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Exploring Numerical Methods for Solving Boundary Value Problem: A Study of Finite Difference and Shooting Methods with MATLAB Implementation

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Abstract

This research work focused on the numerical methods involved in solving boundary value problems. We employed finite difference method and shooting method to solve boundary value problems. We equally implemented the numerical methods in MATLAB through two illustrative examples. The results show that each of the two numerical methods employed is suitable for solving linear boundary value problems of ordinary differential equations.

Keyword: **boundary value, ordinary differential equations, finite difference method**

1.1 Introduction

In the history of mathematics and modeling physical phenomena, Ordinary and partial differential equation has played important roles; they continue to serve a critical tools in the nearest future. one may be interested to find a solution to a differential equation satisfying certain defined conditions in the theory and application of ordinary and partial differential equations. If the conditions are given at only one point of the independent variable, we have initial conditions; whereas if the conditions are given at more than one point of the independent variable, they are called boundary conditions (BC). solution of an nth order differential equation together with n-initial conditions is called an initial value problem (IVB), The problem is called boundary value problem if the n-boundary conditions are considered, (BVP) (Biruk Endeshaw 2019).

When calculations are been carryout, we need not write out the full equations at each step or carrying the variables $x_1, x_2, x_3, x_{4,...,} x_n$ through the calculations, since they remain in the same column. The only difference from system to system occurred in the coefficients of the unknowns and the values on the right side of the equations. A linear system is often replaced by a matrix, which contain all the information about the system that is necessary to determine its solutions, but in a compact form (Fair, 2005).

This work is proposing solution to only boundary value problems (BVP) of ODEs. These are problems in which the value of the unknown functions or its derivative are given at two different points known as boundary value problems. It often in the form

$$p'' = f(t, p, p'), p(c) = \propto p(d) = \beta$$

Where $\propto$ and β are given numbers

With Dirichlet boundary conditions (first kind)

$$p\,'(c) = \propto\,, \quad p(d) = \beta$$

Neumann boundary conditions (second kind)

$$p'(c) = c, \quad p'(d) = d$$

Robbin boundary conditions (third kind)

$$y''^{(t)} + a_1(t)p(t) = t, \quad p'(c) + a_2(c)p(c) = c$$

2.1 Numerical Methods

We will considered the following :

2.2 Boundary Value Problem

Boundary value problems deals with a wide range of applications in applied science and engineering, necessitating the development of faster and more reliable numerical methods.

A large number of numerical methods have been introduced for solving two points boundary value problems such as higher order finite difference methods and extended a decomposition method (Timizi, 2002).

There are two main ways for numerical solutions of boundary value problems, which are indirect methods Jang (2008). A lot of work was done on boundary value problems by Kreyzig (2005). He stated that a large class of important boundary value problem is Sturm Liouville problems.

2.3 Finite Difference Method

Methods involved in the finite difference for solving Boundary value problems replace each of the derivatives in the differential equations with an appropriate difference - quotient approximation. Burden, (2010). We shall consider the linear two points ordinary boundary value problem of the form.

Y''(x) + P(x)y' + q(x)y = r(x), y(a)

= y_o, y(b) = y_n satisfies the following conditions to ensure the existence of unique solution. P(x), q(x) and r(x) are containing on [a,b] and q(x) < 0 on [a,b] for positive q(x), the BVP may not possess a solution. For the sake of convenience, we shall employ equal increment in the independent variable, then x_o, x_1, ..., x_n are the interior mesh points of the interval [a,b] related as $x_1 = x_o$ + ih for I=0,1....n and h is the step size with h = (b-a)/n

Based on Gilat and Subramanian (2011), the finite difference method entails that the derivative in the differential equation is replaced with the finite difference approximation, thereby changing the differential equation into algebraic form through a process called discretization. We used the centre difference formulas to solve finite difference methods since they give better validity.

Madhumangal (2007) stated that Euler's method is less efficient in practical problems because if h is not sufficiently small then the result will be inaccurate.

The Runge - Kutta methods give more accurate result. One advantage of this method is that it requires only one value of the function at some selected points at the sub - interval and it is stable, and easy to program. This method uses numerous calculations of the function to find y_{i+1}. When the function f(x,y) has complicated analytical form the Runge - Kutta method is very tedious.

According to Gilberto (2004), the finite difference method is a technique by which derivatives of function are approximated by a difference in the value of the independent variable say X_o and a small increment (X_o+h). To solve differential equations numerically we can replace the derivative in the equation with finite difference approximations on a discretized domain, this result in the number of algebraic equations can be solved.

Stroud (2003) gave a detail description of the finite difference method, according to him, the approach to construct finite difference formulas for partial derivatives, and uses them to construct finite difference formula that represents an approximation to the differential equation.

The method based on finite difference method transform a given ODE into a system of equations. After solving the system, we can get approximate solution of a nodal points interest at one. Computer Algebra System (CAS) usually incorporate routine for solving boundary value problems (Ramos *et al* 2017).

2.3.1 Shooting method

The shooting techniques are quite broad and can be used to solve wide range of differential equations. The equations do not need to be of a certain type, such as even- order self-adjoint, in order for shooting method to work. He also works much on the application of shooting methods for the solution of second order boundary value problems (Akinlabi, 2021)

According to Jain *et al* (2013), the boundary value problem subject to the boundary condition will have a unique solution if the functions p(x), q(x), and r(x) are continuous on [a, b] and q(x) >o.

Akinlabi, *et al* (2021) said that many writers have tried a variety of methods to achieve higher accuracy more quickly. Application of shooting method for the solution of second order boundary value problems are elaborately discussed by them. The shooting technique and nonhomogeneous multipoint boundary value problems of second order ODEs were also reviewed.

Shooting method for a class of two points singular nonlinear boundary value problems was discuss and explore the shooting for calculating eigenvalues of fourth - order two points boundary value problems (Pareen, 2016).

In shooting method, a given boundary value problem is transform into a system consisting of first order initial value problem ODEs. Furthermore, the resulting system is solved with any available ODEs solver, for example, Runge - Kutta or linear multi - step method. A major difficulty with the shooting method is that sometimes a well - known behaved boundary value problem is transformed, requiring later the integration of the initial value problem which is unstable. More precisely, the true solution of a boundary value problem can be stable to some perturbations in the boundary conditions, but the solution of the initial value problem arising in the shooting method area unstable to perturbations of the initial values (Ramos *et al* 2022).

Turner (2008) stated that the philosophy behind shooting method for the solution of a two-point boundary value problem is that, we embedded the initial value problems within an equation solving routine which is used to the appropriate initial condition so that final boundary conditions are also satisfied.

Fox, 2005 presented that, the most usual application of shooting method has been to particular problems for second order equation where to achieve computations, the equation is integrated from each boundary point and marched at a suitable interior point.

3.0 Methods

In this section, we will derive both the shooting method and the finite difference method for solving two point second-order boundary value problems (BVPs) of ordinary differential equations.

3.1 Boundary Value Problems

The simplest boundary value problem is given by the second order differential equation of the form:

$$-y'' + p(x)y' + q(x)y = r(x) \tag{3.1}$$

Where p(x), q(x) and r(x) are continuous function of x or constants, with one of three boundary conditions.

3.1.1 Boundary Condition of the First Kind

$$y(a) = \gamma_1, \qquad y(b) = \gamma_2 \tag{3.2}$$

3.1.2 Boundary Condition of the second Kind

$$y'(a) = \gamma_1, \qquad y'(a) = \gamma_2 \tag{3.3}$$

3.1.3 Boundary Condition of the Three Kind

$$\left.\begin{aligned} a_0 y(a) - a_1 y'(a) = \gamma_1 \\ b_0 y(b) - b_1 y'(b) = \gamma_2 \end{aligned}\right\} \tag{3.4}$$

A homogeneous boundary value problem possesses only a trivial solution

$y(x) = 0.$

We shall therefore consider those boundary value problems in which a parameter λ occurs either in the DE or the boundary conditions, and we determine values of λ, called eigenvalues, for which the BVP has a non-trivial solution.

Furthermore, in general, a boundary value problem does not always have a unique solution. However, we shall assume that the boundary value problem has a unique solution.

3.2 Linear Shooting Method

Consider the numerical solution of the differential equation (3.1)

$-y'' + p(x)y' + q(x)y = r(x)$

$$a < x > b$$

Subject to the boundary conditions (3.4)

$a_0 y(a) - a_1 y'(a) = \gamma_1,$

$b_0 y(b) - b_1 y'(b) = \gamma_2,$

Where $a_{0,}, b_0, a_1, b_1, \gamma_1$ and γ_2 are constants such that

$a_0 a_1 \geq 0 |a_0| + |a_1| \neq 0$

$b_0 b_1 \geq 0 |b_0| + |b_1| \neq 0$

The boundary value problem (3.1) subject to boundary conditions (3.4) will have a unique solution if the function p(x), q(x) are continuous on [a,b] and $q(x) > 0$.

When the IVP is non-homogenous, then it is sufficient to solve the two initial value problems:

$$-\varphi_1'' + p(x)\varphi' + q(x)\varphi = r(x) \qquad (3.4(i))$$

$$-\emptyset_2'' + p(x)\emptyset_2' + q(x)\emptyset_2 = r(x) \qquad (3.4(ii))$$

With suitable initial conditions at $x = a$.

We write the general solution of the value problem in the form

$$y(x) = \lambda\emptyset_1(x) + (1-\lambda)\emptyset_2(x) \qquad (3.5)$$

And determine λ so that the boundary condition at the other end, that is at x=b is satisfied.

We solve the initial value problems [3.4(i)],[3.4(ii)] up to x=b using initial conditions.

3.2.1 Boundary Condition of the first kind

$\emptyset_1(a) = \gamma_1,\ \emptyset'_1(a) = 0,$

$\emptyset_2(a) = \gamma_1,\ \emptyset'_2(a) = 1,$

From (3.5), we obtain

$y(b) = \gamma_2 = \lambda\emptyset_1(b) + (1-\lambda)\emptyset_2(b)$

Which gives

$$\lambda = \frac{\gamma_2 - \emptyset_2\,(b)}{\emptyset_1\,(b) - \emptyset_2\,(b)}, \emptyset_1 \neq \emptyset_2\,(b) \qquad (3.6)$$

3.2.2 Boundary condition of the second kind

$\emptyset_1(a) = 0,\ \emptyset'_1(a) = \gamma_1,$

$\emptyset_2(a) = 1,\ \emptyset'_2(a) = \gamma_1,$

From (3.5), we obtain

$y'(b) = \gamma_2 = \lambda\emptyset'_1(b) + (1-\lambda)\emptyset'_2(b)$

Which gives

$$\lambda = \frac{\gamma_2 - \emptyset_2'(b)}{\emptyset_1'(b) - \emptyset_2'(b)}, \emptyset_1' \neq \emptyset_2'(b) \qquad (3.7)$$

3.2.3 Boundary conditions of third kind

$\emptyset_1(a) = 0,\ \emptyset_1'(a) = \frac{-\gamma_1}{a_1},$

$\emptyset_2(a) = 1,\ \emptyset_2'(a) = \frac{(a_0-\gamma_1)}{a_1}$

From (3.5), we obtain

$y(b) = \lambda\emptyset_1(b) + (1-\lambda)\emptyset_1'(b),$

$y'(b) = \lambda\emptyset_2'(b) + (1-\lambda)\emptyset_2'(b).$

Substituting in the second condition,

$b_0 y(b) + b_1 y(b) = \gamma_2$ in (3.4), we get

$\gamma_2 = b_0[\lambda\emptyset_1(b) + (1-\lambda)\emptyset_2(b)] + b_1[\lambda\emptyset_1'(b) + (1-\lambda)\emptyset_2'(b)]$

Which gives

$\gamma_2 = \frac{\gamma_2 - [b_0\emptyset_2(b) + b_1\emptyset_2'(b)]}{[b_0\emptyset_1(b)\emptyset_1'(b)] - [b_0\emptyset_2(b)] + b_1\emptyset_2'(b)}$

3.1.4. Runge-Kutta method

The Runge-Kutta method of order four requires four evaluation per step, so it should give more accurate answers than Euler's method with one-fourth the step size if it is to be superior.

The formula for fourth-order Runge-Kuta method is given by:

$$y_{n+1} = y_n + \frac{1}{6}(k_{I1} + 2k_{I2} + 2k_{I3} + k_{I4}), \qquad n = 0,1,2,\dots,N-1, \qquad (3.14)$$

where

$k_{I1} = hf(x_n, y_{1n}, y_{2n}, \dots, y_{m.n}),$

$k_{I2} = hf_1(x_n + \frac{h}{2}, y_{1n} + \frac{k_{I1}}{2}, y_{2n} + k_{21}, \dots, y_{m,n} + \frac{k_{m1}}{2})$,

$k_{I3} = hf_1\left(x_n + \frac{h}{2}, y_{1n} + \frac{k_{I2}}{2}, y_{2n} + k_{22}, \dots, y_{m,n} + \frac{k_{m2}}{2}\right),$

$k_{I4} = hf_1\left(x_n + \frac{h}{2}, y_{1n} + \frac{k_{I3}}{2}, y_{2n} + k_{23}, \dots, y_{m,n} + k_{m3}\right),$

$i = 1,2,\dots,m$

But for the sake of these research work, our focus will be on Runge-Kuta method for a pair of equations shown below.

$V_{n+1} = V_n + \frac{1}{6}(k1 + 2(k2 + k3) + k4)$

$W^{(1)}{}_{n+1} = W_n + \frac{1}{6}(I1 + 2(I2 + I3) + I4)$

where

$k^{(i)}1 = hf(u, v, w)$

$I^{(i)}1 = hg(u_n, v_n, w_n)$

$k^{(i)}2 = hf(u_n + \frac{h}{2}, v_n + \frac{k_1}{2}, w_n + \frac{I_1}{2})$

$I^{(i)}2 = hg(u_n + \frac{h}{2}, v_n + \frac{k_1}{2}, w_n + \frac{I_1}{2})$

$k^{(i)}3 = hf(u_n + \frac{h}{2}, v_n + \frac{k_2}{2}, w_n + \frac{I_2}{2})$

$I^{(i)}3 = hg(u_n + \frac{h}{2}, v_n + \frac{k_2}{2}, w_n + \frac{I_2}{2})$

$k^{(i)}4 = hf(u_n + h, v_n + k_3, w_n + I_3)$

$I^{(i)}4 = hg(u_n + h, v_n + k_3, w_n + I_3)$

3.3 Finite Difference Method for Linear Problems

Methods involving finite difference method for solving boundary value problems replace each of the derivatives in the differential equation with an appropriate difference-quotient approximation.

The particular difference quotient and step size h are chosen to maintain a specified order of truncation error. However, h cannot be chosen too small because of instability of the derivative approximations.

The finite difference method for the linear second-order boundary-value problem;

$y'' = p(x)y' + q(x)y + r(x) \;\; a \leq x \leq b$

$y(a) = \alpha, \qquad y(b) = \beta$

Requires that difference-quotient approximations be used to approximate both

y' and y''.

First, we select an integer $n > 0$ and divide the interval $[a, b]$ into (N+1) equal sub-intervals whose endpoints are the mesh point $x_{\,i} = a + ih$, for $i = 0, 1 \ldots, N + 1$,Where

$h = \frac{b-a}{N+1}$.

At the interior mesh points, x_i, for $i = 1,2 \ldots N$, the differential equation to be approximated is

$$y''(x_i) = p(x_i) + q(x_i) + r(x_i) \qquad (3.15)$$

Expanding y in a third Taylor polynomial about x_i evaluate at x_{i-1}, we have, assuming that $y \in C^4[X_{I-1}, X_{I+1}]$,

$$y(x_{i+1}) = y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(\zeta) \qquad (3.16)$$

For some ζ_i^- in (x_{i+1}, x_i). If these equations are added, we have

$$y(x_{i+1}) + y(x_{i-1}) = 2y_i(x_i) + h^2 y''(x_i) - \frac{h^4}{4!}\left[y^{(4)}(\xi_i^+ + \xi_i^-)\right] \qquad (3.17)$$

And solving for $y''(x_i)$ gives

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y \;\; (x_i) + y(x_{i-1})] - \frac{h^4}{4!}\left[y^{(4)}(\xi_i^+ + \xi_i^-)\right] \qquad (3.18)$$

The intermediate theorem can be used to simplify this to

$$y''(x_i) = \frac{1}{h^2}[y(x_{i+1}) - 2y \;\; (x_i) + y(x_{i-1})] - \frac{h^4}{4!}\left[y^{(4)}\left(y^{(4)}(\xi_i \;\;)\right)\right] \qquad (3.19)$$

For some ξ_i in (x_{i-1}, x_{i+1}).

This is called central difference formula for $y''(x_i)$

A central difference for $y'(x_i)$ is obtain in a similar manner resulting in

$$y'(x_i) = \frac{I}{2h}[y(x_{i+1}) - y(x_{i-1})] - \frac{h^3}{3!}y'''(\eta_i) \quad (3.20)$$

For some η_i in (x_{i-1}, x_{i+1}).

The use of the finite difference formula in (3.1) and writing $y(x_{i+1})$ as y_{i+1}, $y(x_i)$ as $y_i, y_{i-1}\ as$ $y_i, p(x_i)$ as $p_i, q(x_i)$ as q_i and $r(x_i)$ as r_i result in the equation

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i\left(\frac{y_{i+1} - y_{i-1}}{2h}\right) + q_i y_i + r_i \quad (3.21)$$

Further algebraic simplification leads to a tri-diagonal system for the unknowns $y_i \ldots, y_{n-1}$ i.e

$i = 1 \ldots, n-1,$

$$(2 + hp_i)y_{i-1} - (4 + 2h^2 q_i)y_i + (2 - hp_i)y_{i+1} = 2h^2 r_i \quad (3.22)$$

Where $y_0 = y(a) = \alpha$ and $y_n = y(b) = \beta$

Using the boundary value and simplifying gives:

First (for i=1)

$$-(4 + 2h^2 q_i)y_i + (2 - hp_i)y_2 = 2h^2 r_i - a(2 + hp_i) \quad (3.23)$$

Then for i=2…, n-2,

$$(2 + hp_i)y_{i-1} - (4 + 2h^2 q_i)y_i + (2 - hp_i)y_{i+1} = 2h^2 r_i \quad (3.24)$$

And finally, $(for\ i = n-1)$

$$(2 - hp_{n-1})y_{n-2} - (4 + 2h^2 q_{n-1})y_{n-1} = 2h^2 r_i - \beta(2 - hp_{n-1}) \quad (3.25)$$

The system in matrix form $Ay = b$, with

$$A = \begin{bmatrix} -(4+2h^2q_1) & (2-hp_1) & 0 & 0 & 0 \\ (2+hp_2) & -(4+2h^2q_2) & (2-hp_2) & 0 & 0 \\ 0 & (4+2h^2q_3) & -(4+2h^2q_3) & (2-hp_3) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & (2+hp_{n-2}) & -(4+2h^2q_{n-1}) & (2+hp_{n-2}) \\ 0 & 0 & 0 & (2+hp_{n-1}) & -(4+2h^2q_{n-1}) \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 + h^2 r_1 - (2 + hp_1)\alpha \\ 2h^2 r_2 \\ 2hr_3 \\ \vdots \\ 2hr_{n-1} \\ 2h^2 r_{n-1} - (2 - hp_{n-1})\beta \end{bmatrix}.$$

4.0 Results and Discussion

In this section, we will provide numerical results of the findings and discussion based on the outcome of our results.

4.1 Results

We will use the methods derived in chapter three to solve the boundary value problems numerically.

4.1.1 Example I

Solve the boundary value problem

$y'' = y + 1$ with boundary condition

$y(0) = 0, \quad y(1) = 0$

The exact solution is

$y(u) = e^{u} - 1$

$u_n = 0.25, 0.50, 0.75, 1.0$

$h = 0.25$

solution using shooting method

$y'' = y + 1 \qquad\qquad U \in [0,1]$

$y(0) = 0, \quad y(1) = 0$

We assume the general solution of the boundary value problem as

$y(u) = \lambda\emptyset_0(u) + (1-\lambda)\emptyset_1(u)$

Then we solve the two IVPs

$\emptyset_0'' = \emptyset_0 + 1 \qquad \emptyset_0(0) = 0, \emptyset_0' = 0$

$\emptyset_1'' = \emptyset_1 + 1 \qquad \emptyset_1(0) = 0, \emptyset_1'(0) = 1$

We write these IVPs as the following equivalent 1st order systems.

The systems are gotten as follows

Let $\emptyset_0 = w', w' = v$

Differentiate we have

$$\left.\begin{aligned} \emptyset_0' = v' = w^{(1)} \\ w'' = w' = v^{(1)} + 1 \end{aligned}\right\} \qquad (1)$$

Let $\emptyset_1 = v, \ v' = w$

Differentiating we have

$$\left.\begin{aligned} \emptyset_1' = v' = w^{(2)} \\ v'' = w' = v^{(2)} + 1 \end{aligned}\right\} \qquad (2)$$

We now apply Runge-Kutta method for a pair of equations.

The first pair (1) is:

$\emptyset_0' = f(u_n, v_n, w_n) = w_n^{(1)}$

$\emptyset_0'' = g(u_n, v_n, w_n) = v_n^{(1)} + 1$

The second pair (2) is:

$\emptyset_1' = f(u_n, v_n, w_n) = w_n^{(2)}$

$\emptyset_1'' = g(u_n, v_n, w_n) = v_n^{(2)} + 1$

$k1 = hf(u_n, v_n, w_n) = hw_n^{(1)}$

$I1 = hg(u_n, v_n, w_n) = h(v_n^{(1)} + I1)$

$k2 = hf\left(u_n + \frac{h}{2}, v_n + \frac{k1}{2}, w_n + \frac{I1}{2}\right) = h(w_n^{(1)} + \frac{I1}{2})$

$I2 = hg\left(u_n + \frac{h}{2}, v_n + \frac{k1}{2}, w_n + \frac{I1}{2}\right) = h(v_n^{(1)} + \frac{k1}{2})$

$k3 = hf\left(u_n + \frac{h}{2}, v_n + \frac{k1}{2}, w_n + \frac{I2}{2}\right) = h(w_n^{(1)} + \frac{I2}{2})$

$I3 = hg\left(u_n + \frac{h}{2}, v_n + \frac{k1}{2}, w_n + \frac{I1}{2}\right) = h(v_n^{(1)} + \frac{k2}{2})$

$k4 = hf(u_n + h, v_n + k3, w_n + I3) = h(w_n^{(1)} + I3)$

$k4 = hg(u_n + h, v_n + k3, w_n + I3) = h(w_n^{(1)} + k3)$

At $u_0 = 0, i = 1, v_o^{(1)}, w_o^{(1)}, n = 0,1,2,3$

$k1 = hw_0 = 0.25 \times 0 = 0$

$I1 = h(v_0 + 1) = 0.25 \times (0 + 1) = 0.25$

$k2 = h\left(w_0 + \frac{I1}{2}\right) = 0.25 \times \left(0 + \frac{0.25}{2}\right) = 0.0313$

$I2 = h\left(v_0 + \frac{k1}{2}\right) = 0.25 \times \left(0 + \frac{0}{2}\right) = 0.03125$

$k3 = h\left(v_0 + \frac{I2}{2}\right) = 0.25 \times \left(0 + \frac{0.03125}{2}\right) = 0.2539$

$k4 = h(w_0 + I3) = 0.25 \times (0 + 0.2539) = 0.06347$

$I4 = h(w_0 + k3) = 0.25 \times (0 + 0.2539) = 0.06347$

$v_o^{(1)} = v_0 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0 + \frac{1}{6}(0 + 2(0.313 + 0.03125) + 0.06347) = 0.0314$

$w_o^{(1)} = w_0 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 0 + \frac{1}{6}(0 + 2(0.25 + 0.2539) + 0.2578) = 0.2526$

For $n = 1, i = 1$

$k1 = hw_1 = 0.25 \times 0.2526 = 0.0632$

$I1 = h(v_1 + 1) = 0.24 \times (0.0314 + 1) = 0.02579$

$k2 = h\left(w_1 + \frac{I1}{2}\right) = 0.25 \times \left(0.02526 + \frac{0.02579}{2}\right) = 0.0957$

$I2 = h\left(v_1 + \frac{k1}{2}\right) = 0.25 \times [(0.0315 + 1) + \frac{0.0632}{2}] = 0.02658$

$k3 = h\left(w_1 + \frac{I2}{2}\right) = 0.25 \times \left(0.02526 + \frac{0.2658}{2}\right) = 0.0946$

$I3 = h\left(v_1 + \frac{k2}{2}\right) = 0.25 \times [(0.2526 + 1) + \frac{0.0954}{2}] = 0.2698$

$k4 = h(w_1 + I3) = 0.25 \times (0.2526 + 0.02698) = 0.1386$

$I4 = h(v_1 + k3) = 0.25 \times [(0.0314 + 1(0.0964) - (0.0435) - (0.2 + 0.2) = 0.281$

$v_2^{(1)} = v_0 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0.0314 + \frac{1}{6}(0 + 2(0.0632 + 0.0954) + 0.1306 = 0.0962$

$w_2^{(1)} = v_1 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 0.2526 + \frac{1}{6}(0.2579 + 2(0.2658 + 0.0964) + 0.2820) = 0.4633$

For $n = 2, i = 1$

$k1 = hw_2 = 0.25 \times 0.4633 = 0.1158$

$I1 = h(v_2 + 1) = 0.25 \times (0.0962 + 1) = 0.2741$

$k2 = h\left(w_2 + \frac{I1}{2}\right) = 0.25 \times \left(0.4633 + \frac{0.2741}{2}\right) = 0.1501$

$I2 = h\left(v_2 + \frac{k1}{2}\right) = 0.25 \times \left((0.0920 + 1) + \frac{0.1158}{2}\right) = 0.2885$

$k3 = h\left(w_2 + \frac{I2}{2}\right) = 0.25 \times \left(0.4633 + \frac{0.2885}{2}\right) = 0.1519$

$I3 = h\left(v_2 + \frac{k2}{2}\right) = 0.25 \times \left((0.0962 + 1) + \frac{0.1501}{2}\right) = 0.2928$

$k4 = h(w_2 + I3) = 0.25 \times (0.4633 + 0.2928) = 0.1890$

$I4 = h(v_2 + k3) = 0.25 \times [(0.0962 + 1) + 0.1593] = 0.3120$

$v_3^{(1)} = v_2 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0.0962 + \frac{1}{6}(0.1158 + 2(0.1501 + 0.1519) + 0.1890) = 0.2400$

$w_3^{(1)} = w_2 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 0.4633 + \frac{1}{6}(0.2741 + 2(0.2885 + 0.2928) + 0.3120) = 0.7548$

For $n = 3$

$k1 = hw_3 = 0.25 \times 0.7548 = 0.1887$

$I1 = h(v_3 + 1) = 0.25 \times (0.2477 + 1) = 0.3119$

$k2 = h\left(w_3 + \frac{I1}{2}\right) = 0.25 \times \left(0.7548 + \frac{0.3119}{2}\right) = 0.2275$

$I2 = h\left(v_3 + \frac{k1}{2}\right) = 0.25 \times [(0.2477 + 1) + \frac{0.1887}{2}] = 0.3355$

$k3 = h\left(w_3 + \frac{I2}{2}\right) = 0.25 \times \left(0.7548 + \frac{0.3355}{2}\right) = 0.2306$

$I3 = h\left(v_3 + \frac{k2}{2}\right) = 0.25 \times [(0.2477 + 1) + \frac{0.2275}{2}] = 0.3404$

$k4 = h(w_3 + I3) = 0.25 \times (0.7548 + 0.3404) = 0.3596$

$I4 = h(v_2 + k3) = 0.25 \times [(0.2477 + 1) + 0.2737] = 1.0777$

$v_4^{(1)} = v_3 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0.2477 + \frac{1}{6}(0.1887 + 2(0.2275 + 0.2306) + 0.2737) = 0.477$

$w_4^{(1)} = w_3 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 0.7548 + \frac{1}{6}(0.3119 + 2(0.3355 + 0.3404) + 0.2737) = 1.0777$

For $i = 2, n = 0$

$k1 = hw_0 = 0.25 \times 1 = 0.25$

$I1 = h(v_0 + 1) = 0.25 \times (0 + 1) = 0.25$

$k2 = h\left(w_0 + \frac{I1}{2}\right) = 0.25 \times \left(1 + \frac{0.25}{2}\right) = 0.3813$

$I2 = h\left(v_0 + \frac{k1}{2}\right) = 0.25 \times [(0 + 1) + \frac{0.25}{2}] = 0.2813$

$k3 = h\left(w_0 + \frac{I2}{2}\right) = 0.25 \times \left(1 + \frac{0.2813}{2}\right) = 0.2852$

$I3 = h\left(v_0 + \frac{k2}{2}\right) = 0.25 \times [(0 + 1) + \frac{0.2813}{2}] = 0.2852$

$k4 = h(w_0 + I3) = 0.25 \times (1 + 0.2852) = 0.3213$

$I4 = h(v_0 + k3) = 0.25 \times [(0 + 1) + 0.2852] = 0.3213$

$v_1^{(1)} = v_0 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0 + \frac{1}{6}(0.25 + 2(0.2813 + 0.2852) + 0.3213) = 0.2840$

$w_1^{(1)} = w_0 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 1 + \frac{1}{6}(0.25 + 2(0.2813 + 0.2852) + 0.3213) = 1.2840$

For $n = 1$

$k1 = hw_1 = 0.25 \times 1.2840 = 0.3210$

$I1 = h(v_1 + 1) = 0.25 \times (0.2840 + 1) = 0.3210$

$k2 = h\left(w_1 + \frac{I1}{2}\right) = 0.25 \times \left(1.2840 + \frac{0.3210}{2}\right) = 0.3611$

$I2 = h\left(v_1 + \frac{k1}{2}\right) = 0.25 \times \left((0.2840 + 1) + \frac{0.3210}{2}\right) = 0.3611$

$k3 = h\left(w_1 + \frac{I2}{2}\right) = 0.25 \times \left(1.2840 + \frac{0.3611}{2}\right) = 0.3661$

$I3 = h\left(v_1 + \frac{k2}{2}\right) = 0.25 \times \left((0.2840 + 1) + \frac{0.3611}{2}\right) = 0.3661$

$k4 = h(w_1 + I3) = 0.25 \times (1.2840 + 0.3661) = 0.4125$

$I4 = h(v_1 + k3) = 0.25 \times [(0.2840 + 1) + 0.3661] = 0.4125$

$v_2^{(2)} = v_1 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0.2840 + \frac{1}{6}(0.3210 + 2(0.3611 + 0.5510) + 0.4125) = 0.7200$

$w_2^{(2)} = w_1 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 1.2810 + \frac{1}{6}(0.3210 + 2(0.3611 + 0.6610) + 0.4125) = 1.7470$

For $n = 2$

$k1 = hw_2 = 0.25 \times 1.7470 = 0.4368$

$I1 = h(v_2 + 1) = 0.25 \times (0.7470 + 1) = 0.4368$

$k2 = h\left(w_2 + \frac{I1}{2}\right) = 0.25 \times \left(1.7470 + \frac{0.4368}{2}\right) = 0.4913$

$I2 = h\left(v_2 + \frac{k1}{2}\right) = 0.25 \times \left((0.7470 + 1) + \frac{0.4368}{2}\right) = 0.4913$

$k3 = h\left(w_2 + \frac{I2}{2}\right) = 0.25 \times \left(1.7470 + \frac{0.4913}{2}\right) = 0.4913$

$I3 = h\left(v_2 + \frac{k2}{2}\right) = 0.25 \times \left((0.7470 + 1) + \frac{0.4918}{2}\right) = 0.4913$

$k4 = h(w_2 + I3) = 0.25 \times (0.4932 + 1.7470) = 0.5613$

$I4 = h(v_2 + k3) = 0.25 \times [(0.7470 + 1) + 0.4982] = 0.5613$

$v_3^{(2)} = v_2 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 0.7470 + \frac{1}{6}(0.4368 + 2(0.4913 + 0.4982) + 0.5613) = 1.2400$

$w_3^{(2)} = w_2 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 1.7470 + \frac{1}{6}(0.4368 + 2(0.4913 + 0.4982) + 0.5613) = 2.2432$

For $n = 3$

$k1 = hw_3 = 0.25 \times 2.2432 = 0.0608$

$I1 = h(v_3 + 1) = 0.25 \times (1.2432 + 1) = 0.0608$

$k2 = h\left(w_3 + \frac{I1}{2}\right) = 0.25 \times \left(2.2432 + \frac{0.0608}{2}\right) = 0.5684$

$I2 = h\left(v_3 + \frac{k1}{2}\right) = 0.25 \times [(1.2432 + 1) + \frac{0.5684}{2}] = 05684$

$k3 = h\left(w_3 + \frac{I2}{2}\right) = 0.25 \times \left(2.2432 + \frac{0.5684}{2}\right) = 0.6319$

$I3 = h\left(v_3 + \frac{k2}{2}\right) = 0.25 \times [(1.2432 + 1) + \frac{0.6319}{2}] = 0.6319$

$k4 = h(w_3 + I3) = 0.25 \times (2.2432 + 0.6319) = 0.7188$

$I4 = h(v_3 + k3) = 0.25 \times [(1.2432 + 1) + 0.6319] = 0.7188$

$v_4^{(2)} = v_3 + \frac{1}{6}(k1 + 2(k2 + k3) + k4) = 1.2432 + \frac{1}{6}(0.0608 + 2(0.5684 + 0.6319) + 0.6398) = 1.7732$

$w_4^{(2)} = w_3 + \frac{1}{6}(I1 + 2(I2 + I3) + I4) = 2.2432 + \frac{1}{6}(0.0608 + 2(0.5684 + 0.6319) + 0.6398) = 2.7732$

Using the conditions $i = 1, w_0^{(1)} = 0, v_0^{(0)} = 0$, we get

$$\begin{bmatrix} v_1^{(1)} \\ w_1^{(1)} \end{bmatrix} = \begin{bmatrix} 0.0314 \\ 0.2526 \end{bmatrix}, \begin{bmatrix} v_2^{(1)} \\ w_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0.0962 \\ 0.4633 \end{bmatrix}, \begin{bmatrix} v_3^{(1)} \\ w_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0.2477 \\ 0.7548 \end{bmatrix}, \begin{bmatrix} v_4^{(1)} \\ w_4^{(1)} \end{bmatrix} = \begin{bmatrix} 0.4775 \\ 1.0777 \end{bmatrix}.$$

Using the condition $i = 2, v_0^{(2)} = 0, w_0^{(2)} = 1$, we get

$$\begin{bmatrix} v_1^{(2)} \\ w_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0.2840 \\ 1.2840 \end{bmatrix}, \begin{bmatrix} v_2^{(2)} \\ w_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0.7470 \\ 1.7470 \end{bmatrix}, \begin{bmatrix} v_3^{(2)} \\ w_3^{(2)} \end{bmatrix} = \begin{bmatrix} 1.2432 \\ 2.2432 \end{bmatrix}, \begin{bmatrix} v_4^{(2)} \\ w_4^{(2)} \end{bmatrix} = \begin{bmatrix} 1.7730 \\ 2.7730 \end{bmatrix}.$$

$y(x) = \lambda\emptyset_0(x) + (1 - \lambda)\emptyset_1(x)$

$\lambda = \frac{\gamma_2 = \emptyset_2(x)}{\emptyset_1(x) - \emptyset_2(x)}$ $\gamma_2 = 0$

$= \frac{0 - v_4^{(2)}}{v_4^{(1)} - v_4^{(2)}} = \frac{-1.2773}{0.4775 - 1.2773} = 1.3686$

$y(x) = \lambda\emptyset_0(x) + (1 - \lambda)\emptyset_1(x)$

$y(x) = 1.3686\emptyset_0(x) - 0.3686\emptyset_1(x)$

Calculating for the remaining values of x at

$X = 0.25, 0.50, 0.75$ and 1.00

$y(0.25) = 1.3686(0.0314) - 0.3686(0.2840) = -0.0617$

$y(0.50) = 1.3686(0.0962) - 0.3686(0.7470) = -0.1437$

$y(0.75) = 1.3686(0.2477) - 0.3686(1.2432) = -0.1192$

$y(1.00) = 1.3686(0.4775) - 0.3686(1.2773) = -0.1827$

Solution Using Finite Difference Method

Applying the finite difference scheme, we discretize the given equation into algebraic form

$$\frac{y_{n+1} - 2y_1 + y_{n+1}}{h^2} = y_1 + 1$$

Multiply each term by h^2 and collect like terms to have

$y_{n+1} - (2 + h^2)y_1 + y_{n+1} = h^2$

With mesh points

0.00	0.25	0.50	0.75	1.00

The solution at the boundaries is given, so we shall be interested in finding solutions at the interior points, i.e., $i = 1,2\ and\ 3$.

For $i = 1$

$$\Rightarrow -(2 + h^2)y_1 + y_2 = h^2 - \alpha$$

For $i = 2$

$$y_1 - (2 + h^2)y_1 - y_2 = h^2$$

For $i = 3$

$$y_1 - (2 + h^2)y_3 = h^2 - \beta$$

This has given a system of three algebraic equations and it can be represented in matrix form as

$$\begin{pmatrix} -2.0625 & 1 & 0 \\ 1 & -2.0625 & 1 \\ 0 & 1 & -2.0625 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.0625 \\ 0.0625 \\ 0.0625 \end{bmatrix}$$

This can be solved using Gauss elimination method.

The augmented Matrix is given below

$$\left(\begin{array}{ccc|c} -2.0625 & 1 & 0 & 0.0625 \\ 1 & -2.0625 & 1 & 0.0625 \\ 0 & 1 & -2.0625 & 0.0625 \end{array}\right)$$

We will then reduce the matrix into row echelon form

$$\left(\begin{array}{ccc|c} -2.0625 & 1 & 0 & 0.0625 \\ 1 & -1.5778 & 1 & 0.0928 \\ 0 & 1 & -2.0625 & 0.0625 \end{array}\right) \begin{array}{c} pivot\ row \\ R_2 + \frac{R_1}{2.0625} \\ R_3 = 0R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} -2.0625 & 1 & 0 & 0.0625 \\ 0 & -1.5778 & 1 & 0.0928 \\ 0 & 1 & -1.4287 & 0.0121 \end{array}\right) \begin{array}{c} pivot\ row \\ R_3 + \frac{R_2}{1.5778} \\ \ \end{array}$$

Using backward substitution, we have the following

$$-1.4287y_3 = 0.0121$$

$$y_3 = -0.0085$$

$$-1.577y_2 = 0.0928 + 0.0085$$

$$y_2 = -0.0642$$

$$-2.0625y_1 = 0.0625 + 0.0642$$

$$y_1 = 0.0614$$

Tabulating the above solutions for both shooting method (SM) and Finite Difference Method (FDM) results in the following table.

Table 1: Results of Shooting Method (SM) and Finite Difference Method (FDM) for Example 1

X_n	Exact Solution $y(x)$	Numerical Solution		Absolute Error	
		Shooting Method (SM)	Finite Diff. Method (FDM)	SM	FDM
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	0.2840	-0.0617	-0.0352	0.3457	0.3198
0.50	0.6487	-0.1437	-0.0112	0.7924	0.6599
0.75	1.1170	-0.1192	-0.0085	1.2362	1.1255
1.00	1.7183	0.1827	0	1.5356	1.7183

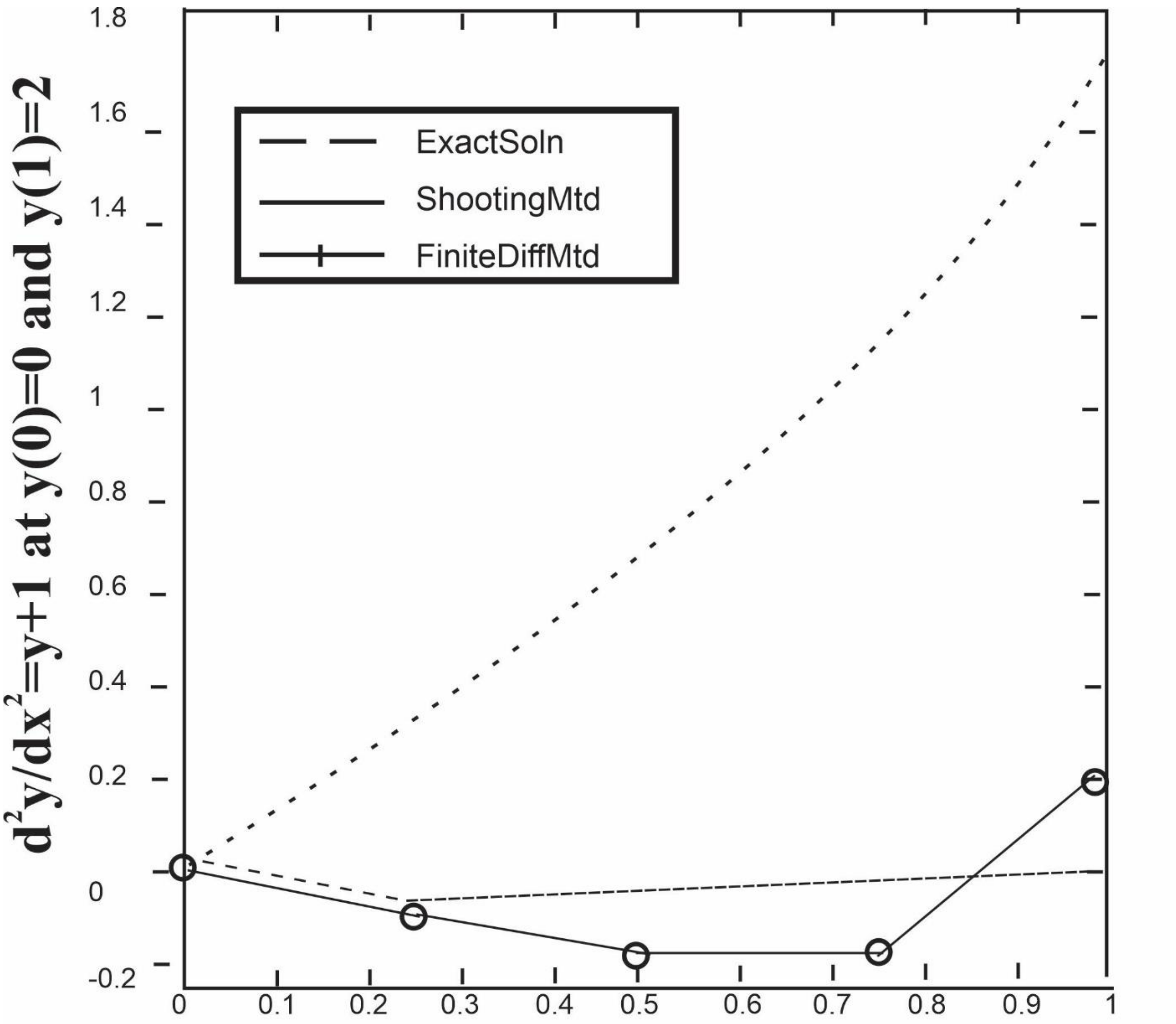

FIGURE 1: Graph of shooting Method (SM) and Finite Differnce Method (FDM) for Example 1

Solution Using Finite Difference Method

Applying the finite difference scheme, we discretize the given equation into algebraic form

$\frac{y_{i+1}-2y_1+y_{i+1}}{h^2} = 4(y_1 + u)$

Multiply each term by h^2 and collect like terms to have

$y_{i-1} - (2 + 4h^2)y_1 + y_{i+1} = -4hu^2$

With mesh points

0.00	0.20	0.40	0.60	0.80	1.00

The solution at the boundaries are given, so we shall be interested in finding solutions at the interior points, that is, $i = 1,2,3$ and 4.

For $i = 1$

$\Rightarrow -(2+4h^2)y_2 + y_3 = -4h^2 - \alpha$

For $i = 2$

$y_1 - (2+4h^2)y_2 - y_3 = -4h^2$

For $i = 3$

$y_2 - (2+4h^2)y_2 - y_3 = -4h^2$

For $i = 4$

$y_1 - (2+h^2)y_3 = -4h^2 - \beta$

This has given a system of three algebraic equations and it can be represented in matrix form as

$$\begin{pmatrix} -2.16 & 1 & 0 & 0 \\ 1 & -2.16 & 1 & 0 \\ 0 & 1 & -2.16 & 1 \\ 0 & 0 & 1 & -2.16 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -0.032 \\ -0.064 \\ -0.096 \\ -2.096 \end{bmatrix}$$

This can be solved using Gauss elimination method.

The augmented matrix is given below

$$\left(\begin{array}{cccc|c} -2.16 & 1 & 0 & 0 & -0.032 \\ 1 & -2.16 & 1 & 0 & -0.064 \\ 0 & 1 & -2.16 & 1 & -0.096 \\ 0 & 0 & 1 & -2.16 & -2.096 \end{array}\right)$$

We will the reduce the matrix into row echelon form

$$\left(\begin{array}{cccc|c} -2.16 & 1 & 0 & 0 & -0.032 \\ 1 & -1.697 & 1 & 0 & -0.064 \\ 0 & 1 & -2.16 & 1 & -0.096 \\ 0 & 0 & 1 & -2.16 & -2.096 \end{array}\right) \begin{array}{l} pivot\ row \\ R_2 + \frac{R_1}{2.16} \\ R_3 + 0R1 \\ R_4 + 0R1 \end{array}$$

$$\left(\begin{array}{cccc|c} -2.16 & 1 & 0 & 0 & -0.032 \\ 1 & 1.697 & 1 & 0 & -0.064 \\ 0 & 1 & -2.7493 & 1 & -0.096 \\ 0 & 0 & 1 & -2.16 & -2.096 \end{array}\right) \begin{array}{l} pivot\ row \\ R_3 + \frac{R_2}{-1.697} \\ R_4 + 0R2 \end{array}$$

$$\left(\begin{array}{cccc|c} -2.16 & 1 & 0 & 0 & -0.032 \\ 1 & 1.697 & 1 & 0 & -0.064 \\ 0 & 1 & -2.7493 & 1 & -0.096 \\ 0 & 0 & 1 & -2.5237 & -2.0612 \end{array}\right) \begin{array}{l} pivot\ row \\ R_4 + \frac{R_3}{-2.7493} \end{array}$$

Using backward substitution, we have the following

$-2.5237y_3 = -2.0612$

$y_4 = 0.8167$

$-2.7493y_2 = -0.096 - 0.8167$

$y_3 = 0.3320$

$1.697y_2 = -0.064 - 0.3320$

$y_2 = -0.3960$

$-2.5237y_1 = -0.032 + 0.3960$

$y_2 = -0.1442$

$y_1 = 0.0614$

Tabulating the above solutions for both shooting method (SM) and Finite Difference Method (FDM) results in the following table.

Table 2: Results of Shooting Method (SM) and Finite Difference Method (FDM) for Example 2

X_n	Exact Solution $y(x)$	Numerical Solution		Absolute Error	
		Shooting Method (SM)	Finite Diff. Method (FDM)	SM	FDM
0.0	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.3131	0.1313	0.0610	0.1818	0.2521
0.4	0.6449	0.3236	-0.1442	0.3213	0.7891
0.6	1.0162	0.0462	0.3320	0.9700	0.6842
0.8	1.4550	0.0595	0.8163	1.3955	0.6387
1.0	2.0000	0.0000	2.0000	2.0000	0.0000

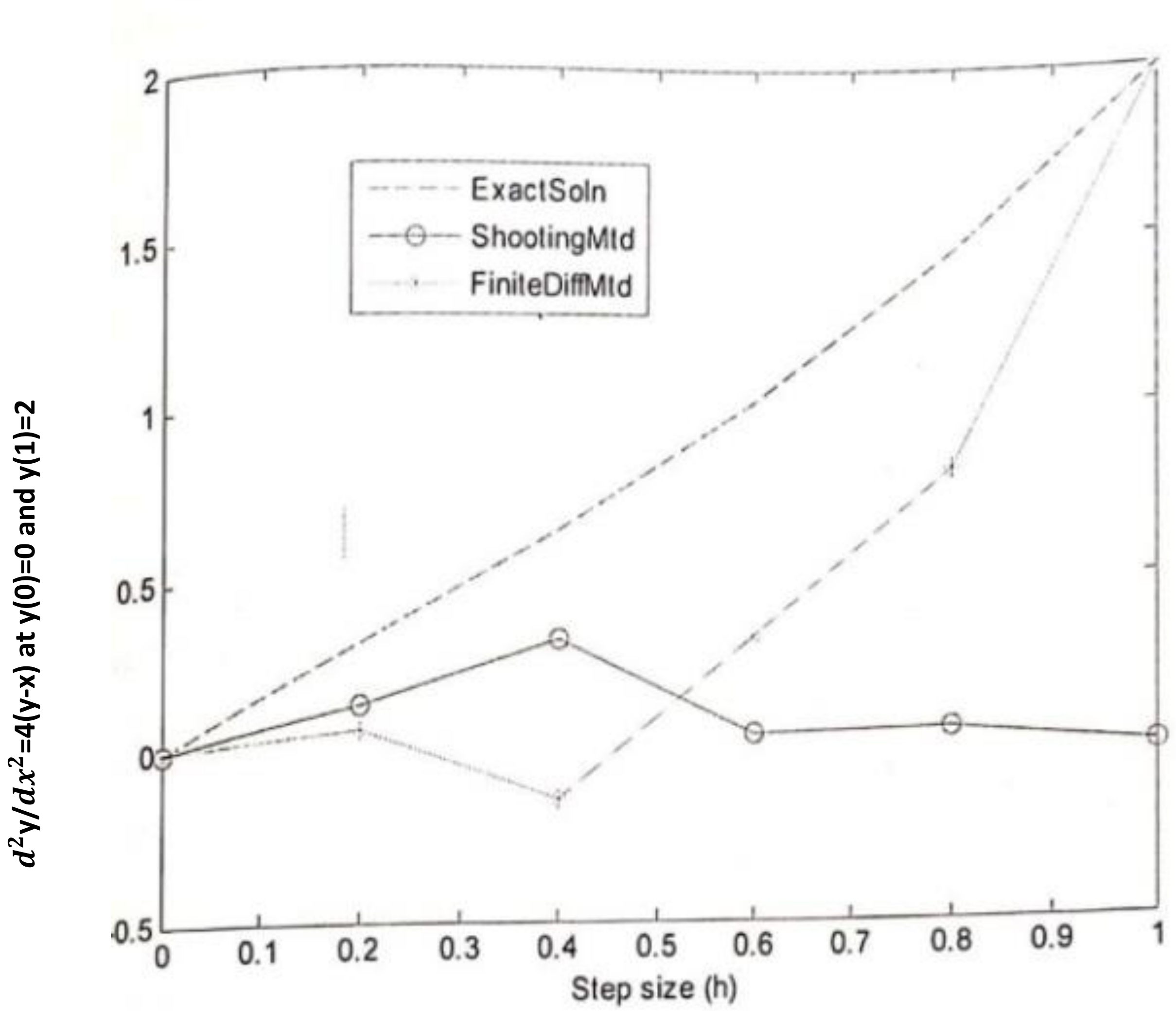

Figure 2: Graph of Shooting Method (SM) and Finite Difference Method (FDM) for Example 2.

5.0 Discussion

From the examples we have solved above, we observed that Example 1 is linear boundary value problem of Dirichlet boundary conditions. We solved Example 1 using shooting method, we observed that in the course our solving, we come across two systems of pair of equations with two variably and we applied the Runge - Kutta method for the pair of equations.

On the other hand, as we applied the finite difference method, we observed that we need not to compute the values at the boundaries since it was explicitly given to be zero at both ends, we only calculated for the interior points and we obtained a system of three algebraic equations which was represented in matrix form and we went further to solve using Gauss elimination method, we got the result shown in Table 1.

5.2 Conclusion

In this research work, we only examined two numerical methods namely; shooting method and finite difference method; however, we did not take a closer look on the finite element method which is another numerical method for solving boundary value problems. The accuracy of the methods highly depends on the step size, the smaller the step size the higher the accuracy. The shooting method requires longer step of computations compared to the finite difference method. From the result obtained, we can say that either of the method can be used in solving boundary value problems even though the shooting method is toilsome when it comes to computation and is based on guesses which brings about uncertainty.

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