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Existence of solutions for nonlinear Caputo-Hadamard fractional differential equations with nonlocal Condition

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Abstract. In this paper we study the existence of solutions for initial value problem for a class of nonlinear Caputo-Hadamard fractional differential equations with nonlocal Condition. Our results are based on Schauder's fixed point theorem and the Banach contraction principle fixed point theorem.

Key words: fractional differential equation; Caputo-Hadamard derivative; existence fixed point, nonlocal Condition.
AMS Subject Classification: 26A33, 34A08.

1 Introduction

In this paper we deal with the existence of solutions for non-local initial value problem (IVP for short), for fractional order differential equation with Caputo-Hadamard fractional derivative:

$$^cD_1^\alpha y(t) = f(t, y(t), I_1^\alpha y(t)),\ t \in J := [1, T],\ 0 < \alpha \leq 1. \tag{1}$$

$$y(1) = y_1 + \varphi(y), \tag{2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_1 \in \mathbb{R}$, and $^cD_1^\alpha$, I_1^α are the Caputo-Hadamard fractional derivative and the Hadamard integral operators and $\varphi : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.
The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $\varphi(y)$ may be given by

$$\varphi(y) = \sum_{i=1}^{p} c_i y(t_i).$$

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where $c_i, i = 1, 2, ..., p$ are given constants and $0 < ... < t_p < T$. Nonlocal conditions were initiated by Byszewski [3] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [4, 5],the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 3.1) and the second one on the Banach contraction principle (Theorem 3.2). Finally, in Section 4 an example is given to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
By $C(J, \mathbb{R})$, we denote the Banach space of continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}$$

, by $AC(J, \mathbb{R})$, we denote the space of absolutely continuous functions from J_0 into $\mathbb{R}$

Definition 2.1 *.([7], [10]) Let $0 < a < b < \infty$, and $h : [a, b] \to R_+$ is a function. The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ of the function h is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (log\frac{t}{s})^{\alpha-1} \frac{h(s)}{s} ds, \ t \in [a, b]$$

where $\Gamma(.)$ is the gamma function.

Definition 2.2 *.([7], [10]) Let $0 < a < b < \infty$, and $h : [a, b] \to R_+$ is a function. The Hadamard fractional derivative of order $\alpha \in]0, 1]$ of the function h is defined by*

$$D_a^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} t \frac{d}{dt} \int_a^t (log\frac{t}{s})^{-\alpha} \frac{h(s)}{s} ds, \ t \in [a, b]$$

Obviously, we can obtient

$$I_a^\alpha (log\frac{t}{a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (log\frac{t}{a})^{\beta+\alpha-1}, \ D_a^\alpha (log\frac{t}{a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (log\frac{t}{a})^{\beta-\alpha-1}$$

The following properties are some of the main ones of the fractional Hadamard integrals and derivatives operators.

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Proposition 2.1 *([10]) Let $\alpha,\ \beta > 0$, and $1 \leq p \leq +\infty$ Then we have*

(1) $I_a^\alpha :\ L^p(J, \mathbb{R}_+) \to L^p(J, \mathbb{R}_+)$, *and if* $f \in L^p(J, \mathbb{R}_+)$, *then*

$$I_a^\alpha I_a^\beta f(t) = I_a^\beta I_a^\alpha f(t) = I_a^{\alpha+\beta} f(t).$$

(2) *The fractional integration operator I_a^α is linear*

(3) *The fractional order integral operator I_a^α maps L^p into itself continuously.*

Definition 2.3 *([10]). Let $0 < a < b < \infty$ and $1 \leq p < +\infty$. The Caputo-Hadamard fractional derivative of order $\alpha \in]0, 1]$ of the function $h \in L^p([a, b], \mathbb{R}_+)$ is defined by*

$$^cD_a^\alpha h(t) = D_a^\alpha [h(t) - h(a)],\ t \in [a, b]$$

If $h \in AC([a, b], \mathbb{R}_+)$, then Caputo-Hadamard fractional derivative has the following equivalent formulation

$$^cD_a^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (log\frac{t}{s})^{-\alpha} h'(s)ds,\ t \in [a, b]$$

Lemma 2.1 *([7], [10]) Let $1 \leq p < +\infty$ and $\alpha > 0$ be such that $n = [\alpha] + 1$.*

(1) If $h \in C([a, b], \mathbb{R}_+)$, then $^cD_a^\alpha (I_a^\alpha h(t)) = h(t),\ t \in [a, b]$.

(2) If $h \in AC([a, b], \mathbb{R}_+)$, then $I_a^\alpha\ (^cD_a^\alpha h(t)) = h(t) - h(a),\ t \in [a, b]$.

The following theorems will be needed.

Theorem 2.1 *(Schauder fixed point theorem) ([10]) Let E a Banach space and Q be a convex subset of E and $F : Q \longrightarrow Q$ is compact, and continuous map. Then F has at least one fixed point in Q.*

3 Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem $(1)-(2)$.

Definition 3.1 . *A function $y \in C(J, \mathbb{R})$ is said to be a solution of IVP $(1) - (2)$ if y satisfies (1) and (2).*

For the existence of solutions for the problem $(1) - (2)$, we need the following auxiliary lemma.

Lemma 3.1 *The solution of the IVP $(1)-(2)$ can be expressed by the integral equation*

$$y(t) = y_1 - g(y) + \frac{1}{\Gamma(\alpha)} \int_1^t (log\frac{t}{s})^{\alpha-1} \frac{f\,(s, y(s), I_1^\alpha y(s))}{s} ds,\ for\ t \in J \tag{3}$$

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Proof. Assume that $y \in C(J, \mathbb{R})$ is a solution of the integral equation (3).
Obviously we obtain $y(1) = y_1$ and $t \mapsto I_1^\alpha y(t) \in C(J, \mathbb{R})$.
The continuity of f and definition of Hadamard integral I_1^α guarantee that $t \mapsto f(t, y(t), I_1^\alpha y(t))$ is continuous as well and

$$I_1^\alpha f(t, y(t), I_1^\alpha y(t))|_{t=1} = 0$$

Since $t \mapsto I_1^\alpha f(t, y(t), I_1^\alpha y(t))$ is continuous, then we have y is differential for $t \in J$ (see (3)) then $y \in AC(J, \mathbb{R})$.
From Lemma (2.1), we have

$${}^cD_1^\alpha I_1^\alpha f(t, y(t), I_1^\alpha y(t)) = f(t, y(t), I_1^\alpha y(t)), \ for\ t \in J$$

On the other hand,

$$\begin{aligned} {}^cD_1^\alpha [y(t) - y_1] &= \frac{1}{\Gamma(1-\alpha)} \int_1^t (log\frac{t}{s})^{-\alpha} [y(s) - y_1]' ds \\ &= \frac{1}{\Gamma(1-\alpha)} \int_1^t (log\frac{t}{s})^{-\alpha} y'(s) ds \\ &= {}^cD_1^\alpha y(t), \ for\ t \in J \end{aligned}$$

By all above, we conclude that $y \in C(J, \mathbb{R})$ is a solution of the problem (1) and (2)
Let us introduce the following assumptions:

(H1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous.

(H2) There exists a constants, $K > 0$ and $0 < L < 1$ such that:
$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq K|u_1 - u_2| + L|v_1 - v_2|$, for any $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $t \in J$.

(H3) there exist a constant $\widetilde{K} > 0$ such that

$$|\varphi(u) - \varphi(\overline{u})| \leq \widetilde{K}|u - \overline{u}| \text{ for any } u, \overline{u} \in C(J, \mathbb{R}),$$

(H4) there exists a constants $\widetilde{M} > 0$ such that

$$|\varphi(y)| \leq \widetilde{M}, \forall y \in C(J, \mathbb{R}).$$

Our first result is based on Schauder fixed point theorem.

Theorem 3.1 *Assume that the assumptions $(H1) - (H4)$ are satisfied. If*

$$\frac{K(logT)^\alpha}{\Gamma(\alpha+1)} + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)} < 1, \tag{4}$$

then the IVP $(1) - (2)$ has at least one solution $y \in C(J, \mathbb{R})$.

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Proof
Transform the problem $(1)-(2)$ into a fixed point problem. Consider the operator

$$\mathcal{H} : C(J, \mathrm{I\!R}) \longrightarrow C(J, \mathrm{I\!R})$$

defined by:

$$(\mathcal{H}y)(t) = y_1 + \varphi(y) + I_1^{\alpha} f(t, y(t), I_1^{\alpha} y(t)),$$

Let

$$r \geq \frac{|y_1| + \widetilde{M} + \frac{f^*(logT)^{\alpha}}{\Gamma(\alpha+1)}}{1 - (\frac{K(logT)^{\alpha}}{\Gamma(\alpha+1)} + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)})}$$

where

$$f^* = sup_{t\in J}|f(t, 0, 0)|$$

and consider the set

$$\mathcal{B}_r = \{y \in C(J, \mathrm{I\!R}) / \|y\|_{\infty} \leq r\}$$

Clearly $\mathcal{B}_r$ is nonempty, bounded, convex and closed.
Now, we shall show that $\mathcal{H}$ satisfies the assumption of Schauder fixed point theorem. The proof will be given in three parts.
Step 1. $\mathcal{H}$ is continuous
Let (y_n) be a sequence such that $y_n \to y$ in $C(J, \mathbb{R})$ if $t \in J$
then

$$\|\mathcal{H}(y_n)(t) - \mathcal{H}(y)(t)\|_{\infty} = 0$$

For $t \in J$, we have

$$\begin{aligned}
|\mathcal{H}(y_n)(t) - \mathcal{H}(y)(t)| &\leq |\varphi(y_n) - \varphi(y) + I_1^{\alpha}[f(s, y_n(s), I_1^{\alpha} y_n(s)) - f(s, y(s), I_1^{\alpha} y(s))]\frac{ds}{s}| \\
&\leq |\varphi(y_n) - \varphi(y)| + I_1^{\alpha}|f(s, y_n(s), I_1^{\alpha} y_n(s)) - f(s, y(s), I_1^{\alpha} y(s))| \\
&\leq \widetilde{K}|y_n - y| + I_1^{\alpha}[K|y_n(s) - y(s)| + L(I_1^{\alpha}|y_n(s) - y(s)|)] \\
&\leq \widetilde{K}|y_n - y| + K(I_1^{\alpha}|y_n(s) - y(s)|) + L(I_1^{2\alpha}|y_n(s) - y(s)|)
\end{aligned}$$

then

$$\begin{aligned}
\|\mathcal{H}(y_n) - \mathcal{H}(y)\|_{\infty} &\leq \widetilde{K}\|y_n - y\|_{\infty} + \frac{K(logT)^{\alpha}}{\Gamma(\alpha+1)}\|y_n - y\|_{\infty} + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)}\|y_n - y\|_{\infty} \\
&\leq (\widetilde{K} + \frac{K(logT)^{\alpha}}{\Gamma(\alpha+1)} + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)})\|y_n - y\|_{\infty}
\end{aligned}$$

$$\|\mathcal{H}(y_n) - \mathcal{H}(y)\|_{\infty} \to 0 \ as \ n \to \infty$$

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Consequently, $\mathcal{H}$ is continuous.
Step 2. $\mathcal{H}\mathcal{B}_r \subset \mathcal{B}_r$,
let $y \in \mathcal{B}_r$ then

$$\begin{aligned}
|\mathcal{H}y(t)| &= |y_1 + \varphi(y) + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1} f(t, y(t), I_1^\alpha y(t))\frac{dt}{s}| \\
&\leq |y_1| + |\varphi(y)| + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1} |f(t, y(t), I_1^\alpha y(t))|\frac{dt}{s} \\
&\leq |y_1| + |\varphi(y)| + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}\Big(|f(t, y(t), I_1^\alpha y(t)) - f(t,0,0)| + |f(t,0,0)|\Big)\frac{dt}{s} \\
&\leq |y_1| + |\varphi(y)| + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1} |f(t, y(t), I_1^\alpha y(t)) - f(t,0,0)|\frac{dt}{s} \\
&+ \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}|f(t,0,0)|\frac{dt}{s}
\end{aligned}$$

By $(H2)$ and $(H3)$ we have

$$\begin{aligned}
|\mathcal{H}y(t)| &\leq |y_1| + \widetilde{M} + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}|f(t,0,0)|\frac{dt}{s} + \frac{K}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}|y(t)|\frac{dt}{s} \\
&\leq | y_1| + \widetilde{M} + \frac{1}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}|f(t,0,0)|\frac{dt}{s} + \frac{K}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1}|y(t)|\frac{dt}{s} \\
&+ \frac{L}{\Gamma(\alpha)}\int_1^t (log\frac{t}{s})^{\alpha-1} I_1^\alpha |y(t)|\frac{dt}{s}
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathcal{H}y\|_\infty &\leq |y_1| + \widetilde{M} + \frac{f^*(logT)^\alpha}{\Gamma(\alpha+1)} + \frac{K(logT)^\alpha}{\Gamma(\alpha+1)}\|y\|_\infty + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)}\|y\|_\infty \\
&\leq |y_1| + \widetilde{M} + \frac{f^*(logT)^\alpha}{\Gamma(\alpha+1)} + \frac{K(logT)^\alpha}{\Gamma(\alpha+1)}r + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)}r \leq r
\end{aligned}$$

Then $\mathcal{H}\mathcal{B}_r \subset \mathcal{B}_r$.

Step 3. $\mathcal{H}$ is compact
Now, we will show that $\mathcal{H}\mathcal{B}_r$ is relatively compact, meaning that $\mathcal{H}$ is compact. Clearly $\mathcal{H}\mathcal{B}_r$ is uniformly bounded because by Step 2, we have $\mathcal{H}(\mathcal{B}_r) = \{\mathcal{H}(y) : y \in \mathcal{B}_r\} \subset \mathcal{B}_r$ thus for each $y \in \mathcal{B}_r$ we have $\|\mathcal{H}(y)\|_\infty \leq r$ which means that $\mathcal{H}(\mathcal{B}_r)$is uniformly bounded. It remains to show that $\mathcal{H}\mathcal{B}_r$ is equicontinuous
Let $t_1, t_2 \in J; t_1 < t_2$, and let $y \in \mathcal{B}_r$, then

$$|\mathcal{H}(y)(t_1) - \mathcal{H}(y)(t_2)| = \frac{1}{\Gamma(\alpha)}\Big|\int_1^{t_2} (log\frac{t_2}{s})^{\alpha-1} f(s, y(s), I_1^\alpha y(s))\frac{ds}{s}$$

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$$\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}|\int_1^{t_2}(log\frac{t_2}{s})^{\alpha-1}f(s,y(s),I_1^\alpha y(s))\frac{ds}{s} \\
&- \int_1^{t_1}(log\frac{t_1}{s})^{\alpha-1}f(s,y(s),I_1^\alpha y(s))\frac{ds}{s}| \\
&= \frac{1}{\Gamma(\alpha)}|\int_1^{t_1}[(log\frac{t_2}{s})^{\alpha-1}-(log\frac{t_1}{s})^{\alpha-1}]f(s,y(s),I_1^\alpha y(s))\frac{ds}{s} \\
&+ \int_{t_1}^{t_2}[(log\frac{t_2}{s})^{\alpha-1}f(s,y(s),I_1^\alpha y(s))\frac{ds}{s}|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{M}{\Gamma(\alpha)}\Big|\int_1^{t_1}[(log\frac{t_2}{s})^{\alpha-1}-(log\frac{t_1}{s})^{\alpha-1}]\frac{ds}{s}+\int_{t_1}^{t_2}[(log\frac{t_2}{s})^{\alpha-1}\frac{ds}{s}\Big| \\
&\leq \frac{M}{\Gamma(\alpha)}|\int_1^{t_1}[(log\frac{t_2}{s})^{\alpha-1}-(log\frac{t_1}{s})^{\alpha-1}]\frac{ds}{s}+\int_{t_1}^{t_2}[(log\frac{t_2}{s})^{\alpha-1}\frac{ds}{s}| \\
&\leq \frac{M}{\Gamma(1+\alpha)}[(log(t_2)^\alpha-(log(t_1)^\alpha+2(log\frac{t_2}{t_1})^\alpha]
\end{aligned}$$

Hence $|\mathcal{H}(y)(t_1)-\mathcal{H}(y)(t_2)|\to 0$ as $|t_1-t_2|\to 0$ where $M>0$ is a constant independent of t_1 and t_2. It implies that $\mathcal{H}\mathcal{B}_r$ is equicontinuous. From Arzela-Ascoli Theorem, we imply that $\mathcal{H}\mathcal{B}_r$ is relatively compact.
As a consequence of Schauders fixed point theorem the IVP $(1)-(2)$ has at least one solution in $\mathcal{B}_r$.

Theorem 3.2 *Assume that the assumptions* $(H1)-(H3)$ *and if*

$$\widetilde{K}+\frac{K(logT)^\alpha}{\Gamma(\alpha+1)}+\frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)}<1, \tag{5}$$

then there exists a unique solution for IVP $(1)-(2)$ *on* J.

Proof
Let $x,y\in C(J,\mathbb{R})$, for $t\in J$, we have

$$\begin{aligned}
|\mathcal{H}(y)(t)-\mathcal{H}(x)(t)| &\leq |\varphi(x)-\varphi(y)+\frac{1}{\Gamma(\alpha)}\int_1^t(log\frac{t}{s})^{\alpha-1}[f(s,y(s),I_1^\alpha y(s))-f(s,x(s),I_1^\alpha x(s))]\frac{ds}{s}| \\
&\leq |\varphi(x)-\varphi(y)|+\frac{1}{\Gamma(\alpha)}\int_1^t(log\frac{t}{s})^{\alpha-1}[K|y(s)-x(s)|+L(I_1^\alpha|y(s)-x(s)|)]\frac{ds}{s} \\
&\leq \widetilde{K}|x-y|+\frac{K}{\Gamma(\alpha)}\int_1^t(log\frac{t}{s})^{\alpha-1}|y(s)-x(s)|\frac{ds}{s}+L(I_1^{2\alpha}|y(s)-x(s)|)
\end{aligned}$$

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then

$$\begin{aligned}\|\mathcal{H}(y)-\mathcal{H}(x)\|_\infty &\leq \widetilde{K}\|x-y\|_\infty+\frac{K(logT)^\alpha}{\Gamma(\alpha+1)}\|y-x\|_\infty+\frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)}\|y-x\|_\infty\\ &\leq (\widetilde{K}+\frac{K(logT)^\alpha}{\Gamma(\alpha+1)}+\frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)})\|y-x\|_\infty\end{aligned}$$

Consequently by (5), the operator $\mathcal{H}$ is a contraction. Hence, by Banach's contraction principal, $\mathcal{H}$ has a unique fixed point $y\in C(J,\mathbb{R})$, which is a solution of the problem $(1)-(2)$.

3.1 Example

Let us consider the following fractional initial value problem,

$$^cD_1^{\frac{1}{2}}y(t)=\frac{e^{-t}}{(e^t+\pi)(1+|y(t)|+|I_1^{\frac{1}{2}}y(t)|)},\ t\in J:=[1,e], \tag{6}$$

$$y(1)-\sum_{i=1}^{n}c_iy(t_i)=1, \tag{7}$$

where $0<t_1<t_2<...<t_n<1$ and $c_i=1,...,n$ are positif constants with

$$\sum_{i=1}^{n}c_i<\frac{1}{5}.$$

Set

$$f(t,y,z)=\frac{e^{-t}}{(e^t+\pi)(1+y+z)},\ (t,y,z)\in J\times[0,+\infty)\times[0,+\infty).$$

and

$$\varphi(y)=\sum_{i=1}^{n}c_iy(t_i)$$

Clearly, the function f is continuous.
For each $y_1,y_2,z_1,z_2\in\times[0,+\infty)$ and $t\in[1,e]$: Then we have

$$\begin{aligned}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+\pi}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right|\\ |f(t,y_1,z_1)-f(t,y_2,z_2)| &= \frac{e^{-t}}{e^t+\pi}\left|\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right|\\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+\pi)(1+y_1+z_1)(1+y_2+z_2)}\end{aligned}$$

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$$
\begin{aligned}
&\le \frac{e^{-t}}{(e^t+\pi)}(|y_1-y_2|+|z_1-z_2|)\\
&\le \frac{e^{-1}}{(e+\pi)}|y_1-y_2|+\frac{e^{-1}}{(e+\pi)}|z_1-z_2|.
\end{aligned}
$$

Hence the condition **(H2)** holds with $K = L = \frac{e^{-1}}{e+\pi}$.
On the other hand, we have

$$
\begin{aligned}
|\varphi(u)-\varphi(\bar{u})| &= \left|\sum_{i=1}^{n} c_i u - \sum_{i=1}^{n} c_i \bar{u}\right|\\
&\le \sum_{i=1}^{n} c_i\,|u-\bar{u}|\\
&< \frac{1}{5}\,|u-\bar{u}|\,.
\end{aligned}
$$

Hence the condition **(H3)** holds with $\widetilde{K} = \frac{1}{5}$.
We shall check that condition (5) is satisfied with $T = e$, and $\alpha = \frac{1}{2}$. Indeed

$$
\widetilde{K} + \frac{K(logT)^{\alpha}}{\Gamma(\alpha+1)} + \frac{L(logT)^{2\alpha}}{\Gamma(2\alpha+1)} = \frac{1}{5} + \Big(\frac{e^{-1}}{e+\pi}\Big)\Big(\frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(2)}\Big) \simeq 0.33 < 1. \qquad (8)
$$

Then by Theorem 3.2, the problem $(6)-(7)$ has a unique solution on $[1, e]$.

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