EOQ Model for Ameliorating Items with Time Dependent Demand and linear Time-Dependent Holding Cost

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Abstract

In many instances, the demand for inventory items fluctuates with time. The demand may increases, decreases, remains constant or even vanishes with time. An economic order quantity model for ameliorating items that describes the above scenario is hereby developed. Stocked items are said to be ameliorative when they incur a gradual increase in quality, quantity or both with time. We develop a model that determines the optimal replenishment cycle time such that the total variable cost is minimized. Numerical examples are given to illustrate the derived model.

Introduction

Demand for inventory items fluctuates with time. The demand for many inventoried items is either periodic, increases, decreases or vanishes with time. In developing countries it is observed that during the harvest period foodstuff flood markets as most of the local farmers stock it in abundance. However as time goes on the stock with some farmers starts getting exhausted and others were forced by daily needs to sale whole or part of their harvest and thus resort to buying food items from the market towards the end of planting session. Hence, the rate of demand for foodstuffs remains partly constant and increases partly with time.

Generally, deterioration is defined as decay, damage, or spoilage which rendered stored items partly or wholly unfit for its original purpose. Food items, chemicals, drugs, electronic components, photographic films, drugs, radio-active substances and so on are some examples of items that incur deterioration during their storage period. Therefore, for effective stock keeping the stockiest must take such loss into account while analyzing the inventory systems.

The research on deteriorating inventory was pioneered by Ghare and Schrader (1963) who developed a simple economic order quantity model for exponentially decaying inventory. In the course of time, Covert and Philip (1073) extended Ghare and Schrader's constant deterioration rate to a two-parameter Weibull distribution. Later, Shah and Jaiswal (1997) and Aggarwal (1981) presented and re-established an order level

inventory model with a constant rate of deterioration respectively. Dave and Patel (1981) considered an inventory model for deteriorating items with time-proportional demand when shortages were not allowed. Later, Sachan (1984) extended the model to all for shortages. Hollier and Mak (1983), Hariga and Benkherouf (1994), Wee (1995a, 1995b) developed their models taking the exponential demand. Earlier, Goyal and Giri (2001), wrote an excellent survey on the recent trends in modeling of deteriorating inventory.

The assumption that holding cost is constant at all times is not always realistic as the changes in value of money and price index are dynamic. In this era of globalization where countries engage in tough economic competition, it is unrealistic to assume that holding cost remains constant over time. Various functions describing different forms of holding costs were considered by researchers like Naddor (1966), Van der Veen (1967), Muhlemann and Valtis-Spanopoulos (1980), Goh (1994), Giri and Chaudhuri (1998), and Roy (2008).

Weiss (1982) considered a variation of the economic order quantity model where cumulative holding cost is a nonlinear function of time. The problem involves an approximation of the optimal order quantity for perishable goods, such as milk, and produce, sold in small to medium size grocery stores where there are delivery surcharges due to infrequent ordering, and managers frequently utilize markdowns to stabilize demand as the product's expiration date nears. Weiss (1982) showed how the holding cost curve parameters can be estimated via a regression approach from the product's usual holding cost (storage plus capital costs), lifetime and markdown policy. The cumulative holding cost for one unit that has been stored

during t units of time is $H(t) = \bar{h}t^{\gamma}$, where $\bar{h}$ and $\gamma \geq 1$ are constants; if $\gamma = 1$ then the problem reduces to the classical EOQ model with $\bar{h}$ being the cost of holding one unit for one time period. Weiss (1982) also showed in a numerical study that the model provides significant improvement in cost vis-à-vis the classical EOQ model, items with higher daily demand rate, lower holding cost, shorter lifetime, and a markdown policy with steeper discounts.

Tripathi (2013) developed an inventory model for non-deteriorating items under permissible delay in payments in which holding cost is a function of time.

Sharma and Vijay (2013) developed a deterministic inventory model for price dependent demand with time dependent deterioration, varying holding cost and shortages allowed. They assumed that credit limit is available for certain time with no interest, but after that time some interest will be charged.

Karmakar and Choudhury (2014) developed an inventory model for deteriorating items with general ramp-type demand rate, partial backlogging of unsatisfied demand and time-varying holding cost. The authors studied the model under two different replenishment policies, that is, (a) the first replenishment policy starting with no shortages and (b) the second policy starting with shortages. The backlogging rate was assumed to be a non-increasing function of the waiting time up to the next replenishment.

Some items when in stock have the property of undergoing increase in quantity or quality or both. Generally, fast growing animals like fishes, poultry, cattle, fruits, etc, provide good examples. Some fruit merchants in Nigeria invest huge amount of money in buying large plantations of orange, banana, pineapple, etc and keep the farms for months waiting for the arrival of times of their exponential demand. Within these months, it is certain that these

items (in the farm) undergo increase in quantity and quality. The items that exhibit such properties are referred to as ameliorating items.

The ameliorative nature of inventory was not given much attention until recently. Hwang (1997) developed an economic order quantity model (EOQ) and partial selling quantity model (PSQ) for ameliorating items under the assumption that the ameliorating time follows the Weibull distribution. Again, Hwang (1999, 2004) developed more inventory models for both ameliorating and deteriorating items separately under the LIFO and FIFO issuing policies. Later Moon *et al* (2005) developed an EOQ model for ameliorating/deteriorating items under inflation and time discounting. The model studied inventory models with zero-ending inventory for fixed order intervals over a finite planning horizon allowing shortages in all but in the last cycle. They also developed another model with shortages in all cycles taking into account the effects of inflation and time value of money. A partial selling inventory model for ameliorating items under profit maximization was studied by Mandal *et al* (2005).

In our present study, we focus our attention on ameliorating inventory where the rate of amelioration is constant but the demand is linearly dependent on time.

Assumptions and Notation

The proposed ameliorating inventory model is developed under the following assumptions and notation:

The inventory system involves only one single item.

Amelioration occurs when the items are effectively in stock.

The cycle length is T.

The initial stock level is I_0.

The inventory carrying cost in a cycle is C_h.

The unit cost of the items know constant C And the replenishment cost is also a constant C_O per replenishment.

The demand rate per unit time R is dependent on time.

The total demand in a cycle is R_t.

The rate of amelioration A is constant.

The ameliorated amount when considered in items of value (say weight) in a cycle is A_m.

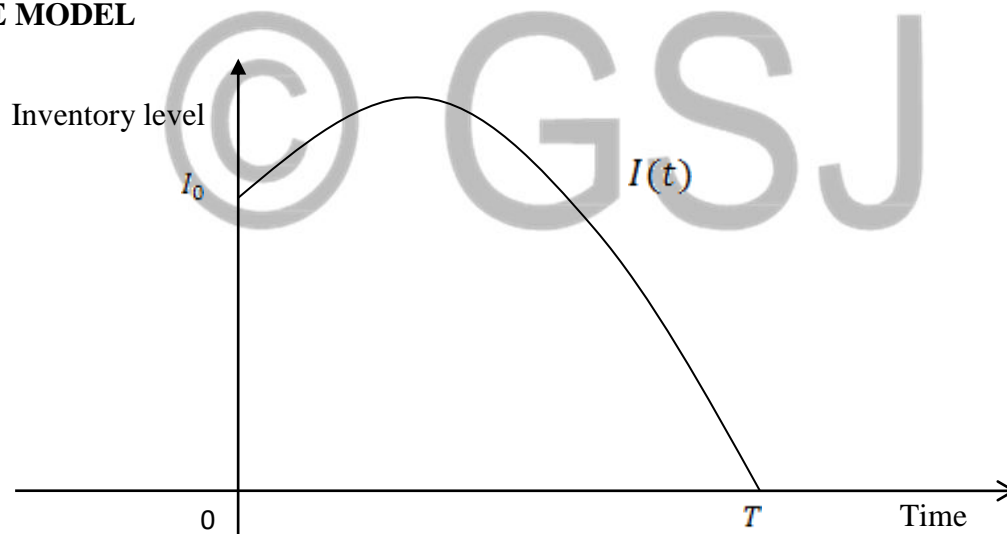Inventory ordering charge per unit i, is a known constant.

THE MODEL

Fig 1: inventory movement with an ameliorating inventory system with time depended demand

During the time interval $(0 \leq t \leq T)$ amelioration occurs at a constant rate A and the demand rate is time dependant. The differential equation that describes the instantaneous state of inventory level I(t) is given by

$$\frac{d}{dt}\big[I(t)\big] - AI(t) = -bt \tag{1}$$

The solution of the above equation is given by:

$$I(t) = \frac{1}{A^2}(Abt + b) + ke^{At} \tag{2}$$

Where k is a constant.

Applying the boundary condition at $t = 0, I(t) = I_0$ and substituting these values in equation (2) yields

$$I_0 = \frac{b}{A^2} + k$$

So that $k = I_0 - \dfrac{b}{A^2}$ \hfill (3)

Substituting equation (3) into equation (2), we obtain;

$$I(t) = \frac{b}{A^2}(At + 1 - e^{At}) + I_0 e^{At} \tag{4}$$

Also when t = T, I(t) = 0. Hence equation (4) becomes-

$$0 = \frac{b}{A^2}(AT + 1 - e^{AT}) + I_0 e^{AT}$$

$$\Rightarrow I_0 e^{AT} = -\frac{b}{A^2}(AT + 1 - e^{AT})$$

$$I_0 = -\frac{b}{A^2}(AT + 1)e^{-AT} + \frac{b}{A^2} \tag{5}$$

Substituting equation (5) into equation (4) gives:

$$I(t) = \frac{b}{A^2}(At + 1 - e^{At}) + \left[-\frac{b}{A^2}(AT + 1)e^{-AT} + \frac{b}{A^2} \right]e^{At}$$

$$= \frac{b}{A^2}\left[(At + 1) - (AT + 1)e^{A(t-T)} \right] \tag{6}$$

From Appendix I the total demand within the time interval $(0 \leq t \leq T)$, is obtained as:

$$R_T = \frac{bT^2}{2} \tag{7}$$

From Appendix II the ameliorated amount is given by;

$$A_m = \frac{bT^2}{2} + \frac{b}{A^2}(AT + 1)e^{-AT} - \frac{b}{A^2} \tag{8}$$

From Appendix III the linearly time dependent holding cost in the period, $(0 \leq t \leq T)$ is given by;

$$C_h(t) = \frac{bCi\sigma_1}{2A^3}\left[A^2T^2 + 2ATe^{-AT} + 2e^{-AT} - 2 \right] + \frac{bCi\sigma_2}{6A^4}\left[2A^3T^3 - 3A^2T^2 - 6ATe^{-AT} - 6e^{-AT} + 6 \right]$$

$$\tag{9}$$

Total variable cost per unit time in a cycle,(TVC(T))=Inventory Ordering cost+ Holding cost-

Cost of Ameliorated Amount that is

$$TVC(T) = \frac{1}{T}(C_0 + C_h - C_{A_m})$$

$$TVC(T) = \begin{bmatrix} \frac{1}{T}\left[C_0 + \frac{bCi\sigma_1}{2A^3}\left[A^2T^2 + 2ATe^{-AT} + 2e^{-AT} - 2\right]\right] \\ + \frac{1}{T}\left[\frac{bCi\sigma_2}{6A^4}\left[2A^3T^3 - 3A^2T^2 - 6ATe^{-AT} - 6e^{-AT} + 6\right]\right] \\ - \frac{C}{T}\left[\frac{bT^2}{2} + \frac{b}{A^2}(AT+1)e^{-AT} - \frac{b}{A^2}\right] \end{bmatrix}$$

$$= \begin{bmatrix} \frac{C_0}{T} + \frac{bCi\sigma_1}{2A^3}\left[A^2T + 2Ae^{-AT} + \frac{2e^{-AT}}{T} - \frac{2}{T}\right] \\ + \frac{bCi\sigma_2}{6A^4}\left[2A^3T^2 - 3A^2T - 6Ae^{-AT} - \frac{6e^{-AT}}{T} + \frac{6}{T}\right] \\ - C\left[\frac{bT}{2} + \frac{be^{-AT}}{A} + \frac{be^{-AT}}{A^2T} - \frac{b}{A^2T}\right] \end{bmatrix} \quad (10)$$

Equation (10) is then differentiated with respect to T to obtain

$$\frac{d}{dT}\left[TVC(T)\right] = \begin{bmatrix} \frac{d}{dT}\left[\frac{C_0}{T}\right] + \frac{d}{dT}\left[\frac{bCi\sigma_1}{2A^3}\left[A^2T + 2Ae^{-AT} + \frac{2e^{-AT}}{T} - \frac{2}{T}\right]\right] \\ + \frac{d}{dT}\left[\frac{bCi\sigma_2}{6A^4}\left[2A^3T^2 - 3A^2T - 6Ae^{-AT} - \frac{6e^{-AT}}{T} + \frac{6}{T}\right]\right] \\ - \frac{d}{dT}\left[C\left[\frac{bT}{2} + \frac{be^{-AT}}{A} + \frac{be^{-AT}}{A^2T} - \frac{b}{A^2T}\right]\right] \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{C_0}{T^2} + \dfrac{bCi\sigma_1}{2A^3}\left[A^2 - 2A^2 e^{-AT} - \dfrac{2(AT+1)e^{-AT}}{T^2} + \dfrac{2}{T^2} \right] \\ + \dfrac{bCi\sigma_2}{6A^4}\left[4A^3 T - 3A^2 - 6A^2 e^{-AT} + \dfrac{6(AT+1)e^{-AT}}{T^2} - \dfrac{6}{T^2} \right] \\ - C\left[\dfrac{b}{2} - be^{-AT} - \dfrac{b(AT+1)e^{-AT}}{A^2 T^2} + \dfrac{b}{A^2 T^2} \right] \end{bmatrix} \qquad (11)$$

For optimal T which minimizes the total variable cost per unit time, we have $\dfrac{d[TVC(T)]}{dT} = 0$

provided $\dfrac{d[TVC(T)]}{dT} \geq 0.$ That is;

$$= \begin{bmatrix} -\dfrac{C_0}{T^2} + \dfrac{bCi\sigma_1}{2A^3}\left[A^2 - 2A^2 e^{-AT} - \dfrac{2(AT+1)e^{-AT}}{T^2} + \dfrac{2}{T^2} \right] \\ + \dfrac{bCi\sigma_2}{6A^4}\left[4A^3 T - 3A^2 - 6A^2 e^{-AT} + \dfrac{6(AT+1)e^{-AT}}{T^2} - \dfrac{6}{T^2} \right] \\ - C\left[\dfrac{b}{2} - be^{-AT} - \dfrac{b(AT+1)e^{-AT}}{A^2 T^2} + \dfrac{b}{A^2 T^2} \right] \end{bmatrix}$$

This simplifies to;

$$\begin{bmatrix} 0 = \dfrac{-C_0}{T^2} + \dfrac{bCi\sigma_1}{2A^3 T^2}\left(A^2 T^2 - 2A^2 T^2 e^{-AT} - 2(AT+1)e^{-AT} + 2 \right) \\ + \dfrac{bCi\sigma_2}{6A^4 T^2}\left(4A^3 T^3 - 3A^2 T^2 - 6A^2 T^2 e^{-AT} + 6(AT+1)e^{-AT} - 6 \right) \\ - \dfrac{Cb}{2A^2 T^2}\left[A^2 T^2 - 2A^2 T^2 e^{-AT} - 2(AT+1)e^{-AT} + 2 \right] \end{bmatrix}$$

$$= \begin{bmatrix} -6A^4 C_0 + 3AbCi\sigma_1\left(A^2 T^2 - 2A^2 T^2 e^{-AT} - 2(AT+1)e^{-AT} + 2 \right) \\ + bCi\sigma_2\left(4A^3 T^3 - 3A^2 T^2 - 6A^2 T^2 e^{-AT} + 6(AT+1)e^{-AT} - 6 \right) \\ - 3A^2 Cb\left[A^2 T^2 - 2A^2 T^2 e^{-AT} - 2(AT+1)e^{-AT} + 2 \right] \end{bmatrix} \qquad (12)$$

However from equation (5), (7) and (8) we have

$$EOQ = I_0 = R_t - A_m$$

$$= \dfrac{b}{A^2}\left(1 - (AT+1)e^{-AT} \right)$$

$$(13)$$

Numerical examples:

The table below gives the solutions of ten different numerical examples with different parameters, using equations (12) and (13). In all the ten cases, the value $\frac{d^2}{dT^2}[TVC(T)] > 0,$ and so the solutions are for minimum values

Table 1: Input values of five different numerical examples with different parameters corresponding to minimum total variable cost per unit time

S/N	A	B	C	C_0	I	σ_1	σ_2
1.	2	10000	25	2500	0.67	400	100
2.	2.5	20000	300	2500	0.23	20	2
3.	9	10000	100	10000	0.44	2000	50000
4.	20	50000	50	1000000	0.01	4000	20000
5.	3	700000	500	25000	0.45	100	500

Table 2: Output values of five different numerical examples with different parameters corresponding to minimum total variable cost per unit time

S/No	T	TVC(T)	EOQ
1.	42 days	543808	141 units
2.	35 days	219854	165 units
3.	81 days	1244667	732 units
4.	72 days	9287263	9112 units
5.	36 days	1002153	666 units

Sensitivity analysis

We then carry out a sensitivity analysis on the first example to see the effect of parameter changes on the decision variables. This has been done by changing the parameters by 1%, 5%, and 25% and taking one parameter at a time, keeping the remaining parameters constants.

The Tables below give the sensitivity analysis of Table 1 and Table 2 to see the effects of parameter changes on decision variables:

Table 3: Sensitivity analysis to see the effect of percentage change in the rate of amelioration, A, on decision variables

b= 10000, C = 25, C_0 = 2500, i = 0.67, σ_1 = 400, σ_2 =100

% change in the value of A	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-5	8	-16
-5	-3	2	-4
-1	0	0	1
1	1	0	2
5	2	-2	4
25	9	-9	25

From Table 3, we can see that T and EOQ increase with increase in A, but TVC(T) decreases. Such phenomenon implies that the higher the rate of amelioration, the higher the rate of accumulation of the inventory and hence, the higher the EOQ and the longer the period it takes to dispose it. The total variable cost decreases as expected since the high rate of amelioration alleviates the invested costs.

Table 4: Sensitivity analysis to see the effect of percentage change in the rate of demand, b, on decision variables

A= 2, C = 25, C_0 = 2500, i = 0.67, σ_1 = 400, σ_2 =100

% change in the value of b	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	4	5	-10
-5	2	1	-6
-1	0	0	0
1	-3	-3	1
5	-5	-5	5
25	-10	-10	8

The result in the Table above shows that the increase in the demand rate translates into a decrease in the cycle period and total variable cost and an increase in order quantity. This is logical since an increased demand confers the stockiest with lots of profit and speeds up the rate at which the stock depletes and hence, the enhanced optimal ordering quantity.

Table 5: Sensitivity analysis to see the effect of percentage change in the unit cost of items, C, on decision variables

$A = 2$, $b = 10000$, $C_0 = 2500$, $i = 0.67$, $\sigma_1 = 400$, $\sigma_2 = 100$

% change in the value of C	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-8	-6	1
-5	0	-2	0
-1	1	0	0
1	3	4	-1
5	7	6	-4
25	10	21	-20

Table 5 shows that all the decision variables are sensitive to change in the cost of items. As the unit cost of items soars high, it invariably induces increase in other costs resulting in low demand and thus elongates the cycle period and hence, a low EOQ.

Table 6: Sensitivity analysis to see the effect of percentage change in the ordering cost, C_0, on decision variables

$A = 2$, $b = 10000$, $C = 25$, $i = 0.67$, $\sigma_1 = 400$, $\sigma_2 = 100$

% change in the value of C_0	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-7	23	12
-5	0	3	0
-1	0	1	0
1	2	-3	-4
5	2	-6	-4
25	11	-19	-22

The tabulated results above conform to the theoretical aspect of Inventory which asserts that the high rate of ordering cost reduces the frequency of placing orders and hence, results in a reduced TVC(T) and EOQ.

Table 7: Sensitivity analysis to see the effect of percentage change in the inventory holding charge, i, on decision variables

$A = 2$, $b = 10000$, $C = 25$, $C_0 = 2500$, $\sigma_1 = 400$, $\sigma_2 = 100$

% change in the value of i	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-13	-8	12
-5	-3	-2	0
-1	0	0	0
1	12	0	-4
5	21	2	-4
25	35	9	-22

In conformity with real life, a high inventory holding charge results in high TVC(T), an elongated T* (since this will induce increase in selling price which in turn will reduce the demand rate and hence elongates the rate of depletion of the items) and a low EOQ as illustrated by the Table above.

Table 8: Sensitivity analysis to see the effect of percentage change in the initial holding cost, σ_1, on decision variables

A = 2, b = 10000, C = 25, C_0 = 2500, I = 0.67, σ_2 =100

% change in the value of σ_1	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-8	-43	22
-5	-5	-27	10
-1	0	8	2
1	2	0	-4
5	7	2	-7
25	18	9	-27

A high initial holding cost results in high TVC(T) and low EOQ and an elongated T* as illustrated by the Table above.

Table 9: Sensitivity analysis to see the effect of percentage change in the trend in holding cost, σ_2, on decision variables

A = 2, b= 10000, C = 25, C_0 = 2500, I = 0.67, σ_1 =400

% change in the value of σ_2	% change in results		
	T^*	$TVC(T^*)$	EOQ
-25	-7	-7	10
-5	0	-3	5
-1	0	0	0
1	2	2	-1
5	5	2	-6
25	10	8	-15

A high trend in holding cost results in high TVC (T), a low EOQ and an elongated T* as illustrated by the Table above.

Reference

Aggarwal, S.C., (1981), Purchase-inventory decision models for inflationary conditions, Interfaces 11 18–23.

Covert, R.P. and Philip, G.C., (1973), An EOQ Model for Items with Weibull Distribution, *AHE Trans 5*(4), 323-326

Dave, U. and Patel, U.K., (1981). (t,Si) policy inventory model for deteriorating items with time proportional demand. *J. of Opl.Res.Soc.*, **32**: 137-142

Ghare, P.M. and Shrader, G.F., (1963), A Model for Exponential Decaying Inventory, *Journal of Industrial Engineering* **14:** 238-243

Giri, B.C., Chaudhuri, K.S., (1998): Deterministic models of perishable inventory with stock-dependent demand rate and non-linear holding cost, *European Journal of Operational Research,* 105(3), 467-474.

Goh, M., (1994): EOQ models with general demand and holding cost functions, *European Journal of Operational Research*,73,50-54.

Goyal S.K. and Giri B.C. (2001), Recent trends in modeling of deteriorating inventory, *European Journal of Operational Research* **134**(1) 1-16

Hariga, M.A., and Benkherouf, L., (1994). Optimal and heuristic inventory replenishment models for deteriorating items with exponential time varying demand. *European J. of Operational Research*, **79:** 123-137

Hollier R.H. and Mak K.L., (1983), Inventory replenishment policies for deteriorating items in a declining market, *International Journal of Production Research,* **21**(1) 813-826

Hwang S.K., (1997), A study of Inventory model for items with Weibull ameliorating, *Computers and Industrial Engineering,* **33:** 313-325

Hwang S.K. (1999), Inventory models for both ameliorating and deteriorating items, *Computers and Industrial Engineering*, **37:** 257-260

Hwang, H.S., Kim, H.G. and Paik, C.H., (2005), Optimal Issuing Policy for Fish-Breeding Supply Center with Items Weibull Ameliorating, *International Journal of the Information Systems for Logistics and Management* **1**(1,) pp. 1-7

Karmakar, B., Choudhury, K.D., (2014), Inventory models with ramp-type demand for deteriorating items with partial backlogging and time –varying holding cost, *Yugoslav Journal of Operation Research,* **2:** 249-266

Mondal B., Bhunia A. K. and Maiti M. (2003), An inventory system of ameliorating items for price dependent demand, *Computers and Industrial Engineering,* **45:** 443-456.

Moon I., Giri B. C. and Ko B. (2005), Economic order quantity model for ameliorating/deteriorating items under inflation and time discounting, *European Journal of Operations Research* 173(2) 1234-1250.

Moon I., Giri B. C. and Ko B. (2006), Erratum to ''Economic order quantity model for ameliorating/deteriorating items under inflation and time discounting'' *European Journal of Operations Research* 174(2) 1345-1347.

Muhlemann, A.P. and Valtis-Spanopoulos, N.P., (1980), A variable holding Cost rate EOQ model, *European Journal of Operational Research,* **4** 132-135

Naddor, E., (1966): Inventory Systems, New York, *John Wiley and Sons*.

Roy, A., (2008), An inventory model for deteriorating items with price dependent demand and time varying holding cost, *Advanced modeling and* optimization, **10**(1):25-37

Sachan, R.S., (1984): On (T, Si) policy inventory model for deteriorating items with time proportional demand, *Journal of the Operational Research Society*, **35:** 1013-1019.

Shah, Y.K. and Jaiswal, M.C.,(1997), An order-level inventory model for a system with constant rate of deterioration, Opsearch **14:** 174–184.

Sharma, S.C., Vijay, V., (2013), An EOQ Model for Deteriorating Items with Price Dependent Demand, Varying Holding Cost and Shortages under Trade Credit, *International Journal of Science and Research, Volume* **4**(1) 445-438.

Tripathi, R.P., (2013), Inventory model with cash flow oriented and time-dependent holding cost under permissible delay in payments, *Yugoslav Journal of Operation Research, JIEC,* **4(1):** 122-132.

Van Der Veen, B., (1967): Introduction to the theory of Operational Research, Philip Technical Library, *Springer-Verlag*, New York.

Wee, H.L., (1995a), A deterministic lot-size inventory model for deteriorating items with shortages and a declining market, *Compt*. *Operat*. *Res*., **26:** 545-558

Wee, H.L., (1995b), Joint Pricing and Replenishment Policies for Deteriorating Inventory with declining markets, *International Journal of Production Economics,* **40:** 163-171

Weiss, H.J., (1982): EOQ models with non-linear holding cost, *European Journal of Operational Research*, 9,56-60.

Appendix I: The total demand within the time interval $(0 \leq t \leq T)$, is obtained as:

$$R_T = \int_0^T btdt$$

$$= b\left|\frac{t^2}{2}\right|_0^T$$

$$= \frac{bT^2}{2} \tag{7}$$

Appendix II: The ameliorated amount within the interval $(0 \leq t \leq T)$ is obtained as:

$$A_m = R_T - I_0$$

$$= \frac{bT^2}{2} - \left[-\frac{b}{A^2}(AT+1)e^{-AT} + \frac{b}{A^2}\right]$$

$$= \frac{bT^2}{2} + \frac{b}{A^2}(AT+1)e^{-AT} - \frac{b}{A^2}$$

Appendix III: The linearly time dependent holding cost in the period, $(0 \leq t \leq T)$ is obtained as:

$$C_h(t) = iC\int_0^T (\sigma_1 + \sigma_2 t)I(t)dt$$

$$C_h(t) = iC\int_0^T (\sigma_1 + \sigma_2 t)\left[\frac{b}{A^2}\left[(At+1) - (AT+1)e^{A(t-T)}\right]\right]dt$$

$$C_h(t) = \frac{bCi\sigma_1}{A^2}\int_0^T \left[(At+1) - (AT+1)e^{A(t-T)}\right]dt + \frac{bCi\sigma_2}{A^2}\int_0^T \left[t(At+1) - (AT+1)e^{A(t-T)}\right]dt$$

$$= \begin{bmatrix} \dfrac{bCi\sigma_1}{A}\int_0^T tdt + \dfrac{bCi\sigma_1}{A^2}\int_0^T dt - \dfrac{bCTi\sigma_1}{A}\int_0^T e^{A(t-T)}dt - \dfrac{bCi\sigma_1}{A^2}\int_0^T e^{A(t-T)}dt \\ + \dfrac{bCi\sigma_2}{A}\int_0^T t^2dt + \dfrac{bCi\sigma_2}{A^2}\int_0^T tdt - \dfrac{bCTi\sigma_2}{A}\int_0^T te^{A(t-T)}dt - \dfrac{bCi\sigma_2}{A^2}\int_0^T te^{A(t-T)}dt \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{bCi\sigma_1 T^2}{2A} + \dfrac{bCi\sigma_1 T}{A^2} - \dfrac{bCTi\sigma_1}{A}\left[\dfrac{1}{A} - \dfrac{e^{-AT}}{A}\right] - \dfrac{bCi\sigma_1}{A^2}\left[\dfrac{1}{A} - \dfrac{e^{-AT}}{A}\right] \\[2mm] + \dfrac{bCi\sigma_2 T^3}{3A} + \dfrac{bCi\sigma_2 T^2}{2A^2} - \dfrac{bCTi\sigma_2}{A}\left[\dfrac{T}{A} - \dfrac{1}{A^2} + \dfrac{e^{-AT}}{A^2}\right] - \dfrac{bCi\sigma_2}{A^2}\left[\dfrac{T}{A} - \dfrac{1}{A^2} + \dfrac{e^{-AT}}{A^2}\right] \end{bmatrix}.$$

$$= \begin{bmatrix} \dfrac{bCi\sigma_1 T^2}{2A} + \dfrac{bCTi\sigma_1 e^{-AT}}{A^2} - \dfrac{bCi\sigma_1}{A^3} + \dfrac{bCi\sigma_1 e^{-AT}}{A^3} \\[2mm] + \dfrac{bCi\sigma_2 T^3}{3A} - \dfrac{bCi\sigma_2 T^2}{2A^2} + \dfrac{bCTi\sigma_2}{A^3} - \dfrac{bCTi\sigma_2 e^{-AT}}{A^3} - \dfrac{bCTi\sigma_2}{A^3} + \dfrac{bCi\sigma_2}{A^4} - \dfrac{bCi\sigma_2 e^{-AT}}{A^4} \end{bmatrix}$$

$$= \dfrac{bCi\sigma_1}{2A^3}\left[A^2 T^2 + 2ATe^{-AT} + 2e^{-AT} - 2\right] + \dfrac{bCi\sigma_2}{6A^4}\left[2A^3 T^3 - 3A^2 T^2 - 6ATe^{-AT} - 6e^{-AT} + 6\right]$$