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Exponential Maps on Complex Matrices

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DECLARATION

This work was carried out at AIMS Rwanda in partial fulfilment of the requirements for a Master of Science Degree.

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at AIMS Rwanda or any other University.

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Prof. Douglas Farenick

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DEDICATION

I dedicate this essay to God Almighty my creator, my strong segment, my wellspring of inspiration, cleverness, learning, and appreciation. He has been the wellspring of my quality all through this program and on his wings simply have I took off. I like manner commit this work to **SUKAKA AGUSA JOHN** and **Dr. Bernard Bainson** who have bolstered me the separation and whose comfort has guaranteed that I give all that it takes to finish what I have started.

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Abstract

This essay examines the matrix exponential function $\exp$, which is a function defined on the additive abelian group $M_n(\mathbb{C},+)$ of $n\times n$ matrices over the field $\mathbb{C}$ of complex numbers with values in the non-abelian multiplicative group $GL_n(\mathbb{C},\cdot)$ of invertible $n \times n$ complex matrices. Because $GL_n(\mathbb{C},\cdot)$ is a non-abelian group, the matrix exponential function $\exp : M_n(\mathbb{C},+) \to GL_n(\mathbb{C},\cdot)$ is not a group homomorphism, except in the case $n = 1$ where the familiar formula $e^{w+z} = e^w e^z$ holds for all complex numbers w and z. However, there are cases in which the matrix exponential does act like a group homomorphism. For example, if $A, B \in M_n(\mathbb{C},+)$ commute multiplicatively (that is, $AB = BA$), then $\exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A)$. This essay proves this fact, and presents several examples in which $\exp(A + B) = \exp(A)\exp(B)$ fails to hold for noncommuting matrices A and B, as well as examples of cases where $\exp(A+B) = \exp(A)\exp(B)$ holds even though A and B do not commute. The matrix exponential function is a substantial interest in different branches of mathematics. This essay presents some applications in which the matrix exponential has a natural role.

Contents

1. Introduction

The concept of matrices is one of the early development of mathematics. The term matrix is known to be the Latin word for womb was first introduced in mathematics by J. J. Sylvester in 1850 to describe an array of numbers (Higham, 2016). Following the introduction of matrices, the concept has recorded remarkable contributions which have seen it to its current form. Authors including Lagrange, Cayley, Schwerdtfeger and many more can be listed as contributors. Matrices have found its way into many fields of study other than mathematics (Smalls, 2007). We can list applications such as in physics, in chemistry, in economics, in data encryption, in geology, in biology and many more (Higham, 2008). In mathematics, matrices appear frequently in algebra in many forms. One can describe data encryption using matrix. Matrix is the key point of linear algebra. In matrix theory there exist different matrix functions such as matrix exponential, matrix inverse, and more (Banerjee and Roy, 2014).

In 1978 Molar Van Loan Sirev mentioned different nineteen Dubious ways of computing matrix exponential (Higham, 2016). In this essay, we study matrix exponential. We define the exponential map on matrices and how we can represent exponential of a complex number using matrix exponential function. We are interested to show that if the matrix exponential is a homomorphism, or not from $M_n(\mathbb{C},+)$ to $GL_n(\mathbb{C},.)$. If not, what are the main condition that can make it be homomorphism? Also, we shall see if $n=1$ the exponential will be homomorphism. In this essay, we seek to give an account of the concept of exponential maps of complex matrices.

In this essay, we used literature research for more understanding. This essay also considers some of the mathematical literature that deals with cases where A and B do not commute because, even in such cases, it is sometimes possible that the matrix exponential maps sums to products hold (Bourgeois, 2014). We also provide explicit computations and give an application on how to solve the differential equation and how to find a nuclear magnetic moment.

This essay is structured as follows: chapter 2 is devoted to explaining some basic terminologies on matrix and exponential map. In chapter 3, the computations of matrix exponential are derived. We present different proofs on the $e^{A+B}=e^Ae^B=e^Be^A$ and mathematical literature for the $e^{A+B}=e^Ae^B$ and some example and counter-example in chapter 4. Chapter 5 gives the applications of the matrix exponential. We then finally conclude.

2. Preliminaries

In this chapter, we review some basic concepts which will be used extensively in subsequent chapters. The first section gives a succinct explanation of some necessary terminologies and ideas of matrices. In the second section, we introduce the exponential map.

2.1 Matrices

There are much literature that give detailed discussions on matrices (Meyer, 2000; Horn and Johnson, 1990). We present in this section some key concepts which are needed for later discussion and allow the reader to see for further details. The most of terminologies used in this section are derived from (Horn and Johnson, 1990; Rod, 2018). We shall use the notation $GL(n, \mathbb{C})$ denotes the set of complex $n \times n$ invertible matrices which forms a group under multiplication. In our case, we will consider $n \times n$ matrices with entries from $\mathbb{C}$ abbreviated by $M(n, \mathbb{C})$. We write a matrix $A \in M(n, \mathbb{C})$ as $A = [a_{ij}]$ where a_{ij} are the entries and the subscripts the i^{th} and the j^{th} represent row and the column respectively.

2.1.1 Definition. A matrix A is said to be diagonal if all non zero entries lie on the main diagonal (a_{ii}).

2.1.2 Example. A vital example of diagonal matrices is the identity matrix I_n with all non zero entries 1.

An interesting observation is the following proposition.

2.1.3 Proposition. All diagonal matrices of equal dimensions commute under multiplication.

2.1.4 Definition. A matrix A is said to be an upper (lower) triangular matrix if all entries below (above) the main diagonal (a_{ii}) are zeros.

2.1.5 Definition. Determinant of a matrix is a number that can be computed from a square matrix. The determinant of a matrix A is denoted by $\det(A)$ or $|A|$. Determinant can be computed as

$$\det A = \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}.$$

Therefore,

$$|A| = A_{i1}(-1)^{i+1} \det A_{i1} + \cdots + a_{in}(-1)^{i+n} \det A_{in}.$$

Where A_{ik} is the sub matrix gained from A by eliminating the i^{th} row and k^{th} column.

The determinant of the matrix is very helpful for solving the system of linear equations and to find the inverse of a matrix and more.

2.1.6 Definition. (Cayley-Hamilton Theorem). The eigenvalues of a matrix A are the roots of the characteristic polynomial associated with A. The set of all eigenvalues of A is called the **spectrum of A** denoted by s(A).

The polynomial defined by $\det(A - \lambda I)$ is called the characteristic polynomial of A. By solving $\det(A - \lambda I) = 0$, we find the eigenvalues. By computing the null space of $(A - \lambda_i I)v$, we get the eigenvectors.

2.1.7 Definition. Let A and B be two matrices. A is said to be similar to B if there is an invertible (non-singular) matrix P such that $A = P^{-1}BP$.

2.1.8 Proposition. Suppose $A \in M(n, \mathbb{C})$. Then A is diagonalisable if A is similar to the diagonal matrix D of its eigenvalues, that is $A = PDP^{-1}$.

Proof. Using the definition of similarity of matrices A and D implies that $D = P^{-1}AP$. By multiplying the invertible matrix P to the left for each sides, that is $PD = PP^{-1}AP = AP$. If we multiply the inverse of matrix P to right of each side, then $PDP^{-1} = APP^{-1}$.

Therefore, $A = PDP^{-1}$. □

The matrix P is formed by the column matrix of eigenvectors .

2.1.9 Definition. A square matrix N is said to be nilpotent if $N^q = 0$ for $q \in \mathbb{Z}_*^+$.

2.1.10 Example. Every triangular matrix (either upper or lower) with zeros along the main diagonal is nilpotent. For example the matrix

$$M = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

is nilpotent since $M^3 = 0$.

2.1.11 Definition. An $n \times n$ matrix J is said to be in **Jordan canonical form** if it is a matrix of the form

$$J = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_i} \end{pmatrix}.$$

Where each block $J_{\lambda i}$ is a square matrix of the form

$$J_{\lambda i} = \begin{pmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{i-1} & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

$J_{\lambda i}$ is called Jordan block of size n with eigenvalues λ_i.

Refer to Golub and Wilkinson (1976), a square matrix A with complex elements, there exists a non singular matrix X such $X^{-1}AX = J$, $AX = XJ$ where J is **Jordan canonical form** of A.

2.1.12 Example. Given matrix $A = \begin{pmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The eigenvalues of A are the solution of the polynomial characteristic $\det(A-\lambda I) = 0$.Thus,$s(A) = \{2, 4\}$. The eigenvector can be computed from $(A - \lambda I)v = 0$ where v is the column vector. For $\lambda = 2$, the eigenvector $v_1 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$. For $\lambda = 4$, the eigenvector $v_2 = \begin{bmatrix} 2 & -2 & 0 \end{bmatrix}^T$. As $\lambda = 4$ has multiplicity 2 means that it must have two linear independent eigenvectors, so for $\lambda = 4$ we take the generalized eigenvector $v_3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

The matrix of eigenvectors $P[v_1v_2v_3] = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, its inverse is $P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{-1}{2} & \frac{-1}{2} \\ 1 & 1 & 1 \end{pmatrix}$.

Jordan canonical form is,

$$J = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

2.1.13 Proposition (Lebovitz (2016)). Let $A \in M(n, \mathbb{C})$, a **matrix norm** of A denoted by $\|A\|$, is the map $A \longmapsto R$ which is a real-valued. $\| A \| = \max_{i \leqslant j \leqslant n} \left(\sum_{i=0} \| a_{ij} \| \right)$.

A matrix norm is defined in different ways, but all definitions share the following properties:

1) $\| A \| > 0$.

2) $\| A \| = 0$ if $A = 0$

3) $\| \alpha A \| = |\alpha| \| A \|$.

4) $\| A + B \| \leqslant \| A \| + \| B \|$.

5) $\| AB \| \leqslant \| A \| \| B \|$.

2.2 The Exponential maps

In this section we introduce the exponential map on real or complex numbers and the exponential matrix. The excerpts here are stated to suit the scope of our study.

2.2.1 The exponential on real or complex numbers. Refer to Ahlfors (1953), the exponential function exp : $\mathbb{C} \to \mathbb{C}$, defined by

$$\exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \text{ for } z \in \mathbb{C},$$

on the closed disk

$$\overline{D}(0, R) := \{z \in |z| \leqslant R\}.$$

Using that definition,

$$|\frac{1}{k!}z^k| \leqslant \frac{1}{k!}R^k.$$

and the series $\sum_{k=0}^{\infty} \frac{R^k}{k!}$ converge for any $R > 0$.Thus, $\exp(\mathrm{z})$ is normally convergent series of continuous functions and $z \longmapsto \exp(\mathrm{z})$ is a continuous function from $\mathbb{C}$ to $\mathbb{C}$. Also is one of the most important of all mathematical functions in real and complex analysis.

The exponential a function has the useful feature of mapping sums to products:

$$e^{z+w} = e^z e^w, \text{ for } z, w \in \mathbb{C}.$$

A key feature of the real and complex numbers is the commutative nature of multiplication: $zw = wz$, for all complex numbers.

2.2.2 The Exponential Matrix. Refer to Bourgeois (2014), the exponential matrix is a function defined from $M(n, \mathbb{C})$ to $GL(n, \mathbb{C})$ as

$$\exp : M(n, \mathbb{C}) \mapsto GL(n, \mathbb{C})$$

$$X \rightsquigarrow \exp(\mathrm{X}) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

If $n = 1$, then $\exp(\mathrm{X}) = e^X$. That is exponential of real or complex numbers.

If $n = 1$, then exponential map is a homomorphism for $M_n(\mathbb{C}, +)$ to $GL_n(\mathbb{C}, .)$. But for $n \geqslant 2$, the exponential map is not a homomorphism, because if we assume that $(A, B) \in M_n(\mathbb{C}, +)$,

$$\exp(\mathrm{A} + \mathrm{B}) = \exp(\mathrm{A})\exp(\mathrm{B})$$

is not always true.

By using the fifth property of matrix norm, we can say

$$\| \frac{1}{k!}A^k \| \leqslant \frac{1}{k!} \| A \|^k, \forall k \in \mathbb{N}.$$

So that the series $\sum_{k=0}^{\infty} \frac{1}{k!}X^k$ converges absolutely for any $X \in M_n(\mathbb{C})$.

In this project we write for short $e^X = \exp(\mathrm{X})$. For $n = 1$ the notation is consistent as e^X.

3. The computation of exponential map

This chapter focuses on the different ways of computing the exponential map. The First section is devoted to the type of matrices which is easier to compute their exponential. We used Smalls (2007) and Meyer (2000).

In the second section, we focus on some methods of computing the exponential maps. We are following the paper of Moler and Van Loan (2003); Bellman (1997); Oberhettinger and Badii (1973).

3.1 Special cases of Computing the matrix Exponential

There exist some matrices which are easier to find their exponential. We are interested in computing the exponential of a diagonal matrix, nilpotent matrices, and matrices which is transformed in Jordan canonical form.

3.1.1 Diagonalizable matrix. If matrix $A = [a_{ij}]$ is diagonal, then exponential of A can be written as $e^A = [e^{\lambda_{ij}}]$.

Proof. Let $D = [\lambda_{ij}]$ denote a diagonal matrix.

The power of a diagonal matrix is equal to the power of entries, that is $D^n = (\lambda_{ij}^n)$, so

$$e^D = \sum_{n=0}^{\infty} \frac{D^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{\lambda_{ij}^n}{k!} \right].$$

Therefore,

$$e^D = \left[e^{\lambda_{ij}} \right].$$

□

3.1.2 Proposition. Let A and P be complex $n \times n$ matrices, if P is invertible, then $e^A = Pe^D P^{-1}$.

Proof. Assume that $A = PDP^{-1}$.

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}, A^3 = PD^2P^{-1}PDP^{-1} = PD^3P^{-1}, ..., A^k = PD^kP^{-1}.$$

In fact,

$$\exp(\mathrm{A}) = \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{3.1.1}$$

Replacing the power of A in (3.1.1), we have: $\exp(\mathrm{A}) = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!}$.

Generally,

$$\exp(\mathrm{At}) = P\sum_{k=0}^{\infty}\frac{t^kD^k}{k!}P^{-1} = P\exp(\mathrm{tD})P^{-1}.$$

□

3.1.3 Exponential of Nilpotent matrix. If M is nilpotent such that $M^q = 0$ with $q \in \mathbb{N}$, then the exponential of M is

$$e^{Mt} = I + tM + \frac{t^2M^2}{2!} + ... + \frac{t^{q-1}M^{q-1}}{(q-1)!} + \frac{t^qM^q}{q!} = \sum_{k=1}^{q}\frac{t^{k-1}M^{k-1}}{(k-1)!}.$$

3.1.4 Exponential of the matrix which can be transformed into Jordan canonical form. Jordan canonical form allows us to write the matrix as diagonal and nilpotent matrices.

3.1.5 Example. Suppose that we want to compute the exponential of matrix $M = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix}$.

The eigenvalues of M are 4 and 2 (multiplicity 2).

Let us first calculate $\exp(\mathrm{J})$ where $J = J_1(\lambda_1) \oplus J_2(\lambda_2)$. That is $J = J_1(4) \oplus J_2(2)$.

$J_1(4)$ has eigenvalue 4 multiplicity 1 its exponential is matrix of 1×1 dimension.Thus ,

$$\exp(\mathrm{J_1(4)} = e^4.$$

$J_2(2)$ has eigenvalue 2 multiplicity two, it's exponential can be written as sum of diagonal and nilpotent matrix respectively.

$$J_2(2) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

$$\exp(\mathrm{J}) = \begin{pmatrix} e^4 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix}.$$

The exponential of the matrix M is

$$\exp(\mathrm{M}) = P\begin{pmatrix} e^4 & 0 & 0 \\ 0 & e^2 & e^2 \\ 0 & 0 & e^2 \end{pmatrix}P^{-1} = \frac{1}{2}\begin{pmatrix} e^4 - 2e^2 & -2e^2 & e^4 - e^2 \\ 3e^2 - e^4 & 4e^2 & e^2 - e^4 \\ e^4 + e^2 & 2e^2 & e^4 + e^2 \end{pmatrix}$$

where P is column matrix of eigenvectors.

3.2 Computation methods

We emphasize on four methods of computing the exponential of the matrix where we look for: Inverse Laplace transform, Polynomial method, a method of eigenvectors and we shall use differential equation.

3.2.1 Inverse Laplace transformation. Refer to Oberhettinger and Badii (1973), suppose that we have $x'(t) = Ax(t)$. Laplace transform says that $\mathcal{L}[x'] = sX(s) - x(0)$,

we can write

$$sX(s) - x(0) = AX(s) \Leftrightarrow sX(s) - AX(s) = x(0).$$

In fact,

$$X(s) = (sI - A)^{-1}x(0).$$

By using inverse Laplace transform, we can write $x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]x(0) = e^{At}x(0)$. Therefore

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right].$$

3.2.2 Example. Given $B = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{pmatrix}$, compute e^{At} using Laplace transformation.

Firstly, $sI - B = \begin{pmatrix} s & 0 & 0 \\ -3 & s & 0 \\ -5 & -1 & s \end{pmatrix}$. We use $\mathcal{L}[e^{Bt}] = (sI - B)^{-1}$, to find the exponential of B.

Does $(sI - B)^{-1}$ exist?

A Matrix has the inverse if and only its determinant is different from zero. Because $(sI - B)$ is lower triangular matrix, its determinant is the product of main diagonal entries. That is $\det(sI - B) = s^3$ for $s \neq 0$, we can use Inverse Laplace transform.

Let write M as $(sI - B)$, its inverse is $M^{-1} = \begin{pmatrix} \frac{1}{s} & 0 & 0 \\ \frac{3}{s} & \frac{1}{s} & 0 \\ \frac{5}{s^2} + \frac{3}{s^3} & \frac{1}{s^2} & \frac{1}{s} \end{pmatrix}$.

We can find $e^{Bt} = \mathcal{L}^{-1}\left(M^{-1}\right)$. From inverse Laplace transformation

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n.$$

We can compute the inverse Laplace transformation for each entries

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, n = 0, t^0 = 1.$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t, n = 1, t^1 = t.$

$\mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} = \frac{3}{2}$ and $\mathcal{L}^{-1}\left\{\frac{1.2!}{s^3}\right\} = \frac{3}{2}t^2, n = 2.$

Therefore, exponential of matrix B can be written as

$$e^{Bt} = \mathcal{L}^{-1}\left(M^{-1}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ 5t + \frac{3}{2}t^2 & t & 1 \end{pmatrix}.$$

3.2.3 Polynomial method . To evaluate the exponential of matrix A using polynomial method, we have to find the characteristics polynomial and use Cayley-Hamilton theorem and we write

$$e^{At} = \sum_{l=0}^{n-1} c_l(t)A^l.$$

Proof. Consider a square matrix A with n dimension, the characteristic polynomial of A is

$$C(\lambda) = det(\lambda I - A) = \lambda^n - \sum_{k=0}^{n-1} c_k \lambda^k. \tag{3.2.1}$$

Cayley-Hamilton theorem says that if we replace λ in characteristic polynomial by matrix A, then we get the zero matrix

$$C(A) = 0. \tag{3.2.2}$$

From(3.2.1), we replace λ by A

$$c(A) = A^n - \sum_{k=0}^{n-1} c_k A^k.$$

Using (3.2.3), we can write $A^n - \sum_{k=0}^{n-1} c_k A^k = 0$. Explicitly, $A^n = \sum_{k=0}^{n-1} c_k A^k$.
Power of matrix A can be writen as

$$A^n = c_0 I + c_1 A + c_2 A^2 + ... + c_{n-1}A^{n-1}.$$

By changing the indices we can write

$$A^k = \sum_{l=0}^{n-1} \gamma_{kl} A^l. \tag{3.2.3}$$

While

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}. \tag{3.2.4}$$

Replace (3.2.3) into (3.2.4), the we can write

$$e^{At} = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{n-1} \gamma_{kl} A^l \frac{t^k}{k!}\right) = \sum_{l=0}^{n-1}\left(\sum_{k=0}^{\infty} \gamma_{kl} \frac{t^k}{k!}\right) A^l.$$

Assume that $c_l(t) = \left(\sum_{k=0}^{\infty} \gamma_{kl} \frac{t^k}{k!}\right)$.

Consequently,

$$e^{At} = \sum_{l=0}^{n-1} c_l(t) A^l. \tag{3.2.5}$$

□

3.2.4 Example. Given $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, by using polynomial method compute e^{At}.

The power of matrix A can be written as linear combination

$$A^k = c_0(t)I + c_1(t)A.$$

From (3.2.5)

$$e^{At} = c_0(t)I + c_1(t)A. \tag{3.2.6}$$

The characteristic equations are determined from the set of equations given by the eigenvalues of A, that is

$$e^{\lambda_i t} = \sum_{k=0}^{n-1} c_k \lambda_i^k.$$

As A is the matrix of order 2, we can write

$$e^{\lambda_i t} = c_0(t) + c_1(t)\lambda_i.$$

The eigenvalue can be computed by using this equation, $\det(A - \lambda I) = 0$.

The characteristic equation is $(2 - \lambda)^2 - 1 = 0$.

The roots are $\lambda_1 = 1$ and $\lambda_2 = 3$.

For $\lambda_1 = 1$, we have

$$e^t = c_0(t) + c_1(t). \tag{3.2.7}$$

For $\lambda_2 = 3$, we have

$$e^{3t} = c_0(t) + 3c_1(t). \tag{3.2.8}$$

The linear system of (3.2.7) and (3.2.8) can be solved by taking (3.2.7) subtract to (3.2.8): $c_1(t) = \frac{e^{3t} - e^t}{2}$ and $c_0(t) = \frac{3e^t - e^{3t}}{2}$.

By replacing the value of $c_1(t)$ and $c_0(t)$ into (3.2.6) we get,

$$e^{At} = \begin{pmatrix} \frac{e^t + e^{3t}}{2} & \frac{e^{3t} - e^t}{2} \\ \frac{e^{3t} - e^t}{2} & \frac{e^{3t} + e^t}{2} \end{pmatrix}.$$

3.2.5 Remark. Assume that we have 2×2 matrix, if the eigenvalues are the same then we have only one algebraic equation as

$$\lambda_1^k = c_0(t) + c_1(t)\lambda_1. \tag{3.2.9}$$

To find the value c_0 and c_1 we derive (3.2.9) with respect to λ_1. That is

$$\frac{d}{d\lambda_1}\left(\lambda_1^k = c_0(t) + c_1(t)\lambda_1\right) \Leftrightarrow k\lambda_1^{k-1} = c_1(t).$$

Having obtained $c_1(t)$, from (3.2.9) we can write

$$c_0(t) = \lambda_1^k - c_1(t)\lambda_1 = \lambda_1^k - k\lambda_1^k = \lambda_1^k(1-k).$$

Therefore exponential can be obtained as

$$e^{At} = k\lambda_1^{k-1}I + \lambda_1^k(1-k)A.$$

Where k is the multiplicity of eigenvalue.

3.2.6 Using Eigenvectors. Suppose that A and P are $n \times n$ complex matrices for which P is column vector of eigenvectors and is invertible.Then $e^{P^{-1}AP} = P^{-1}e^AkP$ (Klain, 2017).

Proof. For integers $k \geqslant 0$, we have $(P^{-1}AP)^k = P^{-1}A^kP$. The exponential can be computed as

$$\begin{aligned} e^{P^{-1}AP} &= I + P^{-1}AP + \frac{P^{-1}AP^2}{2!} + \ldots \\ &= I + P^{-1}AP + \frac{P^{-1}A^2P}{2!} + \ldots \\ = P^{-1}\left(I + A + \frac{A^2}{2!} + \ldots)\right) P &= P^{-1}e^AP. \end{aligned}$$

Therefore,

$$e^{P^{-1}AP} = P^{-1}e^AP.$$

□

If the matrix A is diagonalizable is such that $A = PDP^{-1}$ then, $e^A = Pe^DP^{-1}$.

3.2.7 Example. Find $\exp(\mathrm{At})$, where $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$.

Compute the eigenvalues by using $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0.$$

The characteristics polynomial is $-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$.

We can find the roots of that characteristics polynomial.

Roots are 1 and 2 (with multiplicity two), that are the eigenvalues.

The eigenvectors can be computed around the eigenvalues.

For $\lambda = 1$

$$\begin{pmatrix} 5-1 & -6 & -6 \\ -1 & 4-1 & 2 \\ 3 & -6 & -4-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this system we can use reduction of row.

Let write the matrix of the homogeneous system as

$$\begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}.$$

Let us define, R_1 as the first row, R_2 as the second row and R_3 as the third row.

$$R_1' = \frac{R_1}{2}, R_2' = 2R_2 + R_1', R_3' = -2R_3 + 3R_1'.$$

We write

$$\begin{pmatrix} 2 & -3 & -3 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \xrightarrow[R''_3 = R'_2 - R'_3]{} \begin{pmatrix} 2 & -3 & -3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

After reduction of row, we can write the homogeneous system as $\begin{pmatrix} 2 & -3 & -3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Therefore,

$$2x - 3y - 3y = 0 \tag{3.2.10}$$
$$0x + 3y + z = 0 \tag{3.2.11}$$
$$0x + 0y + 0z = 0. \tag{3.2.12}$$

Let z be a parameter.

From (3.2.11), $3y + z = 0 \Leftrightarrow y = \frac{-1}{3}z$.

From (3.2.10), we can replace the value y and z: $2x - 3y - 3z = 0 \Leftrightarrow 2x - 3(\frac{-1}{3}z) - 3z = 0$. That is,

$$2x + z - 3z = 0 \Leftrightarrow 2x - 2z = 0 \Leftrightarrow x = z.$$

The first eigenvector can be written as,

$$v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} z \\ \frac{-1}{3}z \\ z \end{pmatrix} = \frac{1}{3}z \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, z \in \mathbb{R}.$$

Hence, $v_1 = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$.

We can compute the eigenvector for $\lambda = 2$. By using the same procedure we can write the homogeneous system equation as

$$\begin{cases} -x + 2y + 2z = 0 \\ -x + 2y + 2z = 0 \\ -x + 2y + 2z = 0. \end{cases}$$

All equations are the same,

$$x = 2y + 2z.$$

The second eigenvector has two linear independent vectors which are

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + 2z \\ y \\ z \end{pmatrix}.$$

For $y, z \in \mathbb{R}$.

$$v = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

The matrix formed by linear independent eigenvectors $P = [v_1 v_2 v_3]$ is $P = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

The inverse of P matrix is $P^{-1} = \begin{pmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$.

$$e^{At} = P^{-1} e^{tD} P = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} -3e^t + 4e^{2t} & 6e^t - 6e^{2t} & 6e^t - 6^{2t} \\ e^t - e^{2t} & -2e^t + 3e^{2t} & -2e^t + 2e^{2t} \\ -3e^t + 3e^{2t} & 6^t - 6e^{2t} & 6e^t - 5e^{2t} \end{pmatrix}.$$

3.2.8 Remark. Refer to Lebovitz (2016), there is the short cut for matrix of order 2 while we compute their exponential using the eigenvectors.

Eigenvalues of A	$\exp(At)$
λ_1, λ_2 real distinct	$e^{\lambda_1 t}\frac{1}{\lambda_1-\lambda_2}(A-\lambda_2 I) - e^{\lambda_2 t}\frac{1}{\lambda_1-\lambda_2}(A-\lambda_1 I)$
λ repeated twice	$e^{\lambda t}I + e^{\lambda t}t(A-\lambda I)$
λ is conjugate complex roots, that is $\alpha \pm i\beta$	$e^{\alpha t}\cos(\beta t)I + \frac{1}{\beta}e^{\alpha t}\sin(\beta t)(A-\alpha I)$

3.2.9 Using differential equation. Generally when matrix A is diagonalizable,the general solution of $\vec{x}' = A\vec{x}$ is given by

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v_1} + ... + c_n e^{\lambda_n t}\vec{v_n}.$$

We want to solve the initial value problem,

$$\vec{x}' = A\vec{x}, \vec{x}(0) = x_0.$$

we choose $c_1, .., c_n$ for which $x(0) = c_1\vec{v_1} + ... + c_n\vec{v_n} = x_0$. The constants c_i are the coordinates for the vectors $\vec{x_0}$ in the basis $\vec{v_1}, ..., \vec{v_n}$.

Consequently each term $e^{\lambda_i t}\vec{v_i}$ in the solution is actually $e^{tA}v_i$. In fact j^{th} column of the matrix e^{tA} is given by $e^{tA}\vec{e_j}$ where $\{\vec{e_1}, .., \vec{e_n}\}$ is the canonical basis. To compute the exponential of matrix we proceed those processes with condition $\vec{x_0} = \vec{e_j}$ where $j = 1, .., n$.

This method is very interesting because when matrix A is not diagonalizable this method works. The only difference is that some of the vectors $\vec{v_i}$ are generalized eigenvectors (Wahln, 2016).

3.2.10 Example. Given matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, compute e^{At} by using differential equation.

Characteristic polynomial is $(\lambda+1)(\lambda-3)$, by solving the characteristic equation we get $s(A) = \{-1, 3\}$. In that case we can find the eigenvector by solving $(A - \lambda I)v = 0$.

For $\lambda = -1$, $\vec{v_1} = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Also for $\lambda = 3$, $\vec{v_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

As the general solution of the system $\vec{x}'$ is given by

$$\vec{x}(t) = c_1 e^{-t}v_1 + c_2 e^{3t}v_2.$$

It can be also written as

$$\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The matrix A is of order 2, the solution is founded from $\vec{x}(0) = \vec{e_1}$ and $\vec{x}(0) = \vec{e_2}$.

By solving the initial value problem we found the value of c_1 and c_2.

For $\vec{x}(0) = \vec{e_1}$, we have $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. That is $\begin{cases} c_1 + c_2 = 1 \\ -c1 + c_2 = 0 \end{cases} \Leftrightarrow c_1 = c_2 = \frac{1}{2}$.

For $\vec{x}(0) = \vec{e_2}$, we have $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

That is $\begin{cases} c_1 + c_2 = 0 \\ -c1 + c_2 = 1 \end{cases} \Leftrightarrow c_1 = \frac{-1}{2}, c_2 = \frac{1}{2}$.

We can use those constants to find the exponential A.

$$e^{At}\vec{e_1} = \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t} + e^{3t} \\ -e^{-t} + e^{3t} \end{pmatrix}.$$

$$e^{At}\vec{e_2} = \frac{-1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{2}e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -e^{-t} + e^{3t} \\ e^{-t} + e^{3t} \end{pmatrix}.$$

Significantly,

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^{-t} + e^{3t} & -e^{-t} + e^{3t} \\ -e^{-t} + e^{3t} & e^{-t} + e^{3t} \end{pmatrix}.$$

4. The exponential of complex matrices.

This chapter divided into three main sections. In the first section we use different proofs to prove that the matrix exponential maps sums to the product, if the matrices commute. In the second section, we look for some mathematical literature that deals with the cases where the exponential of matrices which are commuting or not. Finally, we work on the example and counter-example for where matrices commute or not to verify the equality on the exponential maps sums to the product.

4.1 The exponential maps sums to product

4.1.1 Proposition. For the purpose of A and B are commuting matrices of $M_n(\mathbb{C})$ then

$$e^{A+B} = e^B e^A = e^A e^B \tag{4.1.1}$$

Giamarchi (2004) pointed out this formula

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}.$$

This formula is the simplification of Zassenhaus formula where A and B are two matrices and $[A, B] = AB - BA$. what is the importance of the commuting matrices? If those matrices are commuting then $[A, B] = 0$. Therefore

$$e^A e^B = e^{A+B}.$$

Proof. [1] According to (Baker, 2012, page 47), the series of the exponential A and B are absolutely convergent, notably

$\exp(\mathrm{A}) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ and $\exp(\mathrm{B}) = \sum_{s=0}^{\infty} \frac{B^s}{s!}$.

Its product can be written as

$$\exp A \exp B = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)\left(\sum_{s=0}^{\infty} \frac{B^s}{s!}\right) = \sum_{s,k=0}^{\infty} \frac{A^k B^s}{s!k!}.$$

By assuming that A and B commute, we can rearrange and collect terms while expanding $\frac{1}{n!}(A+B)^n$ to get

$$\exp(A+B) = \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} A^k B^{n-k}\right) = \sum_{s,k=0}^{\infty} \frac{A^k B^s}{s!k!}.$$

In fact,

$$\exp(A+B) = \exp A \exp B.$$

□

16

4.1.2 Proposition. Refer to (Smalls, 2007, page 15), if A and B commute, then $e^{At}B = Be^{At}$.

Before proving this proposition, let us recall **PicardLindelöf theorem**.

PicardLindelöf theorem ensures the presence and uniqueness of the solution $x^*(t)$ of the differential equation $\dot{x}(t) = f(x(t), t)$ with initial condition $x(0) = x^*(0)$ for small t.

Proof. Assume that $x_1(t) = e^{At}BX_0$ and $x_2(t) = Be^{At}X_0$ where X_0 denotes the column vector. By differentiating $x_1(t)$ and $x_2(t)$, we have:

$$x_1'(t) = Ae^{At}BX_0 = Ax_1(t)$$

and

$$x_2'(t) = BAe^{At}X_0 = Ax_2(t).$$

Because $x_1(0) = x_2(0) = X_0$, and according to the uniqueness of the solution from PicardLindelöf theorem, that is $x_1(t) = x_2(t)$. This implies that

$$e^{At}B = Be^{At}. \tag{4.1.2}$$

□

Using (4.1.2), we use the same idea to prove the proposition (4.1.1) .

Proof. [2] Let assume that $f(t) = e^{At}e^{Bt}X_0$ and $g(t) = e^{(A+B)t}X_0$. we are interested to show that $f(t) = g(t)$ for all t.

As from the previous proof we have seen that $e^{At}B = Be^{At}$ for $AB = BA$.

If we assume that $h(t) = e^{Bt}X_0$, then $f(t)$ can be written as

$$f(t) = e^{At}h(t). \tag{4.1.3}$$

By differentiating (4.1.3) we get:

$$f'(t) = Ae^{At}e^{Bt}X_0 + e^{At}Be^{Bt}X_0 = Af(t) + Bf(t) = (A + B)f(t)$$

and

$$g'(t) = (A + B)g(t).$$

Since $f(0) = g(0) = X_0$, then by PicardLindelöf theorem we get,

$$f(t) = g(t)$$

for all t. □

Proof. [3] **Infinite matrix power series**

The infinite power series of the matrix is written as:

$$I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + ... + \frac{1}{k!}A^K.$$

The sum of the infinite series is called the matrix exponential.

The matrix exponential of A is absolutely convergent that is $\parallel A \parallel < \infty$. Let $A, BM_n(\mathbb{C})$, then

$$\exp(\mathrm{A}) = I + A + \frac{A^2}{2!} + ...$$

and

$$\exp(\mathrm{B}) = I + B + \frac{B^2}{2!} + ...$$

$\exp(\mathrm{A})\exp(\mathrm{B}) = (I + A + \frac{A^2}{2!} + ...)(I + B + \frac{B^2}{2!} + ...) = I + (A + B) + \frac{1}{2!}(A^2 + 2AB + B^2) + ...$

$\exp(\mathrm{A})\exp(\mathrm{B}) = I + (A + B) + \frac{1}{2!}(A + B)^2 + ...$

Therefore,

$$\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A} + \mathrm{B}).$$

□

Proof. [4] According to (Lee, 2009, page3), the Baker-Campbell-Hausdorff allow us to compute the product of the exponentials of two operators A and B.

$$\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A} + \mathrm{B} + \frac{1}{2}[\mathrm{A}, \mathrm{B}] + \frac{1}{12}[\mathrm{A}, [\mathrm{A}, \mathrm{B}]] + \frac{1}{12}[\mathrm{B}, [\mathrm{A}, \mathrm{B}]] + ...) \quad (4.1.4)$$

Respectively the definition and the property of commutator: $[A, B] = AB - BA$, consequently $[A, 0] = [0, B] = 0$.

If A and B commute, then $[A, B] = AB - BA = 0$. In fact the Equation(4.1.4) becomes

$$\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A} + \mathrm{B} + \frac{1}{2}0 + \frac{1}{12}[A, 0] + \frac{1}{12}[B, 0] + ...)$$

Therefore,

$$\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{A} + \mathrm{B}).$$

□

4.1.3 Definition.

The following definitions are derived from Bourgeois (2014).

- Let $A, B \in M_n(\mathbb{C})$. A and B are simultaneously triangularizable (denoted by ST), $P \in GL_n(\mathbb{C})$ such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices.
- Assume that $A \in M_n(\mathbb{C})$ and $s(A) = \{\gamma, \beta\}$. s(A) is $2i\pi$ congruence -free (denoted by $2i\pi$ CF) if $\gamma - \beta \notin 2i\pi\mathbb{Z}^*$. Refer to Paliogiannis (2003), s(A) said to be $2i\pi$ CF if $s(A) \cap s(s(A) + 2k\pi i) = \emptyset$ for $k \in \mathbb{Z}^*$.

4.2 Some mathematical literature to the exponential maps sums to product

4.2.1 Theorem. *Refer to Paliogiannis (2003), let $A, B \in M_n(\mathbb{C})$ provided that s(A) and s(B) or s(A+B) are $2i\pi$ CF then, $e^A e^B = e^B e^A = e^{A+B}$.*

Proof. If s(A) is $2i\pi$ then, $e^A = I_n$ the same for $s(B)$. While $s(A) \in 2k\pi i$ then $A \in 2\pi i$ the same for $s(B)$. Under those circumstances $s(A+B) \in (2\pi i)$ then $e^{A+B} = I_n$. For this reason $e^A e^B = e^B e^A = e^{A+B}$. □

Generally speaking, to prove this theorem the condition of commuting matrices does not necessarily hold.

4.2.2 Proposition. Bourgeois (2014) argued that the square complex matrices A and B which are 2×2 or 3×3 hold this condition.

$\forall k \in \mathbb{N}$,

$$\exp(\mathrm{kA} + \mathrm{B}) = \exp(\mathrm{kA})\exp(\mathrm{B}) = \exp(\mathrm{B})\exp(\mathrm{kA}) \tag{4.2.1}$$

are ST.

That proposition is not true because the following counter example of Jean -Louis Tu shows :

consider the matrices $A_1 = 2i\pi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B_1 = 2i\pi \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$ those matrices are not ST because they dont have the same eigenvector.

However A is diagonal matrix in $2i\pi\mathbb{Z}$ implies that $e^A = I_3$ also matrix B is diagonalizable its exponential is I_3.

For every $k \in \mathbb{C}$, the characteristic equation of $kA+B$ is $\lambda(\lambda - 2i\pi(k+2))(\lambda - 2i\pi(2k+3))$ by solving it, for $k \in \mathbb{N}$, the matrix $kA + B$ has distinct eigenvalues in $2i\pi\mathbb{Z}$. With this intention, matrix $kA + B$ is diagonalizable due to that, $e^{kA+B} = I_3$. This shows that condition (4.2.1) holds.

4.2.3 Definition. Motzkin and Taussky (1952) introduced the property L, by definition a pair $(X, Y) \in M_n(\mathbb{C}^2)$ is said to have the property L if for a special ordering of the eigenvalues $(\lambda_i)_{i \leqslant n}$, $(\mu_i)_{i \leqslant n}$ of X,Y such that for all $(a, b) \in \mathbb{C}^2$, $s(aX + bY) = (a\lambda_i + b\mu_i)_{i \leqslant n}$.

4.2.4 Proposition. If A and B are commuting matrices, then the pair (A, B) has property L.

Proof. Let λ be an eigenvalue of A and let ξ be an eigenvector corresponding to λ; thus, $\xi \neq 0$ and $A\xi = \lambda\xi$. Therefore,

$$\lambda B\xi = B(\lambda\xi) = B(A\xi) = A(B\xi)$$

implies that $B\xi = 0$ or $B\xi$ is a (nonzero) eigenvector of A corresponding to λ. Therefore, $\mathcal{B}_1 = \{v_1, v_2, \ldots, v_n\}$ is a basis of $\mathbb{C}^n$ whose first element is $v_1 = \xi$, then the matrix representations of A and B have the form

$$A = \begin{bmatrix} \lambda & A_0 \\ 0 & A_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} \mu & B_0 \\ 0 & B_1 \end{bmatrix},$$

where $\mu, \lambda \in \mathbb{C}$, A_1 and B_1 are $(n-1) \times (n-1)$ matrices, A_0 and B_0 are $1 \times (n-1)$ matrices, and 0 is the zero $(n-1) \times 1$ matrix. Thus,

$$AB = \begin{bmatrix} \lambda\mu & * \\ 0 & A_1B_1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} \mu\lambda & * \\ 0 & B_1A_1 \end{bmatrix}.$$

Therefore, the equation $AB = BA$ leads to $A_1B_1 = B_1A_1$. Now repeat this argument to A_1 and B_1 to express A and B (with respect to a new basis) as

$$A = \begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & A_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} \mu_1 & * & * \\ 0 & \mu_2 & * \\ 0 & 0 & B_2 \end{bmatrix},$$

where λ_i and μ_j are scalars and where A_2 and B_2 are commuting $(n-2) \times (n-2)$ matrices. Repeating this argument leads to a basis in which A and B have upper triangular forms (with respect to a specific basis)

$$A = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix} \text{ and } B = \begin{bmatrix} \mu_1 & * & * & * \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \mu_n \end{bmatrix}.$$

Therefore, for every $x, y \in \mathbb{C}$, the eigenvalues of $\lambda A + \mu B$ are given by $x\lambda_j + y\mu_j$, for $j = 1, \ldots, n$. Hence, the pair (A, B) has property L.

4.2.5 Proposition. Given $(A, B) \in M_n(\mathbb{C})^2$. If A and$B$ hold (4.2.1), then has property L.

According to Bourgeois (2014) this proposition is not true always. For $k = 1$, (4.2.1) can be written as $e^A e^B = e^B e^A = e^{A+B}$ and (A,B) has property L in view of (4.2.1), but this is not true. With attention to the pairs $(A_1, -2B_1)$ from the matrices communicated by Jean-Louis Tu has property L. Notably $\exp(\mathrm{A}_1) = \exp(-2\mathrm{B}_1) = I_3$.

With this in mind, one has $\exp(\ \mathrm{kA}_1 - 2\mathrm{B}_1) = I_3$ if and only if $\forall\ k \in \mathbb{N} \backslash \{2, 3, 4\}$ this shows that (4.2.1) does not hold for this pair.

For this reason, Bourgeois (2014) gave another condition.

$\forall \mathbb{N} \geqslant 2, \exists U \subset \mathbb{N}$ such that $\forall\ k \in \mathbb{N} \backslash U$,

$$\exp(\ \mathrm{kA} + \mathrm{B}) = \exp(\ \mathrm{kA})\exp(\mathrm{B}) = \exp(\mathrm{B})\exp(\ \mathrm{kA}). \tag{4.2.2}$$

4.2.6 Theorem. *A pair* $(A,B) \in M_n(\mathbb{C})^2$, *fulfill* (4.2.2) *if and only if* $e^{A+B} = e^A e^B = e^B e^A$ *and has property L.*

Proof. Refer to Bourgeois (2014), there exists $k_0 \in \mathbb{N}$ that satisfies (4.2.2) for every $k \geqslant k_0$. In regards to (4.2.1), the pair $(K_0 A, B)$ has property L and (A,B) too.

Assume that A and B do not commute. According Schmoeger (2000), $s(A)$ and $s(B)$ are not $2i\pi CF$ and, since $n = 2$,A, B are diagonalizable.

An homothecy can be added to A or B and we can assume $A = \begin{pmatrix} 2i\pi\alpha & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \{2i\pi\gamma, 0\}$, where $\alpha, \gamma \in \mathbb{Z}^*$. Also $n = 2$, A and B are ST, we suppose that $B = \begin{pmatrix} 2i\pi\gamma & 1 \\ 0 & 0 \end{pmatrix}$. However $e^{A+B} = e^A e^B$ if and only if $\gamma + \alpha \neq 0$. If $k \in \mathbb{N}$, we have $e^{kA} e^B = e^B e^{kA} = e^{kA+B}$, for $t \neq \frac{-\alpha}{\gamma}$.

4.2.7 Remark. Refer to Bourgeois (2014) there is pair of complex matrices which hold the condition $e^{A+B} = e^A e^B = e^B e^A$ but has not property L.

For instances $A = i\pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \pi \begin{pmatrix} -11i & 6 \\ 16 & 11i \end{pmatrix}$. □

4.2.8 Proposition. According to Bourgeois (2014), assume that $A = diag(\lambda_1, ...\lambda_n) \in M_n(\mathbb{C})$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$ that $B = [b_{jk}] \in M_n(\mathbb{C})$ (where for every $j \leqslant n, b_{jj} \in 2i\pi\mathbb{Z}$) is diagonalizable and that the pair (A,B) has property L. Then the pair (A,B) satisfies (4.2.2).

Proof. Provided that $A = 2i\pi[\lambda_1, ..., \lambda_n]$ for $\lambda_n \in \mathbb{Z}$ and $B = [b_{jk}]$ where $s(B) = 2i\pi[b_{jj}]$, then $e^A = I_n$. Indeed, B is diagonalizable that is $e^B = Pe^D P^{-1}$, $e^B = I_n$. We can use definition of property L which is referenced by Motzkin and Taussky (1952), for every $k \in \mathbb{C}$, $s(\ kA + B) = (\ k\lambda_j + b_{jj})_{j \leqslant n}$.

For $k \in \mathbb{N}$, $kA + B$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$, in that case

$$\exp(\ \mathrm{kA + B}) = I_n.$$

Therefore (4.2.2) holds. □

4.3 Example and Counter-example.

4.3.1 Example. Given $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

A and B commute because are diagonal matrices, their exponential can be written as

$e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix}$ and $e^B = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^2 \end{pmatrix}$.

As matrix A and B are diagonal matrices. Then, $e^A e^B = e^B e^A = \begin{pmatrix} 1 & 0 \\ 0 & e^3 \end{pmatrix}$.

The sum $A + B$ can be written as $\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$, its exponential is $\begin{pmatrix} 1 & 0 \\ 0 & e^3 \end{pmatrix}$.

Therefore, $e^A e^B = e^B e^A = e^{A+B}$. Also A and B commute.

4.3.2 Example. If the matrix A can be written uniquely as $A = D + N$, where D is diagonal and N is Nilpotent matrix and $DN = ND$ then, $e^A = e^{D+N} = e^D e^N$.

Given that $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, the matrix A can be decomposed as the sum of $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ respectively.

The matrix N is nilpotent that is, $N^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The exponential of N and D are computed respectively,

$e^N = I + N = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $e^D = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$.

Also we can compute $e^D e^N = \begin{pmatrix} e & 2e \\ 0 & e \end{pmatrix}$. Hence, $e^A = \begin{pmatrix} e & 2e \\ 0 & e \end{pmatrix}$.

4.3.3 Remark. It is necessary that N and D commute in order to follow that procedure. Follow the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. This matrix can be decomposed as sum of diagonal and nilpotent matrix respectively but diagonal matrix and nilpotent matrix do not commute.That is $e^A \neq e^{D+N} \neq e^D e^N$.

4.3.4 Counter-example.

Let A and B be the matrices communicated by Jean-Louis Tu that is the matrices which holds $\exp(\mathrm{A})\exp(\mathrm{B}) = \exp(\mathrm{B})\exp(\mathrm{A}) = \exp(\mathrm{A}+\mathrm{B})$ but $AB \neq BA$.

$A = 2i\pi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = 2i\pi \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$.

Does A and B commute? $AB = -4\pi^2 \begin{pmatrix} 2 & 1 & 1 \\ 2 & 6 & -4 \\ 0 & 0 & 0 \end{pmatrix}$ and $BA = -4\pi^2 \begin{pmatrix} 2 & 2 & 0 \\ 1 & 6 & 0 \\ 1 & 2 & 0 \end{pmatrix}$.

Therefore $AB \neq BA$ means that A and B do not commute.

A is diagonal matrix,

$$\exp\mathrm{A} = \begin{pmatrix} e^{2i\pi} & 0 & 0 \\ 0 & e^{4i\pi} & 0 \\ 0 & 0 & e^{0} \end{pmatrix}.$$

From Euler's formula $e^{in} = \cos n + i \sin n$, implies that

$e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1$ and $e^{4i\pi} = \cos 4\pi + i \sin 4\pi = 1$.

In fact,

$$\exp\mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

For matrix B, the eigenvalue can be computed by solving this equation $\det(B - \lambda I) = 0$. That is $(2-\lambda)\left(\lambda^2 - 3\lambda + 2\right) = 0$.

The roots are $\lambda = 1$, $\lambda = 2$ (multiplicity two).

The eigenvalues are $2i\pi$ and $4i\pi$ (multiplicity two). The matrix B has distinct eigenvectors for this reason the matrix B is diagonalizable, means that $\exp(\mathrm{B}) = P\exp(\mathrm{D})P^{-1}$.

$$\exp(\mathrm{B}) = P \begin{pmatrix} e^{2i\pi} & 0 & 0 \\ 0 & e^{4i\pi} & 0 \\ 0 & 0 & e^{4i\pi} \end{pmatrix} P^{-1} = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1} = PIP^{-1} = PP^{-1} = I.$$

Consequently,

$$\exp(\mathrm{B})\exp(\mathrm{A}) = \exp(\mathrm{B})\exp(\mathrm{A}) = I.$$

Let looking for sum: $A + B = 2i\pi \begin{pmatrix} 3 & 1 & 1 \\ 1 & 5 & -2 \\ 1 & 1 & 0 \end{pmatrix}$.

Let assume that $A + B = M$, the exponential of M can be computed by following those process.

Firstly, we solve the characteristic equation from $\det(M - \lambda I) = 0$.

The characteristic equation is

$$\lambda\left(-\lambda^2+8\lambda-15\right)=0.$$

The roots are $\lambda_1=0$, $\lambda_2=6$ and $\lambda_3=10$.

By multiplying $2\pi i$ to the roots, the eigenvalues are $\lambda_1=0, \lambda_2=6i\pi$ and $\lambda_3=10i\pi$.

Matrix has distinct eigenvalues, that is matrix M is diagonalizable.

Under those circumstances $e^M=Pe^DP^{-1}$,

$$e^M=P\begin{pmatrix} e^0 & 0 & 0 \\ 0 & e^{10i\pi} & 0 \\ 0 & 0 & e^{6i\pi} \end{pmatrix}P^{-1}=P\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}P^{-1}=I.$$

Therefore,

$$e^Be^A=e^Ae^B=e^{A+B}, AB\neq BA.$$

5. The Applications

The exponential matrix has different applications. In this section, we shall look for how to solve ordinary differential equation and nuclear magnetic resonance (NMR).

5.1 Linear differential equations

The exponential matrix has applications to system of linear differential equations. If the given differential equation is for higher order differential equation with constant coefficient. We transform it into a linear system and solve it by using exponential matrix.

5.1.1 Homogeneous . In first-order ODEs, we say that a differential equation in the form

$$\frac{dy}{dx} = f(x, y).$$

is said to be homogeneous if the function$f(x, y)$ can be expressed in the form $f(\frac{y}{x})$, and then solved by the substitution $z = \frac{y}{x}$ in higher-order ODEs, for example we say that differential equation in the form

$$ay'' + by' + cy = f(x)$$

is said to be homogeneous if $f(x) = 0$.

5.1.2 Example. Solve

$$y''' + 2y'' - y' - 2y = 0.$$

We transform it into system of linear ordinary differential equation by assuming that $y = x_1$, $y' = x_1' = x_2$ and $y'' = x_2' = x_3$. Then

$y''' = -2y'' + y' + 2y$. In fact, $x_3' = 2x_1 + x_2 - 2x_3$.

The system can be written as

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -2x_3 + x_2 + 2x_1. \end{cases}$$

Assume that $U'(t) = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}$ and $U(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

By using matrix we can write the system of ordinary differential equations as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

It can be written as $U'(t) = AU(t)$, implies that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

The characteristic equation can be computed through

$$det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix} = 0.$$

The characteristic equation will be $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$. The roots are $\lambda_1 = 1, \lambda_2 = -1$ and $\lambda_2 = -2$. It is easy to find the eigenvectors associated to each eigenvalue and the matrix P of column eigenvectors is written as

$$P = \begin{pmatrix} 1 & 1 & 4 \\ 1 & -1 & -2 \\ -1 & 1 & 1 \end{pmatrix}.$$

Its inverse is

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 3 & 5 & -2 \\ 0 & -2 & -2 \end{pmatrix}.$$

As the matrix A is invertible $e^{At} = P^{-1}e^{Dt}P$,

$$e^{At} = \frac{1}{2}\begin{pmatrix} e^t + 3e^{-t} - 2e^{-2t} & e^t - 3e^{-t} + 2e^{-2t} & 4e^t - 6e^{-t} + 2e^{-2t} \\ e^t + 5e^{-t} - 6e^{-2t} & e^t - 5e^{-t} + 6e^{-2t} & 4e^t - 10e^{-t} + 6e^{-2t} \\ -2e^{-t} + 2e^{-2t} & 2e^{-t} - 2e^{-2t} & 4e^{-t} - e^{-2t} \end{pmatrix}.$$

We can write $U(t) = e^{At}U_0$ where $U_0 = (C_1, C_2, C3)^T$.

In fact the solution is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}\left(C_1 \begin{pmatrix} e^t + 3e^{-t} - 2e^{-2t} \\ e^t + 5e^{-t} - 6e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{pmatrix} + C_2 \begin{pmatrix} e^t - 3e^{-t} + 2e^{-2t} \\ e^t - 5e^{-t} + 6e^{-2t} \\ 2e^{-t} - 2e^{-2t} \end{pmatrix} + C_3 \begin{pmatrix} 4e^t - 6e^{-t} + 2e^{-2t} \\ 4e^t - 10e^{-t} + 6e^{-2t} \\ 4e^{-t} - e^{-2t} \end{pmatrix} \right).$$

5.1.3 Inhomogeneous. The matrix exponential can also be used to solve the inhomogeneous equation

$$\frac{d}{dt}y(t) = Ay(t) + b(t), y(0) = y_0 \Leftrightarrow y'(t) - Ay(t) - b(t) = 0.$$

Generally if we have O.D.E of the form

$$y' + p(y) + q = 0$$

then we use the integrating factor as

$$\mu = e^{\int p(y)dy}.$$

For our case $\mu = e^{\int -Ad(t)} = e^{-At}$.

By multiplying to the ODE that the integrating factor, we have:

$$e^{-At}y_t' - e^{-At}y(t) = e^{-At}b(t) \Leftrightarrow d(e^{-At}y(t)) = e^{-At}b(t).$$

We can integrate both side by writing

$$\int d(e^{-At}y(t)) = \int e^{-At}b(t)dt \Leftrightarrow e^{-At}y(t) = \int_0^t e^{-uA}b(u)d(u) + C.$$

Thus,

$$y(t) = e^{At}\int_0^t e^{-uA}b(u)d(u) + e^{At}C.$$

Refer to (Pihlak, 2004, page317), the matrix $Y : r \times s$ is called the matrix integral of $Z = Z(X) : pr \times sq$ where $X : p \times q$ if, $Z = \frac{dY}{dX}$. For instance

$$Y = \int_{R^{pq}} dY.$$

$$dY = \int_{R^{pq}} dX.Z.$$

$$dY = \begin{pmatrix} d(Y_{11}) & \dots & d(Y_{1p}) \\ . & & . \\ . & & . \\ . & & . \\ d(Y_{p1}) & \dots & d(Y_{pq}) \end{pmatrix}.$$

We integrate each entry in the domain of $p \times q$.

5.1.4 Example.

Given the system $\begin{cases} x' = x + e^{2t} \\ y' = -2y + e^t. \end{cases}$

From that system we can write the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ and $b(t) = \begin{pmatrix} e^{2t} \\ e^t \end{pmatrix}$.

Exponential of A can be written as

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

The exponential matrix $e^{-uA} = (e^{uA})^{-1}$.

Notably, $e^{-uA} = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^{2u} \end{pmatrix}$.

As

$$y(t) = e^{At}\int_0^t e^{-uA}b(u)du + e^{At}C = e^{At}\int_0^t \begin{pmatrix} e^{-u} & 0 \\ 0 & e^{2u} \end{pmatrix}\begin{pmatrix} e^{2u} \\ e^{u} \end{pmatrix} du + \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

By expanding it, we can write it as

$$y(t) = e^{At}\left[\begin{pmatrix} e^{u} \\ \frac{1}{3}e^{3u} \end{pmatrix}\right]_0^t + \begin{pmatrix} C_1e^{t} \\ C_2e^{-2t} \end{pmatrix}.$$

Therefore,

$$y(t) = \begin{pmatrix} e^{t}\left(e^{t}-1\right) \\ \frac{1}{3}\left(e^{t}-e^{-2t}\right) \end{pmatrix} + \begin{pmatrix} C_1e^{t} \\ C_2e^{-2t} \end{pmatrix}.$$

Simply the solution can be written as

$$y(t) = \begin{pmatrix} e^{2t} + (C_1-1)e^{t} \\ \frac{1}{3}e^{t} + (C_2-\frac{1}{3})e^{-2t} \end{pmatrix}.$$

5.2 Nuclear Magnetic Resonance

According to (Awojoyogbe and Boubaker, 2009, page 278-283). The phenomenon of Nuclear Magnetic Resonance(NRM) is the most used in modern physics and is based on bulk magnetic properties of the materials made up of certain isotopes. Bloch equation is fundamental to nuclear magnetic resonance. The matrix form of Bloch equation is written as

$$\frac{d}{dt}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} = \begin{pmatrix} \frac{-1}{T_2} & \gamma\beta_z & -\gamma\beta_y \\ -\gamma\beta_z & \frac{-1}{T_2} & \gamma\beta_x \\ \gamma\beta_y & -\gamma\beta_x & \frac{-1}{T_1} \end{pmatrix}\begin{pmatrix} M_x \\ M_y \\ M_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{M_0}{T_2} \end{pmatrix}.$$

Where $M(t) = (M_x(t), M_y(t), M_z(t))$ is nuclei magnetization moment, T_2 is the relaxation spin-spin time, T_1 is the relaxation spin-Lattice time, γ is the gyromagnetic ratio, $\beta(t) = (\beta_x(t), \beta_y(t), \beta_z(t))$ is the magnetic field experience by the nuclei and M_0 is steady state nuclear magnetization moment.

In compact form we can write Bloch equation as

$$M'(t) = RM(t) + f(t). \tag{5.2.1}$$

Solom equation is used to make easier the equation (5.2.1), refer to (Higham, 2008, page 37) the two-dimensional nuclear resonance spectroscopy is a tool for determining the structure and dynamic of molecules in solution. The basic theory for the nuclear overhauser effect experiment specifies that a matrix of intensities $M(t)$ is related to symmetric, diagonal dominant matrix R, known as the relaxation matrix.

$$\frac{dM_x}{dt} = \frac{-M_x}{T_2}$$
$$\frac{dM_y}{dt} = \frac{-M_y}{T_2}.$$

Which can be written in matrix form as

$$\begin{pmatrix} \frac{dM_x}{dt} \\ \\ \frac{dM_y}{dt} \end{pmatrix} = -\begin{pmatrix} \frac{1}{T_2} & 0 \\ 0 & \frac{1}{T_2} \end{pmatrix} \begin{pmatrix} M_x \\ M_y \end{pmatrix}.$$

In compact form it can be written as $M'(t) = -RM(t)$.

Where $M(t) = (M_x(t), M_y(t))$ and $R = \begin{pmatrix} \frac{1}{T_2} & 0 \\ 0 & \frac{1}{T_2} \end{pmatrix}$.

By assuming that $M_0 = I_2$, we can say that

$$M(t) = e^{-Rt}.$$

6. The Conclusion

This essay was based on showing that the exponential matrix is not a homomorphism in general case and determining the main conditions such that the exponential matrix is a homomorphism. Also, this work was concentrated on computing the exponential matrix using different methods and we provided some applications.

Firstly, we have seen some technical terms which help us in some computations of this project, we also discussed the exponential function as it is defined on complex matrices and complex number. The characteristic feature of taking sums to products does not hold in general, that is the matrix exponential is not a homomorphism, it does hold for commuting matrices also if the pairs of matrices are $2i\pi\mathbb{Z}$ CF.

Secondly, we have discussed on computations of the exponential matrix using the different method and we have provided some matrix which is easier to compute their exponential. Also, we have done some examples for each method.

Next, we concentrated on showing that the exponential matrix sums to products for commuting matrices, this essay has provided a number of worked examples and counterexamples. This essay has described weaker properties (such as Property L) that commuting matrices possess and some noncommuting matrices also possess, and we have described what the literature tells us about the equation $e^{A+B} = e^A e^B$.

Finally, this essay has described applications on how to solve an ordinary differential equation and how to use nuclear magnetic resonance for finding the nuclear magnetic moment.

We are unable to determine which method is the best for computing the exponential matrix and without further investigation, we are unable to determine causes of the weak property L. We are unable to use Bloch equation to find a nuclear magnetic moment. For further research, we can extend this work by finding which method is the best for computing the matrix exponential and its necessarily to show other applications in mathematics, physics and more.

References

Ahlfors, L. V. (1953). Complex analysis: an introduction to the theory of analytic functions of one complex variable. *New York, London*, page 177.

Awojoyogbe, O. B. and Boubaker, K. (2009). A solution to bloch nmr flow equations for the analysis of hemodynamic functions of blood flow system using m-boubaker polynomials. *Current Applied Physics*, 9(1):278–283.

Baker, A. (2012). *Matrix groups: An introduction to Lie group theory*. Springer Science & Business Media.

Banerjee, S. and Roy, A. (2014). *Linear algebra and matrix analysis for statistics*. Crc Press.

Bellman, R. (1997). *Introduction to matrix analysis*, volume 19. Siam.

Bourgeois, G. (2014). Property I and commuting exponentials in dimension at most three. *Bulletin of the Australian Mathematical Society*, 89(1):70–78.

Giamarchi, T. (2004). Quantum physics in one dimension. published paper.

Golub, G. H. and Wilkinson, J. H. (1976). Ill-conditioned eigensystems and the computation of the jordan canonical form. *SIAM review*, 18(4):578–619.

Higham, N. (2016). Charlie van loan and the matrix expontial. published paper.

Higham, N. J. (2008). *Functions of matrices: theory and computation*, volume 104. Siam.

Horn, R. A. and Johnson, C. R. (1990). *Matrix analysis*. Cambridge university press.

Klain, D. (version 2017). The matrix exponential (with exercises). lecture note.

Lebovitz, N. (2016). Putzer's algorithm. lecture note.

Lee, H. K. (2009). A baker–campbell–hausdorff solution by differential equation. *Journal of Physics A: Mathematical and Theoretical*, 42(13):135202.

Meyer, C. D. (2000). *Matrix analysis and applied linear algebra*, volume 71. Siam.

Moler, C. and Van Loan, C. (2003). Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM review*, 45(1):3–49.

Motzkin, T. and Taussky, O. (1952). Pairs of matrices with property I. *Transactions of the American Mathematical Society*, 73(1):108–114.

Oberhettinger, F. and Badii, L. (1973). Inverse laplace transforms. In *Tables of Laplace Transforms*, pages 207–409. Springer.

Paliogiannis, F. (2003). On commuting operator exponentials. *Proceedings of the American Mathematical Society*, 131(12):3777–3781.

Pihlak, M. (2004). Matrix integral. *Linear algebra and its applications*, 388:315–325.

Rod, P. (2018). Determinant of a matrix. ¡http://www.mathsisfun.com/algebra/matrix-determinant.html¿.

Schmoeger, C. (2000). Remarks on commuting exponentials in banach algebras, ii. *Proceedings of the American Mathematical Society*, 128(11):3405–3409.

Smalls, N. N. (2007). The exponential function of matrices.

Wahln, E. (2016). The matrix exponential. lecture note.