{tttttt} + \cdots 0(\Delta t)^7 \quad (42)$$

$$C_{i,j,k}^{n} = (x,y,z,t) + \cdots \quad (43)$$

Rearranging equation (3) we get

$$4C_{i+1,j,k}^{n+1} + 4C_{i-1,j,k}^{n+1} - 8C_{i,j,k}^{n+1} + 4C_{i,j+1,k}^{n} + 4C_{i,j-1,k}^{n} + 4C_{i,j,k+1}^{n} + 4C_{i,j,k-1}^{n} + C_{i+1,j+1,k}^{n} - C_{i+1,j-1,k}^{n} + C_{i-1.j-1,k}^{n} - C_{i-a,j+1,k}^{n} + 2qC_{i+1,j,k}^{n} - 2qC_{i-1,j,k}^{n} + 2qC_{i,j+1,k}^{n} - 2qC_{i,j-1,k}^{n} - 4qC_{i,j,k}^{n+1} - 4qC_{i,j,k}^{n} - 16C_{i,j,k}^{n} = 0 \quad (44)$$

Substituting finite differences equivalents into each component of the above equation as follows, Adding equations (26) and (28) and multiplying by 4 while we let $\Delta x = \Delta y = \Delta z = \Delta t = q$ gives

$$4(C_{i+1,j,k}^{n+1} + C_{i-1,j,k}^{n+1}) = 8C(x,y,z,t) + 8qC_t + 4q^2 C_{xx} + 4q^2 C_{tt} + \cdots \quad (45)$$

Adding equation (29) and (30) and multiplying by 4 gives

$$4(C^n_{i,j+1,k}+C^n_{i,j-1,k})=8C(x,y,z,t)+4(\Delta y)^2C_{yy}+\tfrac{1}{3}(\Delta y)^4C_{yyyy}+\tfrac{1}{90}(\Delta y)^6C_{yyyyyy}+\cdots \tag{46}$$

Adding equation (31) and (32) and multiplying by 4 gives

$$4(C^n_{i,j,k+1}+C^n_{i,j,k-1})=8C(x,y,z,t)+4(\Delta z)^2C_{zz}+\frac{1}{3}(\Delta z)^4C_{zzzz}+\frac{1}{90}(\Delta z)^6C_{zzzzzz}+\cdots \tag{47}$$

Subtracting equation (35) from (34) gives

$$(C^n_{i+1,j+1}-C^n_{i+1,j-1})=2(\Delta y)C_y+2(\Delta x)(\Delta y)C_{xy}+\tfrac{1}{3}(\Delta x)^3C_{xxx}+\cdots \tag{48}$$

Subtracting equation (39) from (38) and multiplying by $2q$

$$2q(C^n_{i+1,j,k}-C^n_{i-1,j,k})=4q(\Delta x)C_x+\tfrac{2q}{3}(\Delta x)^3C_{xxx}+\tfrac{q}{30}(\Delta x)^5C_{xxxxx}+\cdots \tag{49}$$

Subtracting equation (41) from (40) and multiplying by $-2q$ gives

$$-2q(C^n_{i,j+1,k}-C^n_{i,j-1,k})=4q(\Delta y)C_y+\frac{2q}{3}(\Delta y)^3C_{yyy}+\frac{4q}{3}(\Delta x)^3C_{xxx}+\frac{q}{30}(\Delta y)^5C_{yyyyy}+\cdots \tag{50}$$

Subtracting equation (36) from (37)

$$(C^n_{i-1,j-1,k}-C^n_{i-1,j+1,k})=-2y\Delta y+2(\Delta x)(\Delta y)C_{xy}+\frac{1}{3}(\Delta x)^3C_{xxx}+\cdots \tag{51}$$

Adding equation (27) and (33) and multiplying by $-4q$ gives

$$\begin{aligned}-4q(C^{n+1}_{i,j,k}+C^n_{i,j,k})\\&=-8qC(x,y,z,t)-4q(\Delta t)C_t-4q(\Delta t)^2C_{tt}-\frac{q}{3}(\Delta t)^4C_{tttt}-\frac{q}{90}(\Delta t)^6C_{tttttt}\dots\end{aligned} \tag{52}$$

Multiplying equation (33) by -16 gives

$$-16(C^n_{i,j,k})=-16(x,y,z,t) \tag{53}$$

Multiplying equation (42) by -8 gives

$$-8(C_{i,j,k}^{n+1}) = -8(x,y,z,t+\Delta t) = -8C(x,y,z,t) - 8(\Delta t)C_t - 4(\Delta t)^2 C_{tt} - \frac{4}{3}(\Delta t)^3 C_{tt} - \frac{1}{3}(\Delta t)^4 C_{tttt} - \frac{1}{15}(\Delta t)^5 C_{ttttt} - \frac{1}{9}(\Delta t)^6 C_{tttttt} + \cdots 0\,(\Delta t)^7 \qquad (54)$$

Adding the RHS of equations (45), (46), (47), (48), (49), (50), (51), (52), (53) and (54) while letting $(\Delta x) = (\Delta y) = (\Delta x) = (\Delta t) = q$ will yield,

$$4q^2C_{xx} + 4q^2C_{yy} + 4q^2C_{zz} + 2q^2C_{xy} + 2q^2C_{xy} + 4q^2C_x + 4q^2C_y + 4q^2C_t + 4q^2C_{tt} + 8qC_t + \frac{1}{90}q^6C_{xxxxxx} - 8C(x,y,z,t) + \frac{1}{3}q^4C_{yyyy} + \frac{1}{90}q^6C_{yyyyyy} + \frac{1}{3}q^4C_{zzzz} + \frac{1}{90}q^6C_{zzzzzz} + 8(x,y,z,t) + 2qC_y + \frac{1}{3}q^3C_{xxx} + \frac{2}{3}q^4C_{xxx} + \frac{1}{30}q^6C_{xxxxxx} + \frac{2}{3}q^4C_{yyy} + \frac{4}{3}q^4C_{xxx} + \frac{1}{30}q^6C_{yyyyy} - 2yq + \frac{1}{3}q^3C_{xxx} - 8qC(x,y,z,t) - 4q^2C_t + 4q^3C_{tt} + \frac{1}{3}q^5C_{tttt} + \frac{1}{90}q^7C_{tttttt} - 16(x,y,z,t) - 8qC_t - 4q^2C_{tt} - \frac{4}{3}q^3C_{ttt} - \frac{1}{3}q^4C_{tttt} + 8(x,y,z,t) + 8(x,y,z,t) - \frac{1}{15}q^5C_{ttttt} - \frac{1}{9}q^6C_{tttttt} - 8(x,y,z,t) = 0 \qquad (55)$$

Dividing equation (55) by $4q^2$ we obtain

$$C_{xx} + C_{yy} + C_{zz} + C_{xy} + C_x + C_y - C_t + \frac{1}{12}q^2C_{xxxx} + \frac{1}{360}q^4C_{xxxxxx} + \frac{1}{12}q^2C_{yyyy} + \frac{1}{360}q^4C_{yyyyyy} + \frac{1}{12}q^2C_{zzzzz} + \frac{1}{360}q^4C_{zzzzzz} + \frac{1}{2q}C_y + \frac{1}{12}qC_{xxx} + \frac{1}{6}q^2C_{xxx} + \frac{1}{120}q^4C_{xxxxx} + \frac{1}{6}q^2C_{yyy} + \frac{1}{3}q^2C_{xxx} + \frac{1}{120}q^4C_{yyyy} + \frac{y}{2q} + \frac{1}{12}qC_{xxx} + \frac{2}{q}C(x,y,z,t) + qC_{tt} + \frac{1}{12}q^3C_{tttt} + \frac{1}{360}q^5C_{tttttt} = 0 \qquad (56)$$

Thus the error E_{ijk}^n for this scheme in equation (56) is

$$\frac{1}{12}q^2C_{xxxx} + \frac{1}{360}q^4C_{xxxxxx} + \frac{1}{12}q^2C_{yyyy} + \frac{1}{12}q^2C_{zzzzz} + 0((\Delta x)^3, (\Delta y)^3, (\Delta z)^3, (\Delta t)^3) \qquad (57)$$

Conclusion

We note that the first seven terms of equation (24) and (56) are for the recovered PDE for our (3+1) Dimensional Advection-Diffusion equation and all the other terms are the truncation error which reduce to zero as $\Delta x \to \Delta y \to \Delta z \to \Delta t \to q \to 0$. Since the (3+1) Dimensional Advection-Diffusion equation has been recovered from the algebraic equation of the ADE and ADI scheme, we therefore conclude that the ADE and ADI scheme is consistent with the (3+1) Dimensional Advection-Diffusion differential equation (1)

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