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DRIVEN TWO-LEVEL QUANTUM SYSTEMS

By

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ABSTRACT

Two-Level Quantum System

Two-level quantum systems play a significant role in quantum mechanical processes like quantum computers and quantum information systems. In particular, the evolution of two-level system under the influence of driven fields is of great importance. Unfortunately, the exact solutions due to non commutability of quantum operator are hard to come by, and approximations usually have to be made. Barns et al. (2012) showed that a single general axis driven term and its corresponding evolution operator is determined by a single real function which is restricted to obey some initial conditions and experimental constraints. Any function satisfying these restrictions produced an exact analytical solution. We reproduced their result, which allows us to systematically find an unlimited number of analytically solvable control fields and present exact analytical formulas for the corresponding evolution operators.

Messina et al. (2014) showed that general exact solvable Hamiltonian and its corresponding evolution operator is determined by a single real function which is restricted to obey some constraints and initial conditions. Any function that satisfies these conditions and constraints generates exact solvable Hamiltonians and its corresponding evolution operator.

Introduction

1.1 Quantum Two-Level System

In quantum mechanics a two-level system $(also\ known\ as\ two - state\ system)$isa quantum system that can exist in any quantum superposition of two independent quantum states. The Hilbert Space describing such a system in two dimensions. Any two-level quantum systems can also be seen as a quantum bit$(Qubit)$.

Two-level system is the simplest quantum system that is of interest, since the dynamic of one level system is trivial. The mathematical frame work required for the analysis of quantum two-level system is that of linear differential equation and linear algebra of two dimensional space. As a result, the dynamic of a two-level system can be solved analytically without approximation. The generic behavior of the system is that the amplitude of wave function oscillates between the two levels. The quantum two-level system or a collection of such systems exhibit all the challenges and subtle futures for which the quantum theory is notorious. The two-level system can't be used as a description of absorption or decay because such process requires coupling to a continuum. Such processes involve exponential decay of amplitude, but the solution of quantum two-level system is oscillatory. Examples of two-level system are the polarization state of a photon, the spin of an electron etc.

Modern researchers of the quantum world want to implement quantum concepts and theories to make a quantum computer which can solve problems in small time easily, which would take a long time on a classical computer. For instant, however it is not so easy to apply such concepts and theories. To implement quantum mechanical concepts, we need to use quantum mechanical devices and circuits. The fabrication of such devices is complicated but possible. Some quantum mechanical devices are already made by scientists, such as Josephson junctions which can used as an artificial two-level system (this device is fabricated classically, but it behaves like a quantum mechanical device), superconducting quantum interference device $(SQUID)$ and semi-conductor quantum dots, etc.

The building block of quantum computers is a quantum bit or qubit. A qubit is different from classical bit. Classically a bit is represented by a two state system, like 0 or 1 strictly and a physical example of such system is a charging and discharging of a capacitor. There is no state which is in intermediate state or superposition of these two states. But quantum mechanically, it is quite strange because quantum mechanics is probabilistic. In quantum mechanics we have two state $|0\rangle$ and $|1\rangle$ as well as the superposition$\frac{|0\rangle+|1\rangle e^{\iota\varphi}}{2}$ of these states. The implementation of the superposition of the two-level system is quite difficult to achieve.

The idea of quantum two-level system with just two relevant quantum states is used to represent a family of physical system. For example, two-level systems have been used to describe spin and atomic collision. Moreover, some mesoscopic systems achieve in superconducting quantum circuits [1] and superconductor quantum dots can implemented as two-level system effectively.

Other examples of quantum two-level system are the spin- ½ particles in a magnetic field, the path of a photon in a beam splitter, the polarization of a photon etc. Quantum two-levelsystems exist in a quantum superposition of two independent quantum states. These two-level quantum systems allow simple confirmation of non-classical predictions of quantum mechanics and form the basis of quantum information.

1.2 The Driven TLS

A two-level system driven by an external field is known as driven two-level system. Driven two-level systems have great importance in the implementation of quantum computing. Driven field can control the two-level system and is thus very important for qubit control processes. When a two-level system interacting with a bath (environment) and the qubit states lose their coherence, then we can use driven fields to control this system. When an external field applied to a two-level system, the Hamiltonian has explicit time dependence. But unfortunately, the finding of evolution operator for a time dependent Hamiltonian is highly non-trivial. This is simply because the Hamiltonian at different times dose not commute, leading to the time ordering problems. We now study the driven two-level system in detail.

1.3 Approximate Solution of Quantum TLS Using Perturbation Theory

The driven TLS is one of the very few non-trivial time dependent problems whose dynamic can be solved for analytically. In this system we consider only two levels represented as $|0\rangle$ and $|1\rangle$. The Hamiltonian for this system can be expressed as the sum of the unperturbed part denoted as H_0 and a time-defendant perturbation part, $\mathrm{V}(t)$. Where the term $\mathrm{V}(t)$ is time dependent perturbation part of the Hamiltonian. The state of the system can be described as a superposition of these two states. Thus, at any given moment in time the state of a system can be written as;

$$|\Psi\rangle = C_0(t)e^{-iE_0t/\hbar}\ |0\rangle + C_1(t)e^{-iE_1t/\hbar}\ |1\rangle \quad (1.3.1)$$

This is the linear combination of both these states. If the Hamiltonian doesn't depend on time then C_0 *and* C_1 are constant and all the evolution is curried by the $e^{-iEt/\hbar}$ factor.

The time dependence in C_0 *and* C_1 is due to the time dependent perturbation $V(t)$.

Now the time dependent Schrodinger equation is

$$H|\Psi\rangle = i\hbar\ |\dot{\Psi}\rangle$$

$$\left(H_0 + V(t)\right)|\Psi\rangle = i\hbar\ |\dot{\Psi}\rangle \quad (1.3.2)$$

Where $V(t)$ is some time dependent perturbation or driven to move the system from $|0\rangle$ to $|1\rangle$. Use Eq. (1.3.1) in Eq. (1.3.2) we get

$$\Rightarrow C_0\ (H_0 + V(t))|0\rangle e^{-iE_0t/\hbar} + C_1\ (H_0 + V(t)|1\rangle e^{-iE_1t/\hbar} = i\hbar\ \frac{d}{dt}\,(C_0(t)e^{-iE_0t/\hbar}|0\rangle$$

$$+ C_1(t)e^{-iE_1t/\hbar}|1\rangle)$$

=> $$C_0\,(H_0 + V(t))|0\rangle\, e^{-\frac{iE_0 t}{\hbar}} + C_1\,(H_0 + V(t)|1\rangle e^{-\frac{iE_1 t}{\hbar}} = i\hbar\ [\,\dot{C}_0|0\rangle + C_0|0\rangle\,(-\frac{iE_0}{\hbar})]$$

$$e^{-\frac{iE_0 t}{\hbar}} + i\hbar[\,\dot{C}_1|1\rangle + C_1|1\rangle(-iE_1/\hbar)\,]\, e^{-\frac{iE_1 t}{\hbar}}$$

We cancel the static terms on both sides that lead to

$$C_0 V(t)|0\rangle e^{-iE_0 t/\hbar} + C_1\, V(t)\, |1\rangle e^{-iE_1 t/\hbar} = i\hbar[\,\dot{C}_0|0\rangle e^{-iE_0 t/\hbar} + \dot{C}_1|1\rangle e^{-\frac{iE_1 t}{\hbar}}] \quad (1.3.3)$$

Apply $\langle 0|$ on both side

$$C_0\langle 0|V(t)|0\rangle e^{-iE_0 t/\hbar} + C_1\langle 0|V(t)\,|1\rangle e^{-iE_1 t/\hbar} = i\hbar\,[\,\dot{C}_0\langle 0|0\rangle e^{-\frac{iE_0 t}{\hbar}} + \dot{C}_1\langle 0|1\rangle e^{-\frac{iE_1 t}{\hbar}}\,]$$

$$C_0\langle 0|V(t)\,|0\rangle e^{-iE_0 t/\hbar} + C_1\langle 0|V(t)\,|1\rangle e^{-iE_1 t/\hbar} = i\hbar\dot{C}_0 e^{-iE_0 t/\hbar} \quad (1.3.4)$$

We define

$V_{00}(\mathrm{t}) = \langle 0|V(t)\,|0\rangle$and $V_{01}(\mathrm{t}) = \langle 0|V(t)\,|1\rangle$

Calculating forC_0 we get

$$\dot{C}_0 = \frac{-i}{\hbar}\,[C_0 V_{00}(\mathrm{t}) + C_1 V_{01}(\mathrm{t}) e^{-i(E_1 - E_0)t/\hbar}] \quad (1.3.5)$$

If we apply$\langle 1|$ on both side of Eq. (1.3.3), it turns out

$$\dot{C}_1 = \frac{-i}{\hbar}\ [C_1 V_{11}(t) + C_0 V_{10}(t) e^{i(E_1 - E_0)t/\hbar}] \quad (1.3.6)$$

Usually define $\hbar\omega_0 = E_1 - E_0$ .The Eq. (1.3.5) and Eq. (1.3.6) are coupled ordinary differential equations, we can only solve this exactly in special cases. One special case is given below.

(1) Solve for V (t) with V_{00}(t) = $V_{11}(t)$ = 0 and $V_{10}(t)$ = $\frac{U_{10}}{2}e^{-i\omega t}$, V_{01}(t) = $\frac{U_{01}}{2}e^{-i\omega t}$

With the boundary conditions$C_0(0) = 1$,$C_1(0) = 0$

This is a typical kind of situation in whichU_{10}, $U_{01}$are the potentials required for the transition of the system with the driving frequency ω .

For this typical case the Eq. (1.3.5) and Eq. (1.3.6) becomes

$$\dot{C}_0 = \frac{-i}{\hbar} C_1 V_{01}(t)\, e^{-i(\omega_0)t} \quad (1.3.7)$$

$$\dot{C}_1 = \frac{-i}{\hbar} C_0 V_{10}(t) e^{i(\omega_0)t} \quad (1.3.8)$$

Plugging the value of $V_{10}(t) and V_{01}$(t) then Eq. (1.3.7) and Eq. (1.3.8) becomes,

$$\dot{C}_0 = \frac{-i}{\hbar}\frac{U_{01}}{2} e^{-i(\omega_0+\omega)t} C_1 \quad (1.3.9)$$

$$\dot{C}_1 = \frac{-i}{\hbar}\frac{U_{10}}{2} e^{i(\omega_0-\omega)t} C_0 \quad (1.3.10)$$

(1.3.10) =>$\ddot{C}_1 = \frac{-i}{\hbar}\frac{U_{10}}{2} e^{i(\omega_0-\omega)t} \quad [i(\omega_0 - \omega)C_0 + \dot{C}_0]$

$$\ddot{C}_1 = i(\omega_0 - \omega)\frac{-i}{\hbar}\frac{U_{10}}{2} e^{i(\omega_0-\omega)t} C_0 + \frac{-i}{\hbar}\frac{U_{10}}{2} e^{i(\omega_0-\omega)t}\frac{-i}{\hbar}\frac{U_{01}}{2} e^{-i(\omega_0+\omega)t} C_1$$

$$\ddot{C}_1 = i(\omega_0 - \omega)C_1^{\cdot} - \frac{|U_{10}|^2 e^{-2i\omega t}}{4\hbar^2} C_1 \quad (1.3.11)$$

Trial solution of Eq. (1.3.11) is,

$$C_1 = A\, e^{rt} \quad (1.3.12)$$

Where $r^2 - i(\omega_0 - \omega)r + \frac{|U_{10}|^2 e^{-2i\omega t}}{4\hbar^2} = 0$

Using quadratic formula for calculating *r*

$$r = \frac{1}{2}[i(\omega_0 - \omega) \pm \sqrt{-(\omega_0 - \omega)^2 - 4(\frac{|U_{10}|^2 e^{-2i\omega t}}{4\hbar^2})}\ ]$$

$$r = \frac{1}{2}[i(\omega_0 - \omega) \pm \sqrt{-(\omega_0 - \omega)^2 - (\frac{|U_{10}|^2 e^{-2i\omega t}}{\hbar^2})}\ ]$$

Let, $2\omega_r = \sqrt{(\omega_0 - \omega)^2 - (\frac{|U_{10}|^2 e^{-2i\omega t}}{\hbar^2})}$

$$r = \frac{i(\omega_0 - \omega)}{2} \pm i\omega_r \qquad (1.3.13)$$

The solution of the equation is,

$$C_1 = A\, e^{\frac{i(\omega_0 - \omega)t}{2}} e^{i\omega_r t} + B\, e^{\frac{i(\omega_0 - \omega)t}{2}} e^{-i\omega_r t} \qquad (1.3.14)$$

Checking boundary conditions $C_1(0) = 1$, which gives $B = -A$ then Eq. (1.3.14) gives

$$C_1 = A\, e^{\frac{i(\omega_0 - \omega)t}{2}} e^{i\omega_r t} + A\, e^{\frac{i(\omega_0 - \omega)t}{2}} e^{-i\omega_r t}$$

$$C_1 = A\, e^{\frac{i(\omega_0 - \omega)t}{2}} \quad [e^{i\omega_r t} - e^{-i\omega_r t}\ ] \qquad (1.3.15)$$

Now C_0 will be

$$C_0 = \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} \dot{C}_1$$

$$C_0 = \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} \frac{d}{dt} [A\, e^{\frac{i(\omega_0-\omega)t}{2}} \quad [e^{i\omega_r t} - e^{-i\omega_r t}\,]\,]$$

$$C_0 = A \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} [e^{\frac{i(\omega_0-\omega)t}{2}} (i\omega_r e^{i\omega_r t} + i\omega_r e^{-i\omega_r t}) + \frac{i(\omega_0-\omega)}{2} \times [e^{i\omega_r t} - e^{-i\omega_r t}\,]\, e^{\frac{i(\omega_0-\omega)t}{2}}\,]$$

$$C_0 = A \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} [i\omega_r (e^{\frac{i(\omega_0-\omega)t}{2}} (e^{i\omega_r t} + e^{-i\omega_r t}) + \frac{i(\omega_0-\omega)}{2} (e^{i\omega_r t} - e^{-i\omega_r t}) e^{\frac{i(\omega_0-\omega)t}{2}}\,]$$

$$C_0 = A \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} \quad [i\omega_r\, (e^{\frac{i(\omega_0-\omega)t}{2}} (2\cos\omega_r t) + \frac{i(\omega_0-\omega)}{2} (2i\sin\omega_r t)\, e^{\frac{i(\omega_0-\omega)t}{2}}\,]$$

$$C_0 = A \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} e^{\frac{i(\omega_0-\omega)t}{2}} [2i\omega_r \cos\omega_r t + \frac{i(\omega_0-\omega)}{2} 2i\sin\omega_r t]$$

$$C_0(t) = A \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} e^{\frac{i(\omega_0-\omega)t}{2}} [2i\omega_r \cos\omega_r t - (\omega_0-\omega)\sin\omega_r t] \qquad (1.3.16)$$

Checking boundary condition, $C_0(0) = 1$

$$1 = A\, \frac{2i\hbar}{U_{10}} e^0 e^0\, [2i\omega_r \cos 0 - (\omega_0-\omega)\sin 0]$$

$$1 = A\, \frac{2i\hbar}{U_{10}}\, 2i\omega_r$$

$$A = -(U_{10}/4\hbar\omega_r) \qquad (1.3.17)$$

Put this value of A in Eq. (1.3.16)

$$C_0 = -(U_{10}/4\hbar\omega_r) \frac{2i\hbar}{U_{10}} e^{-i(\omega_0-\omega)t} e^{\frac{i(\omega_0-\omega)t}{2}} [2i\omega_r \cos\omega_r t - (\omega_0-\omega)\sin\omega_r t]$$

$$C_0 = -\frac{i}{2\omega_r} e^{-i(\omega_0-\omega)t} e^{\frac{i(\omega_0-\omega)t}{2}} [2i\omega_r \cos\omega_r t - (\omega_0 - \omega)\sin\omega_r t]$$

$$C_0 = \frac{1}{2\omega_r} e^{-i(\omega_0-\omega)t} e^{\frac{i(\omega_0-\omega)t}{2}} [2\omega_r \cos\omega_r t + i(\omega_0 - \omega)\sin\omega_r t] \qquad (1.3.18)$$

This is the expression for C_0.

If we put the value of A from Eq. (1.3.17) in Eq.(1.3.15), the C_1will be,

$$C_1 = -(U_{10}/4\hbar\omega_r) e^{\frac{i(\omega_0-\omega)t}{2}} \quad [e^{i\omega_r t} - e^{-i\omega_r t}]$$

$$C_1 = -(U_{10}/4\hbar\omega_r) e^{\frac{i(\omega_0-\omega)t}{2}} \quad [2i \sin\omega_r t]$$

$$C_1 = -\frac{\mathrm{i}U_{10}}{2\hbar\omega_r} \sin\omega_r t \, e^{\frac{i(\omega_0-\omega)t}{2}} \qquad (1.3.19)$$

This is the expression for C_1.

Now we can prove that,

$|C_0|^2 + |C_1|^2 = 1$

Where $|C_0|^2$ is the probability that the system is in the state $|0\rangle$, and $|C_1|^2$is the probability that the system is in the state $|1\rangle$. And thesum of these two probabilities is always equal to 1.Now we will check that after time *"t"* the system goes from state $|0\rangle$ to state $|1\rangle$, the probability is,

$P_{0\to1} = |\frac{U_{10}}{2\hbar\omega_r}|^2 (\sin\omega_r t)^2$

Where$|\frac{U_{10}}{2\hbar\omega_r}|^2$ is the intensity of the source.

$$|\frac{U_{10}}{2\hbar\omega_r}|^2 = |\frac{U_{10}}{2\hbar(\omega-\omega_0)}|^2$$

Where $\omega_r$is the driving frequency.

1.4 Periodically Driven Two-Level Systems;

In contrast to time-independent quantum theory, exactly solvable quantum two-level systems with time-dependent potentials are extremely rare.The problem of time dependently driven two-level dynamics is of enormous practical importance in the theory of nuclear magnetic resonance, quantum optics or in low temperature glass systems to name only a few. The driven two-level systems have a long history, and reviews are available [2]. A pioneering piece of work must be attributed to Rabi who considers the two-level system in a circularly polarized magnetic field- a problem that he could solve exactly. He thereby elucidated how to measure simultaneously both sign as well as the magnitude of the magnetic moments. However as Block and Siegert experience soon after[3], this problem is no longer exactly solvable in analytical close form when the field is linearly polarized, rather than circularly. We set for the wave function

$$\Psi(t) = c_1(t)e^{\frac{i\Delta t}{2\hbar}}\binom{1}{0} + c_2(t)e^{-\frac{i\Delta t}{2\hbar}}\binom{0}{1} \qquad (1.4.1.1)$$

Where $|C_1(t)|^2 + |C_2(t)|^2 = 1$. With $2\hbar\lambda$=-μE_0 and $\varphi = \frac{\pi}{2}$ yielding a pure $\cos\omega t$ perturbation, the Schrodinger equation therefore takes the form of

$$i\hbar\frac{d}{dt}\begin{pmatrix} c_1(t)e^{\frac{i\Delta t}{2\hbar}} \\ c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \end{pmatrix} = \begin{pmatrix} -\Delta/2 & -2\hbar\lambda\cos\omega t \\ -2\hbar\lambda\cos\omega t & \Delta/2 \end{pmatrix}\begin{pmatrix} c_1(t)e^{\frac{i\Delta t}{2\hbar}} \\ c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \end{pmatrix} \quad (1.4.1.2)$$

$$i\hbar\begin{pmatrix} \dot{c}_1(t)e^{\frac{i\Delta t}{2\hbar}} + c_1(t)e^{\frac{i\Delta t}{2\hbar}}(\frac{i\Delta}{2\hbar}) \\ \dot{c}_2(t)e^{-\frac{i\Delta t}{2\hbar}} - c_2(t)e^{-\frac{i\Delta t}{2\hbar}}(\frac{i\Delta}{2\hbar}) \end{pmatrix} = \begin{pmatrix} -\frac{\Delta}{2}c_1(t)e^{\frac{i\Delta t}{2\hbar}} + -2\hbar\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \\ -2\hbar\lambda\cos\omega t c_1(t)e^{\frac{i\Delta t}{2\hbar}} + \frac{\Delta}{2}c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \end{pmatrix}$$

Comparing both sides we get two coupled differential equations.

$$i\hbar(\dot{c}_1(t)e^{\frac{i\Delta t}{2\hbar}} + c_1(t)e^{\frac{i\Delta t}{2\hbar}}(\frac{i\Delta}{2\hbar})) = -\frac{\Delta}{2}c_1(t)e^{\frac{i\Delta t}{2\hbar}} + -2\hbar\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \quad (1.4.1.3)$$

$$i\hbar(\dot{c}_2(t)e^{-\frac{i\Delta t}{2\hbar}} - c_2(t)e^{-\frac{i\Delta t}{2\hbar}}(\frac{i\Delta}{2\hbar})) = -2\hbar\lambda\cos\omega t c_1(t)e^{\frac{i\Delta t}{2\hbar}} + \frac{\Delta}{2}c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \quad (1.4.1.4)$$

$$i\hbar\dot{c}_1(t)e^{\frac{i\Delta t}{2\hbar}} + i\hbar(\frac{i\Delta}{2\hbar})c_1(t)e^{\frac{i\Delta t}{2\hbar}} = -\frac{\Delta}{2}c_1(t)e^{\frac{i\Delta t}{2\hbar}} - 2\hbar\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \quad (1.4.1.5)$$

$$i\hbar\dot{c}_2(t)e^{-\frac{i\Delta t}{2\hbar}} - i\hbar\left(\frac{i\Delta}{2\hbar}\right)c_2(t)e^{-\frac{i\Delta t}{2\hbar}} = -2\hbar\lambda\cos\omega t c_1(t)e^{\frac{i\Delta t}{2\hbar}} + \frac{\Delta}{2}c_2(t)e^{-\frac{i\Delta t}{2\hbar}} \quad (1.4.1.6)$$

Solving Eq. (1.4.1.6)

$$i\hbar\dot{c}_1(t)e^{\frac{i\Delta t}{2\hbar}} = -\frac{\Delta}{2}c_1(t)e^{\frac{i\Delta t}{2\hbar}} + \frac{\Delta}{2}c_1(t)e^{\frac{i\Delta t}{2\hbar}} - 2\hbar\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{2\hbar}}$$

$$i\hbar\dot{c}_1(t)e^{\frac{i\Delta t}{2\hbar}} = -2\hbar\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{2\hbar}}$$

$$\dot{c}_1(t) = 2i\lambda\cos\omega t c_2(t)e^{-\frac{i\Delta t}{\hbar}} \quad (1.4.1.7)$$

With $\Delta = \hbar\omega_0$ Eq. (1.4.1.7) becomes

$$\dot{c}_1(t) = 2i\lambda\cos\omega t c_2(t)e^{-i\omega_0 t} \quad (1.4.1.8)$$

$$\dot{c}_1(t) = 2i\lambda(\frac{e^{i\omega t}+e^{-i\omega t}}{2})e^{-i\omega_0 t}c_2 \quad (1.4.1.9)$$

$$\dot{c}_1(t) = i\lambda\left(e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right)c_2 \quad (1.4.1.10)$$

Similarly

$$\dot{c}_2(t) = i\lambda\left(e^{-i(\omega-\omega_0)t} + e^{i(\omega+\omega_0)t}\right)c_1 \quad (1.4.1.11)$$

This is the so-called coupled differential equations. Clearly such equations are generally not solvable in analytical closed form. Hence, although the problem is simple, the job of finding an analytical solution presents a hard task! Here we use the rotating wave approximation (RWA), assuming that $\omega\ is\ closed\ to\ \omega_0$(near resonance), and λ is not very large. Then anti-rotating wave term $e^{i(\omega+\omega_0)t}$ is rapidly varying, as compare to the slowly varying rotating-wave term $e^{-i(\omega-\omega_0)t}$. Therefore it can't transfer much population from state $|1\rangle$ to state $|2\rangle$. neglecting this anti-rotating wave term Eq. (1.4.1.10) and Eq. (1.4.1.11) takes the form of

$$\frac{dc_1}{dt} = i\lambda e^{i\delta t}c_2 \quad (1.4.1.12)$$

$$\frac{dc_2}{dt} = i\lambda e^{-i\delta t}c_1 \quad (1.4.1.13)$$

From these equations we can find for $c_1(t)$a linear second order differential equation with constant coefficients- which can be solve readily for arbitrary conditions. For example setting $c_1(0) = 1\ and c_2(0) = 0$ we obtain

$$c_1(t) = e^{i\delta t}[\cos\left(\frac{1}{2}\Omega t\right) - \frac{i\delta}{\Omega}\sin\left(\frac{1}{2}\Omega t\right)] \quad (1.4.1.14)$$

$$c_1(t) = \frac{e^{i\delta t}2i\lambda}{\Omega}\sin\left(\frac{1}{2}\Omega t\right) \quad (1.4.1.15)$$

Where $\delta = \omega - \omega_0$ the detuning parameter and Ω is denotes the celebrated Rabi frequency

$$\Omega = \sqrt{(\delta^2 + 4\lambda^2)} \tag{1.4.1.16}$$

The populations as function of time are then given by

$$|C_1(t)|^2 = (\frac{\delta}{\Omega})^2 + (\frac{2\lambda}{\Omega})^2 cos^2\left(\frac{1}{2}\Omega t\right) \tag{1.4.1.17}$$

$$|C_2(t)|^2 = (\frac{2\lambda}{\Omega})^2 sin^2\left(\frac{1}{2}\Omega t\right) \tag{1.4.1.18}$$

Note that at short time t, the excitation in the upper state is independent on the detuning, $|C_2(t)|^2 \to \lambda^2 t^2 for \Omega t \ll 1$. This behavior is in accordance with the perturbation theory, valid at small time. Moreover the population at resonance completely cycle the population between the two states, while with $\delta \neq 0$, the lower states is never completely depopulated.

Single Axis Driven Two-Level Quantum Systems

Since the foundation of quantum mechanics to the present time, many physicists want to solve analytically driven two-level systems. It is extremely challenging to find an exact solution for the Schrodinger equation and the corresponding time evolution operator. The most popular driven problem is the Landau-Zener problem [4, 5], Rabi problem [6], and Jaynes Cumming model [7]. Unfortunately, these analytically exact solutions are scare. Analytically solved pulses are very attractive in light of the advantages they offer in the design of qubit control operation.

In this chapter we present a theoretical approach to the driven time-dependent two-level problem. Here we derive an algorithm that produces an unlimited number of the solutions to the analytically solvable driven time-dependent two-level systems, and its evolution operator by applying a single axis controlled field. We use a real function which obeys certain experimental constraints and initial condition and derive the single axis driven Hamiltonian and the corresponding evolution operator.

We generate an algorithm which gives unlimited exact solution to the driven two-level systems, and its evolution operator. The corresponding driven term and its evolution operator can be derived from a real function $\lambda(t)$ which obey certain experimental constraints and gives initial conditions. Also we find out how the properties of $\lambda(t)$ translate to a control field and its corresponding time evolution operator. Here we use 'reverse engineering' approach in which we derived the particular evolution operator of driven two-level system by a controlled field and the corresponding controlled field is restricted to a few experimental imposed constraints and initial conditions.

2.1 Finding Driven Time-Dependent Two-Level Hamiltonian And Its Evolution Operator Using "Reverse Engineering'' Approach;

Consider time-dependent Hamiltonian for the two-level system. Let the Hamiltonian consist of single axis time dependent driven term$\gamma(t)$. So the general form of Hamiltonian is

$$H = \frac{\gamma(t)}{2}\sigma_z + \frac{h}{2}\sigma_x \tag{2.2.1}$$

Where $\gamma(t)$ is the single axis control field, h is the separation between the two energy level, σ_x and σ_z are Pauli operators. The above Hamiltonian describes any single axis driven two-level quantum system. $\gamma(t)$,Is the energy splitting between the states. The time evolution operator for this system is given by

$$U = \begin{pmatrix} u_{11} & u^*{}_{21} \\ u_{21} & u^*{}_{11} \end{pmatrix} \tag{2.2.2}$$

, and

$|u_{11}|^2 + |u_{21}|^2 = 1$

Here we will use the "reverse engineering" approach, to find out the time evolution Eq. (2.2.2) and the corresponding driving term in Eq. (2.2.1). Now consider a general state $|\Psi(t)\rangle\,(t) = d_+(t)|+\rangle(t) + d_-(t)|-\rangle(t)$. Using Schrodinger equation to find out the relation between$d_+(t)$ and $d_-(t)$.

$$i\hbar\frac{d\Psi}{dt} = H\Psi$$

Put $\Psi(t)$ in above equation

$$i\hbar\frac{d}{dt}[d_+(t)|+\rangle(t) + d_-(t)|-\rangle(t)] = \frac{\gamma(t)}{2}\sigma_z + \frac{h}{2}\sigma_x[d_+(t)|+\rangle(t) + d_-(t)|-\rangle(t)] \tag{2.2.3}$$

$$i\hbar\dot{d}_+|+\rangle(t) + i\hbar d_+(t)\frac{d|+\rangle(t)}{dt} + i\hbar d\_(t) + i\hbar d_-(t)\frac{d|-\rangle(t)}{dt}$$
$$= \frac{\gamma(t)}{2}\sigma_z d_+(t)|+\rangle(t) + \frac{\gamma(t)}{2}\sigma_z d_-(t)|-\rangle(t) + \frac{h}{2}\sigma_x d_+(t)|+\rangle(t)$$
$$+ \frac{h}{2}\sigma_x d_-(t)|-\rangle(t)$$

$$i\hbar\dot{d}_+|+\rangle(t) + \frac{h}{2}d_+|+\rangle(t) + i\hbar\dot{d}_-|-\rangle(t) - \frac{h}{2}d_-|-\rangle(t)$$

$$= \frac{\gamma(t)}{2}\sigma_z d_+(t)\left|+\rangle(t) + \frac{h}{2}\sigma_x d_+(t)\right|+\rangle(t) + \frac{\gamma(t)}{2}\sigma_z d_-(t)\left|-\rangle(t) + \frac{h}{2}\sigma_x d_-(t)\right|-\rangle(t)$$

$$i\hbar\dot{d}_+|+\rangle(t) + \frac{h}{2}d_+|+\rangle(t) + i\hbar\dot{d}_-|-\rangle(t) - \frac{h}{2}d_-|-\rangle(t)$$

$$= \left[\frac{\gamma(t)}{2}e^{iht}d_-(t) + \frac{h}{2}d_+(t)\right]\left|+\rangle(t) + \left[\frac{\gamma(t)}{2}e^{-iht}d_+(t) + \frac{h}{2}d_-(t)\right]\right|-\rangle(t) \quad (2.2.4)$$

Where$\sigma_z\,|\pm\rangle = \pm|\pm\rangle$ and $\sigma_x|\pm\rangle = |\mp\rangle$. Similarly $|+\rangle(t) = e^{i(\omega t/2)}|+\rangle$ and $|-\rangle(t) = e^{-i(\omega t/2)}|-\rangle$.

From Eq. (2.2.4) we get the following couple differential equations for, d_+ and d_- .

$$\dot{d}_\pm(t) = -i\frac{\gamma(t)}{2}e^{\pm iht}d_\mp(t) \quad (2.2.5)$$

Using Eq. (2.2.5) to write the corresponding time evolution operator in the following form

$$d_\pm(t) = -i\frac{1}{\sqrt{2}}e^{\pm iht/2}\,(u_{11} \pm u_{21}) \quad (2.2.6)$$

From Eq. (2.2.5) and (2.2.6) we have

$$\ddot{d}_+ + \left(-ih - \frac{\dot{\gamma}}{\gamma}\right)\dot{d}_+ + \left(\frac{\gamma^2}{4}\right)d_+ \quad (2.2.7)$$

Solving Eq. (2.2.7) to find a particular expression for $\gamma(t)$. But we use different approach, inwhich we consider Eq. (2.2.7) as a differential equation for $\gamma(t)$ for a known d_+. for arbitrary d_+, $\gamma(t)$ is given by

$$\gamma(t) = \pm\frac{\dot{d}_+e^{-iht}}{\sqrt{c - \frac{1}{4}{d_+}^2e^{-i2ht} - ih/2\int_0^t dt'e^{-i2ht'}d_+^2(t')}} \quad (2.2.8)$$

Where 'c' is constant of integration. Using Eq. (2.2.5) to find the relation for d_- $interm\ of$ d_+

$$d_- = \pm 2i\sqrt{c - \frac{1}{4}{d_+}^2 e^{-i2ht} - ih/2\int_0^t dt'\, e^{-i2ht'} d_+^2(t')} \quad (2.2.9)$$

We start the evolution from t=0, so we impose that $d_- = d_+ = 1/a2$, this implies that c=0. We can take the function d_+ and use Eq. (2.2.8) and Eq.(2.2.9) to find d_- and $\gamma(t)$. Here the unitary is preserved and $|d_+|^2 + |d_-|^2$ = 1. To satisfy this an ansatz can be used, such as

$$d_+ = e^{F—k+ht)} \cos\theta, \qquad d_- = e^{-k}\sin\theta \quad (2.2.10)$$

Where $F, k\ and\ \theta$ are real and arbitrary functions. From these functions we can derive the time evolution operator in matrix form. The unitary matrix elements in terms of $F, k\ and\ \theta$ is given by

$$u_{11} = \frac{1}{\sqrt{2}} e^{i(\frac{ht}{2}-k)}\left[e^{iF}\cos\theta + \sin\theta\right], u_{21} = \frac{1}{\sqrt{2}} e^{i(\frac{ht}{2}-k)}\left[e^{iF}\cos\theta - \sin\theta\right] \quad (2.2.11)$$

The initial conditions on $d_+\ and\ d_-$ can be translated to $F, k\ and\ \theta$. So $\theta(0) = \frac{\pi}{4} and \quad F(0) = k(0) = 0$. Using Eq. (2.2.10) in Eq.(2.2.9) and find the relation between $F, k\ and\ \theta$. So

$$\dot{F} + h = \dot{k}(1 - \tan^2(\theta)\dot{\theta} = \dot{k}\tan F\ tan\theta \quad (2.2.12)$$

We can also express $\gamma(t)$ in terms of these functions from Eq. (2.2.8)

$$\gamma(t) = 2\dot{k}\sec F\tan\theta \quad (2.2.13)$$

So the modified initial conditions for these functions are $\dot{\theta}(0) = 0, \dot{F}(0) = -h\ and\ \gamma(t) = 2\dot{k}(0). \dot{k}(0)$ is not restricted by these relations. From Eq. (2.2.12) it is straight forward to find $\theta\ and\ k\ interm\ of\ F$. For$\theta$

$$\sin 2\theta = \sec F e^{h\int_0^t dt' \tan F'} \quad (2.2.14)$$

If we know $F\ then\ we\ can\ find\ easily\ \theta, \gamma(t), k\ and\ unitary\ evolution\ U$. Let's take an arbitrary F

$$F = \arctan\frac{\dot{\lambda}}{\lambda h} \tag{2.2.15}$$

Where λ(t) is the new real arbitrary function and this is the single arbitrary function from which, we can obtain the single axis driven term and the corresponding time evolution operator for a single axis driven two-level Hamiltonian. For $\theta\ insert\ F\ in\ Eq.$ (2.2.14)

$$\sin 2\theta = \sec F\ e^{\int_0^t dt' \tan\frac{\dot{\lambda}}{\lambda h}} \tag{2.2.16}$$

$$\sin 2\theta = \sec F\ e^{\int_0^t \tan\frac{\dot{\lambda}}{\lambda h}}$$

$$\sin 2\theta = \lambda \sec F$$

$$\text{Sin}\ 2\theta = \sqrt{\lambda^2 + \frac{\dot{\lambda}^2}{h^2}}$$

Where $\sec F = \sqrt{1 - tan^2(F)}$ and $\int \frac{d\lambda}{\lambda} = ln\lambda$. This is the final expression for θ.

Similarly put $\theta\ and\ F$ in Eq. (2.2.12) and Eq. (2.2.13) we get the following relations

$$\dot{k} = \frac{1}{2}\frac{h\lambda(\ddot{\lambda}+h^2\lambda)}{h^2\lambda^2+\dot{\lambda}^2}\left[1 + \frac{h}{\sqrt{h^2(1-\lambda^2)-\dot{\lambda}^2}}\right] \tag{2.2.17}$$

$$\gamma(t) = \frac{(\ddot{\lambda}+h^2\lambda)}{\sqrt{h^2(1-\lambda^2)-\dot{\lambda}^2}} \tag{2.2.18}$$

The initial conditions on $F, k\ and\ \theta$ can be translated into λ(t) such as

$$\lambda(0)= 1\ ,\ \dot{\lambda}(0)=0\ ,\ \ddot{\lambda}(0)= -h^2 \tag{2.2.19}$$

Where λ(t) is restricted to the following inequality

$$\lambda(t) \leq h^2(1 - \lambda^2)$$

This simple method is generating an unlimited number solutions of the single axis driven two-level problems. For any $\lambda(t)$ that obey the initial conditions and the above inequality is an exact solution of the two-level system. Note this method can only use for single axis driven two-level systems.

Examples

Consider a non-trivial example for λ(t) that should be able to obey the initial conditions and inequalities. If we consider the function such as $\lambda = \cos ht$, this function will give γ =0, because it does not obey the initial conditions and inequality. So we can't take the function like this, because it does not provide the exact solution for a driven two-level system. For this purpose, we consider the following functions which obey the requirements.

Example 1

Let λ(t) is given by

$$\lambda(t)e^{-\frac{\left(\frac{2}{\alpha}\right)sin^2 h\sqrt{\alpha}ht}{2}} \qquad \alpha \leq 2 \tag{2.2.20}$$

This function obeys the corresponding initial condition and inequality. Insert$\lambda, \dot{\lambda}$ and $\ddot{\lambda}$in Eq. (2.2.18)

$$\dot{\lambda} = -\frac{2he^{-\frac{2sin^2\left(\frac{\sqrt{\alpha}ht}{2}\right)}{2}}\cosh(\frac{\sqrt{\alpha}ht}{2})\sin h\left(\frac{\sqrt{\alpha}ht}{2}\right)}{\sqrt{\alpha}} \tag{2.2.21}$$

And

$$\ddot{\lambda} = (\frac{h^2e^{-\frac{2sin^2\left(\frac{\sqrt{\alpha}ht}{2}\right)}{\alpha}}\left(\left(4cosh^2\frac{\sqrt{\alpha}ht}{2}\right)-\alpha\right)sinh^2\frac{\sqrt{\alpha}ht}{2}-\alpha cosh^2\frac{\sqrt{\alpha}ht}{2}}{\alpha}) + h^2(e^{-\left(\frac{2}{\alpha}sinh^2\frac{\sqrt{\alpha}ht}{2}\right)}) \tag{2.2.22}$$

$$\gamma(t) = \frac{(\frac{h^2e^{-\frac{2sin^2\left(\frac{\sqrt{\alpha}ht}{2}\right)}{\alpha}}\left(\left(4cosh^2\frac{\sqrt{\alpha}ht}{2}\right)-\alpha\right)sinh^2\frac{\sqrt{\alpha}ht}{2}-\alpha cosh^2\frac{\sqrt{\alpha}ht}{2}}{\alpha})+h^2(e^{-\left(\frac{2}{\alpha}sinh^2\frac{\sqrt{\alpha}ht}{2}\right)})}{\sqrt{h^2(1-\left(e^{\left(\frac{2}{\alpha}sinh^2\frac{\sqrt{\alpha}ht}{2}\right)}\right)^2)-(-\frac{2he^{-\frac{2sin^2\left(\frac{\sqrt{\alpha}ht}{2}\right)}{2}}\cosh(\frac{\sqrt{\alpha}ht}{2})\sin h\left(\frac{\sqrt{\alpha}ht}{2}\right)}{\sqrt{\alpha}})^2}} \tag{2.2.23}$$

The required single axis driven is given by

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$$\gamma(t) = \frac{h\left[\frac{1}{\alpha}sinh^2(x) - 2sinh^2\left(\frac{x}{2}\right)\right]}{\sqrt{e^{\frac{4}{\alpha}sinh^2\left(\frac{x}{2}\right)} - \frac{1}{\alpha}sinh^2(x) - 1}} \tag{2.2.24}$$

Where$x = \sqrt{\alpha}ht$. α, Is control the driven field (controlling parameter for a given pulse)and change the shape of the driven field $\gamma(t)$.

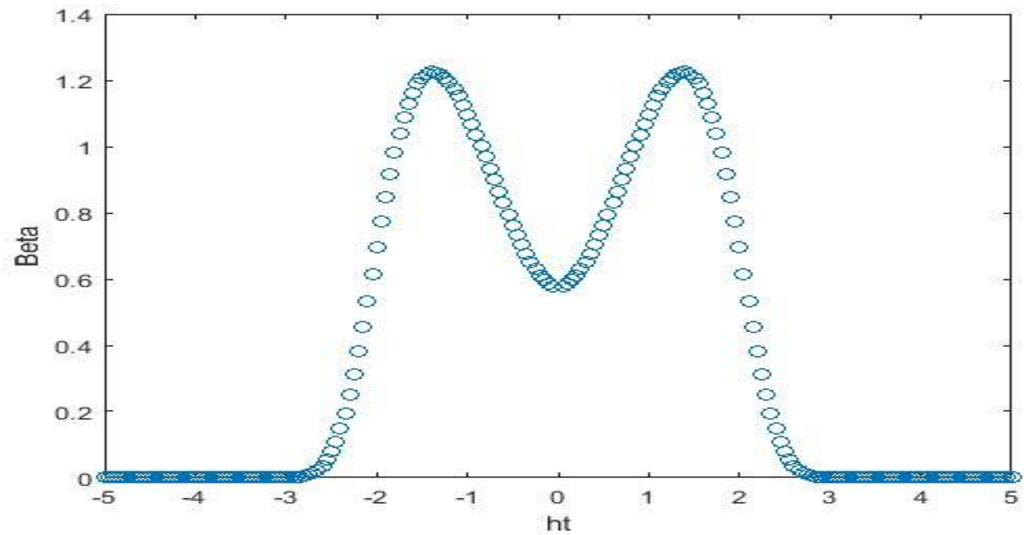

Figure 1: Single-axis driven pulse for $\alpha = 5/3$

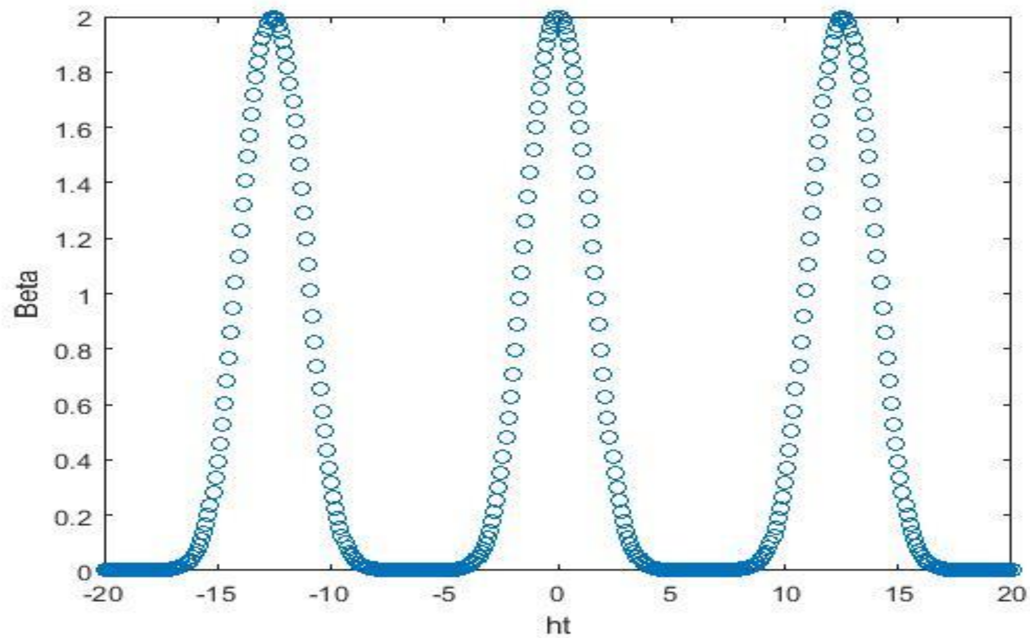

Figure 2: Single-axis driven pulse for $\alpha = -\frac{1}{4}$

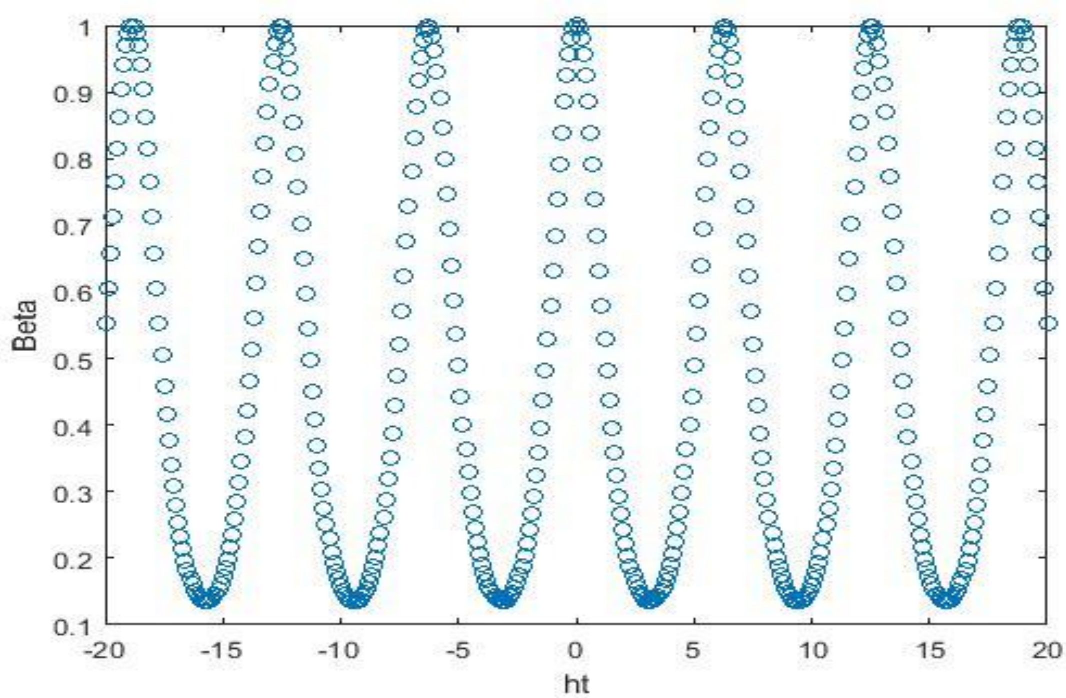

Figure 3: Single-axis driven pulse for $\alpha = -1$

Example 2

Consider another non trivial example, which obey the initial condition and inequality. Let the function is given by

$$\lambda(t) = \frac{1}{1+\alpha}[e^{-\left(\frac{h^2t^2}{2}\right)} + \alpha\cos(ht)] \tag{2.2.25}$$

Where the function is like Gaussian function and the parameter α is the tuning parameter and change the width and magnitude of the function. Insert$\lambda, \dot{\lambda}$ *and* $\ddot{\lambda}$in Eq. (2.2.18)

$$\dot{\lambda} = \frac{-\sin(ht) - h^2te^{-\left(\frac{h^2t^2}{2}\right)}}{\alpha+1} \tag{2.2.26}$$

$$\ddot{\lambda} = \frac{h^2e^{-\left(\frac{h^2t^2}{2}\right)}(\alpha e^{\left(\frac{h^2t^2}{2}\right)}\cos(ht) - h^2t^2 + 1}{\alpha+1} \tag{2.2.27}$$

And

$$\gamma(t) = \frac{\frac{h^2e^{-\left(\frac{h^2t^2}{2}\right)}(\alpha e^{\left(\frac{h^2t^2}{2}\right)}\cos(ht) - h^2t^2 + 1}{\alpha+1} h^2[\frac{1}{1+\alpha}[e^{-\left(\frac{h^2t^2}{2}\right)} + \alpha\cos(ht)]]}{\sqrt{h^2(1 - \frac{1}{1+\alpha}\left[e^{-\left(\frac{h^2t^2}{2}\right)} + \alpha\cos(ht)]\right) - (\frac{-\alpha h\sin(ht) - h^2te^{-\left(\frac{h^2t^2}{2}\right)}}{\alpha+1})^2}} \tag{2.2.28}$$

The required single axis driven is given by

$$\gamma(t) = \frac{h^3t^2e^{-\left(\frac{h^2t^2}{2}\right)}}{\sqrt{1 - (1+h^2t^2)e^{h^2t^2} + 2\alpha(1 - e^{-\left(\frac{h^2t^2}{2}\right)}[\cos(ht) + ht\sin(ht)])}} \tag{2.2.29}$$

The plot of Eq.(2.2.29) is given in the below figure for different values of α

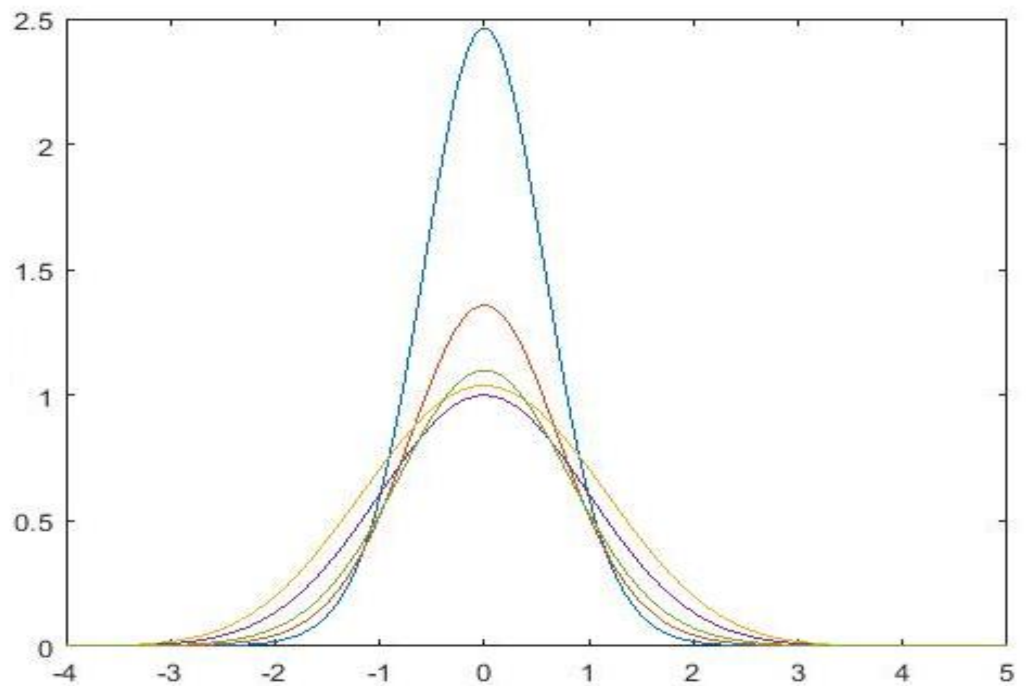

Figure 4: Color lines represent the graph between α *and* ht for different $\alpha's$. For $\alpha = -\frac{1}{4}, 0, \frac{1}{2}, 1, 2$, for large α the amplitude is large and for small the amplitude is small.

2.2 How to choose $\lambda(t)$?

Now the question is that how to choose$\lambda(t)$? We are use a method to choose $\lambda(t)$ systematically. We are generated λ's for real function $P(\lambda)$ which obey the following inequality

$$0 \leq P(\lambda) \leq 1 - \lambda^2 \qquad (2.2.30)$$

Solving the equation $\dot{\lambda}^2 = h^2\, P(\lambda)\; for\; \lambda$

$$\dot{\lambda}^2 = h^2 P(\lambda) \qquad (2.2.31)$$

$$\dot{\lambda} = h\sqrt{P(\lambda)}$$

$$ht = \int_\lambda^1 \frac{d\lambda}{\sqrt{P(\lambda)}} \qquad (2.2.32)$$

Now define a new function $X(\lambda) = \int_\lambda^1 \frac{d\lambda}{\sqrt{P(\lambda)}}$ where

$$X(\lambda) = \int_\lambda^1 \frac{d\lambda}{\sqrt{P(\lambda)}} = cos^{-1}(\lambda) \qquad (2.2.33)$$

λWill obey the following initial conditions

$$X(\lambda) \to \sqrt{2 - 2\lambda}\ \text{As}\lambda \to 1 \qquad (2.2.34)$$

Obeying the initial condition by λ, we will the required relation for λ

$$\lambda(\text{t}) = X^{-1}(ht) \qquad (2.2.35)$$

Now let's check the above formalism, we can find the required λ which follow the initial conditions in Eq. (2.2.19). Let's choose the following form for X(λ),

$$X(\lambda) = \frac{1}{\alpha}\tan\text{h}^{-1}(\alpha\sqrt{2 - 2\lambda})(2.2.36)$$

Comparing the above equation with $ht = X(\lambda)$ and solve for λ. We get the following form for λ(t)

$$\lambda(t) = 1 - \frac{1}{2\alpha^2} \tan h^2 \alpha ht \qquad (2.2.37)$$

This equation follows the required initial conditions and inequality. Switch $\lambda, \dot{\lambda}$ and $\ddot{\lambda}$ in Eq. (2.2.18) we get the final form of single axis driven $\gamma(t)$

$$\gamma(t) = \frac{h[14\alpha^2 - 1 + (2\alpha^2 - 1)\cosh(2\alpha ht)]sinh^2(\alpha ht)}{2\coth(\alpha ht)\sqrt{4\alpha^2[1 - sech^4(\alpha ht) - tanh^2(\alpha ht)]}} \qquad (2.2.38)$$

The shape of this function shown in the below figure for different values of α

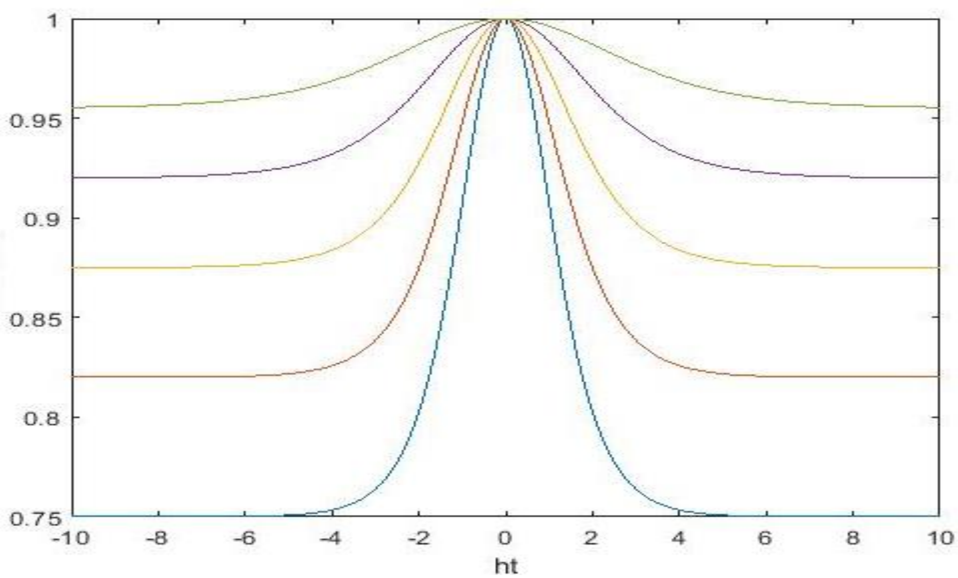

Figure 5: Single-axis driven pulse for $\alpha = \frac{1}{\sqrt{2}}$, 0.6, 0.5, 0.4, 0.3

Exact Solvable Hamiltonian For Quantum TLS

Investigating the quantum dynamics of a spin -1/2 particle subjected to an arbitrary time dependent classical magnetic field is the updated version of the old seminal problem. One example is that of a periodically driven two-level system exactly treated by Rabi [6] in the theory of nuclear magnetic resonance. Another example concerns the physical behavior of a quantum system around an avoided crossing region, analyzed with no approximation by Landau [4] and Zener [5] for a two-level system.

Unfortunately, only a few examples of analytically solvable two-level evolution have been reported up to now. Therefore, we apply a new strategy to generate solvable Hamiltonians and its evolution operators. We treated in the same way as Barnes and Das Sarma [8]i.e., to connect analytically Hamiltonian and the resulting time evolution operator in terms of an arbitrary input parametric function, the method here is much simpler and transparent, and the connection is direct, thus allowing us to have explicit expression for both the Hamiltonian and the evolution operator.

3.1 Method To Generate Solvable Hamiltonians

Consider a quantum two-level system describe by a general time dependent Hamiltonian

$$H = \begin{pmatrix} \Omega & \omega \\ \omega^* & -\Omega \end{pmatrix} \qquad (3.2.1)$$

Where $\Omega(\omega)$ is a real (generally complex) function of t. This is the most general form of the Hermitian two-by-two operator. The unitary evolution operator generated by this Hamiltonian is an element of SU(2) [10],[Appendix A], which is always represented in terms of two complex function x *and* y, as

$$U = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \; |x|^2 + |y|^2 = 1 \qquad (3.2.2)$$

Since the evolution operator satisfy the time defendant Schrodinger equation

$$HU = i\hbar\dot{U}$$

$$H = i\hbar\dot{U}U^{\dagger} \quad (3.2.3)$$

Since $\dot{U} = \begin{pmatrix} \dot{x} & \dot{y} \\ -\dot{y}^* & \dot{x}^* \end{pmatrix}$ and $U^{\dagger} = \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$

$$\begin{pmatrix} \Omega & \omega \\ \omega^* & -\Omega \end{pmatrix} = i\hbar \begin{pmatrix} \dot{x} & \dot{y} \\ -\dot{y}^* & \dot{x}^* \end{pmatrix} \begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix}$$

$$\begin{pmatrix} \Omega & \omega \\ \omega^* & -\Omega \end{pmatrix} = i\hbar \begin{pmatrix} \dot{x}x^* + \dot{y}y^* & -y\dot{x} + x\dot{y} \\ -\dot{y}^*x^* + y^*\dot{x}^* & x\dot{x}^* + y\dot{y}^* \end{pmatrix}$$

$$\begin{pmatrix} \Omega & \omega \\ \omega^* & -\Omega \end{pmatrix} = i\hbar \begin{pmatrix} \dot{x}x^* + \dot{y}y^* & x\dot{y} - y\dot{x} \\ y^*\dot{x}^* - \dot{y}^*x^* & x\dot{x}^* + y\dot{y}^* \end{pmatrix} \quad (3.2.4)$$

Therefore, we get

$$\Omega = i\hbar (\dot{x}x^* + \dot{y}y^*) \quad (3.2.5)$$

$$\omega = i\hbar (x\dot{y} - y\dot{x}) \quad (3.2.6)$$

Using Eq. (3.2.6) to derive the value of y in term of ω *and* x, see appendix B

$$y = \frac{x}{i\hbar} \int_0^t \frac{\omega}{x^2}$$

Where $\int_0^t \frac{\omega}{x^2} = Z$

$$y = \frac{x}{i\hbar} Z \quad (3.2.7)$$

Using Eq. (3.2.5) to derive the value of x in term of Z *and* $\boldsymbol{\Omega}$, see appendix B

$$\Omega = i\hbar (\dot{x}x^* + \dot{y}y^*)$$

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{\omega}{\hbar^2} Z^*] x$$

Use $\omega = x^2 \dot{Z}$

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{x^2}{\hbar^2} \dot{Z} Z^*] x$$

Where Z is constraint to satisfy the following condition

$$|x|^2\left(1+\frac{|Z|^2}{\hbar^2}\right)=1$$

$$\dot{x}=[\frac{\Omega}{i\hbar}-\frac{\dot{Z}Z^*}{(\hbar^2+|Z|^2)}]x$$

Solving by integration, finally we get,

$$x=\frac{\hbar}{\sqrt{\hbar^2+|Z|^2}}e^{-\frac{i}{\hbar}\int_0^t\Omega-\int_0^t\frac{\dot{Z}Z^*}{(\hbar^2+|Z|^2)}} \quad (3.2.8)$$

Once 'x'has been determined, the remaining functions are obtained from the relations, see Appendix B

$$y=\frac{1}{i\hbar}xZ \quad (3.2.9)$$

$$\omega=x^2\dot{Z} \quad (3.2.10)$$

The result means that if transverse field $\omega(t)$ is given by this relation for an arbitrary chosen $Z(t)$ and an arbitrary longitudinal **Ω** (t), the resulting unitary evolution operator is exactly given by the parameters$x(t)$ and $y(t)$. The relations are exact and no approximation is involved.

Examples

Here we consider a few examples as mentioned by A Messina and H Nakazato[9] that is helpful to understand how this method works as well as its usefulness in practice.

Example 1 A Real Function

Let we choose a real function *Z*, Z = Z^* and specifies a transverse field as

$$\omega=\frac{\hbar^2\dot{Z}}{\hbar^2+|Z|^2}e^{\frac{-2i}{\hbar}\int_0^t\Omega} \quad (3.2.11)$$

The evolution operator are characterized by x and y, which is given by, see Appendix B

$$\omega=x^2\dot{Z}$$

$$x = \frac{\hbar e^{\frac{-i}{\hbar}\int_0^t \Omega}}{\sqrt{\hbar^2 + |Z|^2}} \tag{3.2.12}$$

$$y = \frac{-iZe^{\frac{-i}{\hbar}\int_0^t \Omega}}{\sqrt{\hbar^2 + |Z|^2}} \tag{3.2.13}$$

$$Z = \hbar \tan[\int_0^t \omega e^{\frac{2i}{\hbar}\int_0^t \Omega}] \tag{3.2.14}$$

The consistency requires that the integrand inside the brackets should be a real function, which implies that the interaction Hamiltonian in the interaction picture commutes at different times; that is, this case correspond to a well-known solvable case [10]. This will happens if we applied the driven whose ω make the integrand real and we get an exact solvable case theoretically.

Example 2 A Complex Function

Consider a complex function $Z(t)$ parametrized by a real function $\phi(t)$ and a real constant parameter $c$as

$$Z = csin\phi e^{i\phi}, \qquad \phi(t). \tag{3.2.15}$$

The general formula tells us that when the transverse field is represented as

$$\omega = \frac{\hbar^2 c\dot{\phi}}{\hbar^2 + c^2 sin^2\phi} e^{-\frac{2i}{\hbar}\int_0^t \Omega + 2i\int_0^t \frac{\hbar^2\dot{\phi}}{\hbar^2 + c^2 \sin\phi^2}} \tag{3.2.16}$$

The value of x and y is, see Appendix B

$$x = \frac{\hbar}{\sqrt{\hbar^2 + c^2 \sin\phi^2}} e^{-\frac{i}{\hbar}\int_0^t \Omega - i\int_0^t \frac{c^2\dot{\phi}\sin\phi^2}{\hbar^2 + c^2 \sin\phi^2}} \tag{3.2.17}$$

$$y = \frac{-ic\, sin\phi\, e^{i\phi}}{\sqrt{\hbar^2 + c^2 \sin\phi^2}} e^{-\frac{i}{\hbar}\int_0^t \Omega - i\int_0^t \frac{c^2\dot{\phi}\sin\phi^2}{\hbar^2 + c^2 \sin\phi^2}} \tag{3.2.18}$$

Observe that Eq. (3.2.16) requires that (c > 0 and $\dot{\phi}$> 0 are assume for simplicity)

$|\omega| = \frac{\hbar^2 c\dot{\phi}}{\hbar^2 + c^2 sin^2\phi}$, $\phi_\omega = -\frac{2i}{\hbar}\int_0^t \Omega + 2i\int_0^t \frac{\hbar^2\dot{\phi}}{\hbar^2 + c^2 \sin\phi^2}$. In other words, if the magnitude and phase of the transverse field $\omega \equiv |\omega| e^{i\phi_\omega}$ are inter-connected as

$$\frac{|\omega|}{c} = \frac{\Omega(t)}{\hbar} + \frac{\dot{\phi_\omega}(t)}{2}, \quad (3.2.19)$$

We are led to exact solution. This certainly enlarge the domain of solvability, since if we take the c=∞ limit while keeping finite $Z \neq 0\ and |\omega| \neq 0$ to nullify the right hand side of Eq.(3.2.19), we obtain the case where the interaction Hamiltonian in the interaction picture is commutable at different times, leading to a known exact solution [10].

Conclusion

At the beginning of this thesis I introduced what is quantum two-level system and why they are of interest in modern researches. Then I solve the quantum two-level system using perturbation theory. After that I present the problem of periodically driven quantum two-level system and find the time dependent probability coefficients and the corresponding populations in either state, which is an easy job.

In this thesis the main problem was ‘’analytically solvable driven two-level quantum system’’. I reproduce that how to solve a single axis driven Hamiltonian using ‘’reverse engineering’’ approach to find the time evolution operator and then the corresponding driven Hamiltonian. We use a single real function which is restricted to obey certain boundary condition and some experimental constraints. By choosing a real function which follows the required initial condition and inequalities to find a relation for single axis driven term and the corresponding time evolution operator for two-level Hamiltonian.

The other main problem in this thesis is analytically solvable Hamiltonian for quantum two level systems. I started from a general two by two Hamiltonian and a general two by two unitary evolution operator (belonging to SU (2) group), and use reverse engineering approach to find exact solvable Hamiltonians. We use an input function that obey certain boundary condition and satisfy some constraints to generate solvable Hamiltonians. I reproduce how to generate solvable Hamiltonian using reverse engineering approach.

Appendix A

SU (2); Stand for special unitary two dimensionality.

Let I can write the most general unitary unimodular matrix as

$$U = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix},$$

Where $x\ and\ y$ are complex functions satisfying the unimodular condition

$$|x|^2 + |y|^2 = 1$$

I can easily establish the unitary property as follows

$$UU^{\dagger} = \begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}\begin{pmatrix} x^* & -y \\ y^* & x \end{pmatrix} = 1$$

The two complex functions $x\ and\ y$ are known as Cayley-Klein parameters. Historically the connection between a unitary unimodular matrix and a rotation was known long before the birth of quantum mechanics. Infact the Cayley-Klein parameters were used to characterized the complicated motion of gyroscopes in rigid body kinematics.

Without appealing to the interpretation of unitary unimodular matrices in term of rotations, we can directly check the group properties of multiplication operations with unitary unimodular matrices. Note in particular that

$$U(x_1y_1)U(x_2y_2) = U(x_1x_2 - y_1y_2^*, x_1y_2 + x_2^*y_1)$$

Where the unimodular condition for the product matrix is

$$|x_1x_2 - y_1y_2^*|^2 + |x_1y_2 + x_2^*y_1|^2 = 1$$

For the inverse of U we have

$$U^{-1}(x, y) = U(x^*, -y),$$

The group is known as SU (2). In contract the group defines by multiplication operations with general 2×2 unitary matrices is known as U(2). The most general unitary matrix in two dimensions has four independent parameters and can be written as $e^{i\gamma}$ times a unitary unimodular matrix;

$$U = e^{i\gamma}\begin{pmatrix} x & y \\ -y^* & x^* \end{pmatrix}, \ |x|^2 + |y|^2 = 1 \ \text{ and } \gamma^* = \gamma$$

SU(2) is called subgroup of U (2).

Appendix B

B-1 Finding the value of x and y

The equation

$$\omega = i\hbar \ (x\dot{y} - y\dot{x})$$

$$\frac{\omega}{i\hbar} = (x\dot{y} - y\dot{x})$$

$$\frac{1}{i\hbar}\frac{\omega}{x^2} = \frac{(x\dot{y} - y\dot{x})}{x^2}$$

$$d\left(\frac{y}{x}\right) = \frac{1}{i\hbar}\frac{\omega}{x^2}$$

$$\frac{y}{x} = \frac{1}{i\hbar}\int_0^t \frac{\omega}{x^2}$$

$$y = \frac{x}{i\hbar}\int_0^t \frac{\omega}{x^2}$$

Let we define Z $=\int_0^t \frac{\omega}{x^2}$, then

$$y = \frac{x}{i\hbar} Z \qquad \text{(B-1.1)}$$

Now I go to calculate the value of *x*

$$\Omega = i\hbar \, (\dot{x}x^* + \dot{y}y^*)$$

$$(\dot{x}x^* + \dot{y}y^*) = \frac{\Omega}{i\hbar}$$

$$\dot{x}x^* = \frac{\Omega}{i\hbar} - \dot{y}y^*$$

$$\dot{x}x^* = \frac{\Omega}{i\hbar} - \frac{x}{i\hbar}\frac{\omega}{x^2}\frac{ix^*}{\hbar} Z^*$$

Since y= $\frac{x}{i\hbar}\int_0^t \frac{\omega}{x^2}$ therefore $\dot{y} = \frac{x}{i\hbar}\frac{\omega}{x^2}$ and also y= $\frac{x}{i\hbar}$Z therefore $y^* = \frac{ix^*}{\hbar} Z^*$

Then

$$\dot{x}x^* = \frac{\Omega}{i\hbar} - \frac{\omega}{\hbar^2}\frac{x^*}{x}Z^*$$

Where $x \; and \; x^*$ are complex functions and are different from each other only by signs, we therefore;

$$\dot{x}x^* = \frac{\Omega}{i\hbar} - \frac{\omega}{\hbar^2}Z^*$$

So $$\dot{x} = \frac{\Omega}{i\hbar}x - \frac{\omega}{\hbar^2}xZ^*$$

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{\omega}{\hbar^2}Z^*]x$$

Use $\omega = x^2\dot{Z}$

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{x^2}{\hbar^2}\dot{Z}Z^*]x$$

Where Z is constraint to satisfy the following condition

$$|x|^2\left(1 + \frac{|Z|^2}{\hbar^2}\right) = 1$$

So

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{\hbar^2}{\hbar^2(\hbar^2+|Z|^2)}\dot{Z}Z^*]x$$

$$\dot{x} = [\frac{\Omega}{i\hbar} - \frac{\dot{Z}Z^*}{(\hbar^2+|Z|^2)}]x$$

Solving by integration, finally we get,

$$x = \frac{\hbar}{\sqrt{\hbar^2+|Z|^2}}e^{-\frac{i}{\hbar}\int_0^t \Omega - \int_0^t \frac{\dot{Z}Z^*}{(\hbar^2+|Z|^2)}} \quad \text{(B-1.2)}$$

Once 'a' has been determined, the remaining functions are obtained from the relations

$$y = \frac{1}{i\hbar} xZ$$

$$\omega = x^2 \dot{Z}$$

B-2 Examples

1) A Real Function

The function;

$$\omega = \frac{\hbar^2 \dot{Z}}{\hbar^2 + |Z|^2} e^{\frac{-2i}{\hbar} \int_0^t \Omega} \quad \text{(B-2.1)}$$

The evolution operator are characterized by $x \; and \; y$, which is given by

$$\omega = x^2 \dot{Z}$$

So $$x^2 = \frac{\hbar^2}{\hbar^2 + |Z|^2} e^{\frac{-2i}{\hbar} \int_0^t \Omega}$$

And $$x = \frac{\hbar e^{\frac{-i}{\hbar} \int_0^t \Omega}}{\sqrt{\hbar^2 + |Z|^2}} \quad \text{(B-2.2)}$$

Similarly $y = \frac{1}{i\hbar} xZ$

And $$y = \frac{-iZ e^{\frac{-i}{\hbar} \int_0^t \Omega}}{\sqrt{\hbar^2 + |Z|^2}} \quad \text{(B-2.3)}$$

Notice that the above Eq. (B-2.1) can be integrated to give $'Z'$ in terms of $\Omega and\ \omega$.

$$\omega = \frac{\hbar^2 \dot{Z}}{\hbar^2 + |Z|^2} e^{\frac{-2i}{\hbar} \int_0^t \Omega}$$

$$\omega e^{\frac{2i}{\hbar} \int_0^t \Omega} = \frac{\hbar^2 \dot{Z}}{\hbar^2 + |Z|^2}$$

$$\omega e^{\frac{2i}{\hbar} \int_0^t \Omega} = \frac{\hbar^2 \dot{Z}}{\hbar^2 (1 + \frac{|Z|^2}{\hbar^2})}$$

$$\omega e^{\frac{2i}{\hbar}\int_0^t \Omega} = \frac{\dot{Z}}{(1+\frac{|Z|^2}{\hbar^2})}$$

$$\int_0^t \omega e^{\frac{2i}{\hbar}\int_0^t \Omega} = \int_0^t \frac{\dot{Z}}{(1+\frac{|Z|^2}{\hbar^2})}$$

Integrating by parts w.r.t "t"

$$\int_0^t \omega e^{\frac{2i}{\hbar}\int_0^t \Omega} = \tan^{-1}(\frac{Z}{\hbar})$$

$$\tan \int_0^t \omega e^{\frac{2i}{\hbar}\int_0^t \Omega} = \frac{Z}{\hbar}$$

So $$Z = \hbar \tan[\int_0^t \omega e^{\frac{2i}{\hbar}\int_0^t \Omega}] \quad \text{(B-2.4)}$$

2) A Complex Function

The function;

$$\omega = \frac{\hbar^2 c\dot{\phi}}{\hbar^2+c^2 sin^2\phi} e^{-\frac{2i}{\hbar}\int_0^t \Omega + 2i\int_0^t \frac{\hbar^2\dot{\phi}}{\hbar^2+c^2\sin\phi^2}} \quad \text{(B-2.5)}$$

Since $\boldsymbol{\omega = x^2 \dot{Z}}$

Therefore $$x = \frac{\hbar}{\sqrt{\hbar^2+c^2\sin\phi^2}} \boldsymbol{e}^{-\frac{i}{\hbar}\int_0^t \Omega - i\int_0^t \frac{c^2\dot{\phi}\sin\phi^2}{\hbar^2+c^2\sin\phi^2}} \quad \text{(B-2.6)}$$

Similarly $y = \frac{1}{i\hbar} xZ$

$$y = \frac{-ic\, sin\phi\, e^{i\phi}}{\sqrt{\hbar^2+c^2\sin\phi^2}} \boldsymbol{e}^{-\frac{i}{\hbar}\int_0^t \Omega - i\int_0^t \frac{c^2\dot{\phi}\sin\phi^2}{\hbar^2+c^2\sin\phi^2}} \quad \text{(B-2.7)}$$

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