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Frequently Hypercyclic Behaviour of Sequence Operators in a Hypercyclic C_0-Semigroup

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Abstract

Let $\sum_j \{T^j_{1+\varepsilon}\}_{\varepsilon\geq -1}$ be a hypercyclic strongly continuous semigroup of sequence operators. Then each $T^j_{(1+\varepsilon)} (\varepsilon > -1)$ is hypercyclic as a single sequence operator, and it shares the set of hypercyclic vectors with the semigroup. This answers in the affirmative a natural question concerning hypercyclic C_0-semigroups. The analogous result for frequent hypercyclicity is also obtained.

Keywords: Hypercyclic vectors; C_0-semigroups; frequent hypercyclicity; Hypercyclic semigroups

1.Introduction

A continuous linear operator $T^j : X \to X$ on a topological vector space X is said to be hypercyclic if there is a vector $x \in X$ (called a hypercyclic vector) whose orbit under T^j,

$Orb(T^j, x) := \{(T^j)^n x: n \in \mathbb{N}\}$, is dense in X.

Ansari (1995) proved that all the powers of a hypercyclic operator T^j are also hypercyclic. Moreover, they share the same hypercyclic vectors with T^j. Recall that Ansari (1997) and Bernal González (1999) showed that every infinite-dimensional separable Banach space admits a hypercyclic sequence of operator. This result was also extended to the non-normable Fréchet case by Bonet and Peris (1998). For more details about hypercyclic sequence of operators see the surveys Bonet, Martínez-Giménez & Peris (2003), Grosse-Erdmann (1999) and Grosse-Erdmann (2003).

In the continuous case, a one-parameter family $\sum_j \mathcal{T}^j = \sum_\mathrm{j}\left\{\mathrm{T}^\mathrm{j}_{1+\varepsilon}\right\}_{\varepsilon\geq -1}$ of continuous linear sequence of operators in $L(X)$ is a strongly continuous semigroup (or C_0-semigroup) sequence of operators in $L(X)$ if $\sum_j T_0^j = I$, $\sum_j T^j_{(1+\varepsilon)} T^j_s = \sum_j T^j_{(1+\varepsilon)+s}$ for all $\varepsilon \geq -1, \mathrm{s} \geq 0$, and $lim_{(1+\varepsilon)\to s} \sum_j T^j_{(1+\varepsilon)} x = \sum_j T^j_s x$ for all $s \geq 0$, $x \in X$. For further information about C_0-semigroups we refer the reader to the books (Engel & Nagel, 2000, Pazy 1992).

A C_0-semigroup $\sum_j \mathcal{T}^j = \sum_\mathrm{j}\left\{T^j_{(1+\varepsilon)}\right\}_{\varepsilon\geq -1}$ is said to be hypercyclic if $Orb(\sum_j \mathcal{T}^j \ , x) := \sum_\mathrm{j}\left\{T^j_{(1+\varepsilon)}\mathrm{x}: \varepsilon \geq -1\right\}$ is dense in X for some $x \in X$. The investigation of hypercyclic semigroups was initiated by Desch, Schappacher and Webb (1997). So far, several specific examples of hypercyclic strongly continuous semigroups have been studied, see for example (deLaubenfels, Emamirad & Protopopescu (2000), Desch, Schappacher & Webb (1997) and Emamirad (1998)). In (2003) Bermúdez, Bonilla and Martinón proved that every separable infinite-dimensional Banach space admits a hypercyclic semigroup. This result was extended to Fréchet spaces (= ω).

Given $\sum_j T^j \in L(X)$, let us denote by $HC(T^j)$ the set of all hypercyclic vectors of T^j, and analogously, denote by $HC(\mathcal{T}^j)$ the set of hypercyclic vectors of a C_0-semigroup $\mathcal{T}^j$. It is easy to see that if $\sum_j \mathcal{T}^j = \sum_\mathrm{j}\left\{\mathrm{T}^\mathrm{j}_{(1+\varepsilon)}\right\}_{\varepsilon\geq -1}$ is a C_0-semigroup and some sequences of operator $\mathrm{T}^\mathrm{j}_{(1+\varepsilon)}$ in the semigroup is hypercyclic, then the semigroup $\mathcal{T}^j$ itself is hypercyclic.

When one analyzes the converse situation (from the continuous to the discrete case), as a consequence of an old result of Oxtoby and Ulam (1941) it is possible to establish that, if $x \in HC(\mathcal{T}^j)$, then there exists a residual set $G \subset \mathbb{R}^+$ such that $x \in HC\left(\mathrm{T}^j_{(1+\varepsilon)}\right)$ for all $t \in G$ (see, e.g., Conejero, 2004). The point here is whether $G = \mathbb{R}^+$. That is, if $\sum_j \mathcal{T}^j = \sum_j \left\{\mathrm{T}^j_{(1+\varepsilon)}\right\}_{\varepsilon \geq -1}$ is a hypercyclic C_0-semigroup, is every sequence operator $\mathrm{T}^j_{(1+\varepsilon)}, \varepsilon > -1$, hypercyclic? This problem was explicitly stated in (Bermúdez, Bonilla & Martinón 2003). The main result is the solution to this problem in the affirmative. To do this we will adapt an argument due to León-Saavedra and Müller (2004) on rotations of hypercyclic operators. This approach is not new: several authors have tried to use similar arguments to the ones in León-Saavedra and Müller (2004) for the C_0-semigroups context without success e.g., Conejero & Peris (2003), Desch & Schappacher (2006). The key point in the proof, proceeding by contradiction, is to construct a pair of continuous maps $f_j : HC(\mathcal{T}^j) \to \mathbb{T}$ and $g_j : \mathbb{D} \to HC(\mathcal{T}^j)$ such that $f_j \circ g_j|_{\mathbb{T}}$ is homotopically nontrivial. Such a point has resisted previous attempts (notice that the homotopy in Desch& Schappacher 2006) does not yield any contradiction, which results in a serious gap, and it is finally solved here.

A new trend in hypercyclicity was recently opened by the work of Bayart and Grivaux. Motivated by Birkhoff's ergodic theorem, they introduced the notion of frequent hypercyclicity, Bayart, & Grivaux (2004), Bayart & Grivaux (2006) by quantifying the frequency with which an orbit meets open sets. To be precise, let us define the lower density of a set $A_j \subset \mathbb{N}$ by $\underline{dens}(A_j) := \liminf_{N\to\infty} \#\{n \leq N: n \in A_j\}/N$. the sequence of operator $T^j \in L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ such that, for every non-empty open subset $U \subset X$, the set $\left\{n \in \mathbb{N}: \left(T^j\right)^n x \in U\right\}$ has positive lower density. Each such a vector x is called a frequently hypercyclic vector for T^j, and the set of all frequently hypercyclic vectors is denoted by $FHC(T^j)$.

Analogously, if we define the lower density of a measurable set $M \subset \mathbb{R}_+$ by $\underline{Dens}(M) := \liminf_{N\to\infty} \mu^j(M \cap [0,N])/N$, where μ^j is the Lebesgue measure on $\mathbb{R}_+$, then a C_0-semigroup $\sum_j \mathcal{T}^j = \sum_j \left\{\mathrm{T}^j_{(1+\varepsilon)}\right\}_{\varepsilon \geq -1}$ in $L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $\{1+\varepsilon \in \mathbb{R}_+ : T^j_{(1+\varepsilon)} \in$

$U\}$ has positive lower density. As before, we denote by $FHC(\mathcal{T}^j)$ the set of all hypercyclic vectors of $\mathcal{T}^j$. In both cases, frequent hypercyclicity is stronger than hypercyclicity. See also Badea, & Grivaux, Bonilla & Grosse-Erdmann, Grosse-Erdmann & Peris (2005) for further details concerning frequently hypercyclic operators and C_0-semigroups.

We prove that, if a C_0-semigroup $\sum_j \mathcal{T}^j = \sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{\varepsilon\geq -1}$ is frequently hypercyclic, then every single operator $T^j_{1+\varepsilon} \neq I$ is frequently hypercyclic.

From now on, X stands for an F^j-space over $\mathbb{K}$, where $\mathbb{K}$ denotes the field of either real or complex numbers; by an F^j-space we mean a metrizable and complete topological vector space. Let $U_0(X)$ be a base of open balanced neighbourhoods of the origin in X. Within this context, any C_0-semigroup $\sum_j \mathcal{T}^j = \sum_j \left\{T^j_{(1+\varepsilon)}: X \to X\right\}_{\varepsilon\geq -1}$ is locally equicontinuous, i.e., for any $(1+\varepsilon)_0 > 0$, the family of operators $\left\{T^j_{(1+\varepsilon)}: (1+\varepsilon) \in [0, (1+\varepsilon)_0]\right\}$ is equicontinuous. We would like to point out that there is no simplification in the proofs if we assume that X is a Banach space, and that the results remain valid for general topological vector spaces X if we assume that $\sum_j \mathcal{T}^j = \sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{\varepsilon\geq -1}$ is locally equicontinuous.

2. Hypercyclic Operators and Semigroups

We begin with some technical results. The first one is an adaptation to F^j-spaces of a result of Costakis and Peris (2002), using ideas of Wengenroth (2002).

Lemma 2.1. *Let* $\sum_j \mathcal{T}^j = \sum_j \left\{\mathrm{T}^j_{(1+\varepsilon)}\right\}_{\varepsilon\geq -1}$ be a hypercyclic semigroup in $L(X)$. *Then* $\mathrm{T}^j_{(1+\varepsilon)} - \lambda_j I$ has dense range for all $\varepsilon \geq -1$ *and* $\lambda_j \in \mathbb{K}$.

Proof. Fix arbitrarily $\lambda_j \in \mathbb{K}$ and $(1+\varepsilon)_0 > 0$. We assume $L := \overline{(T^j_{(1+\varepsilon)_0} - \lambda_j I)(X)} \neq X$, and consider the quotient map $q^j : X \to X/L$, which satisfies $q^j \circ \circ \ (T^j_{(1+\varepsilon)_0} - \lambda_j I) = 0$. Inductively, this yields $q^j \circ \left(T^j_{(1+\varepsilon)_0}\right)^n = \lambda_j^n q^j$ for all $n \in \mathbb{N}$. Consider $x \in HC(T^j)$, and define $M := q^j(Orb(\mathcal{T}^j, x)) = \left\{\lambda_j^n q^j(T^j_s x): n \in \mathbb{N}_0, s \in [0, (1+\varepsilon)_0]\right\}$, which is dense by the definition of q^j. Now we distinguish two cases.

The case $|\lambda_j| \leq 1$. Since $T_s^j x: s \in [0,(1+\varepsilon)_0]$ is bounded in X, M must be bounded, so that it cannot be dense. A contradiction.

The case $|\lambda_j| > 1$. Fix an arbitrary $y \in L$ with $q^j(y) \neq 0$. There exists an $\varepsilon > -1$ such that $q^j\left(T_{(1+\varepsilon)}^j x\right) \neq 0$. We pick $U \in \mathcal{U}_0\left(\frac{X}{L}\right)$ satisfying $q^j\left(T_{(1+\varepsilon)}^j x\right) \notin U$. The equicontinuity of $\{T_s^j x: s \in [0,(1+\varepsilon)_0]\}$ yields the existence of $V \in \mathcal{U}_0(X)$ such that $q^j\left(T_{(1+\varepsilon)}^j(V)\right) \subset U$, $(1+\varepsilon) \in [0,(1+\varepsilon)_0\,]$. Fix $(1+\varepsilon)' > (1+\varepsilon)$ with $\left(T_{(1+\varepsilon)'}^j x\right) \in V$. We write $(1+\varepsilon)' = \mathrm{m}(1+\varepsilon)_0$, for some $m \in \mathbb{N}$ and $(1+\varepsilon) \in [0,(1+\varepsilon)_0\,]$. Since $|\lambda_j| > 1$, we have $\lambda_j^m q^j\left(T_{(1+\varepsilon)}^j x\right) \notin \mathrm{U}$. On the other hand,

$$\lambda\lambda_j^m q^j\left(T_{(1+\varepsilon)}^j x\right) = q^j\left(T_{m(1+\varepsilon)_0+(1+\varepsilon)}^j x\right) = q^j\left(T_{(1+\varepsilon)}^j\left(T_{(1+\varepsilon)'}^j x\right)\right) \in q^j\left(T_{(1+\varepsilon)}^j(V)\right) \subset U,$$

which is a contradiction.

Corollary 2.2. *Let* $\sum_j \mathcal{T}^j = \sum_{\mathrm{j}}\left\{T_{(1+\varepsilon)}^j\right\}_{\varepsilon\geq-1}$ be a hypercyclic semigroup in $L(X)$. *If* $\varepsilon > -1, \left((\lambda_j)_1, (\lambda_j)_2\right) \neq (0,0)$ and $x \in HC(\mathcal{T}^j)$, then $(\lambda_j)_1 x + (\lambda_j)_2 T_{(1+\varepsilon)}^j x \in HC(\mathcal{T}^j)$.

Theorem 2.3. Let $\sum_j \mathcal{T}^j = \sum_{\mathrm{j}}\left\{T_{(1+\varepsilon)}^j\right\}_{\varepsilon\geq-1}$ be a hypercyclic semigroup in $L(X)$, and let $\mathrm{x} \in HC(\mathcal{T}^j)$. Then $x \in HC\left(T_{(1+\varepsilon)_0}^j\right)$ for every $(1+\varepsilon)_0 > 0$.

Proof. Without loss of generality, we may assume that $(1+\varepsilon)_0 = 1$. Indeed, we can consider the semigroup $\sum_j \tilde{\mathcal{T}}^j = \sum_{\mathrm{j}}\left\{\tilde{T}_{(1+\varepsilon)}^j\right\}_{\varepsilon\geq-1}$ in $L(X)$, with $\tilde{T}_{(1+\varepsilon)}^j := T_{(1+\varepsilon)(1+\varepsilon)_0}^j$ for every $\varepsilon \geq -1$. Clearly, $x \in HC(\mathcal{T}^j)$ and $\tilde{T}_1^j = T_{(1+\varepsilon)_0}^j$.

Let $\mathbb{T} := \{z \in \mathbb{C}: |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbb{C}: |z| \leq 1\}$the closed unit disc, and let $\mathbb{R}_+ := \{(1+\varepsilon) \in \mathbb{R} : \varepsilon \geq -1\}$.

We define the map $\rho: \mathbb{R}_+ \to \mathbb{T}$ by $\rho(1+\varepsilon) := e^{2\pi i(1+\varepsilon)}$. For every pair $u^j, v^j \in X$ *let*

$$F_{u^j,v^j}^j := \Big\{\lambda_{\mathrm{j}} \in \mathbb{T}: \exists((1+\varepsilon)_{\mathrm{n}})_{\mathrm{n}} \subset \mathbb{R} \text{ with } \lim_{\mathrm{n}}(1+\varepsilon)_{\mathrm{n}} = \infty,\ \lim_{\mathrm{n}} \mathrm{T}_{(1+\varepsilon)_{\mathrm{n}}}^{\mathrm{j}} u^j = v^j,$$

$$\text{and } \lim_{\mathrm{n}} \rho(1+\varepsilon)_{\mathrm{n}} = \lambda_{\mathrm{j}}\Big\}.$$

The proof is divided into several steps.

Step 1. If $u^j \in HC(\mathcal{T}^j)$, then $F_{u^j v^j}^j = \emptyset$ *for all* $v^j \in X$. Since u^j is hypercyclic for $\mathcal{T}^j$, we can find an unbounded increasing sequence $\{(1+\varepsilon)_k\}_k$ in $\mathbb{R}_+$, such that $T^j_{(1+\varepsilon)_k} u^j$ converges to v^j. By passing to a subsequence, if necessary, we may assume that $(\rho(1+\varepsilon)_k)_k$ is convergent. Its limit is an element of $F^j_{u^j,v^j}$.

Step 2. If $lim_k\, v^j_k = v^j, (\lambda_j)_k \in F^j_{u^j v^j_k}$, and $lim_k (\lambda_j)_k = \lambda_j$, then $\lambda_j \in F^j_{u^j,v^j}$. (In particular, $F^j_{u^j,v^j}$ is a closed set for each $u^j, v^j \in X$.) Indeed, for each k we select $(1+\varepsilon)_k > k$ such that $lim_k \left(T^j_{(1+\varepsilon)_k} u^j - v^j_k\right) = 0$ and $lim_k \left|\sum_j \left(\rho(1+\varepsilon)_k - (\lambda_j)_k\right)\right| = 0$. It is easy to see that $lim_k T^j_{(1+\varepsilon)_k} u^j = v^j$ and that $lim_k \rho(1+\varepsilon)_k = \lambda_j$.

Step 3. If $u^j, v^j, w^j \in X$, $\lambda_j \in F^j_{u^j,v^j}$, and $\mu^j \in F^j_{v^j,w^j}$, then $\lambda_j \mu^j \in F^j_{u^j,w^j}$. Given $U \in \mathcal{U}_0(X)$ and $\varepsilon > 0$, take $U' \in \mathcal{U}_0(X)$ such that $U' + U' \subset U$. Find $(1+\varepsilon)_1$ such that $T^j_{(1+\varepsilon)_1} v^j - w^j \in U'$ and $\left|\sum_j \left(\rho(1+\varepsilon)_1 - \mu^j\right)\right| < \varepsilon$. Pick $V \in \mathcal{U}_0(X)$ and $(1+\varepsilon)_2 > 0$ satisfying $T^j_{(1+\varepsilon)_1}(V) \subset U'$, $T^j_{(1+\varepsilon)_2} u^j - v^j \in V$, and $\left|\sum_j \left(\rho(1+\varepsilon)_2 - \lambda_j\right)\right| < \varepsilon$. Then

$$T^j_{(1+\varepsilon)_1+(1+\varepsilon)_2} u^j - w^j = T^j_{(1+\varepsilon)_1}\left(T^j_{(1+\varepsilon)_2} u^j - v^j\right) + \left(T^j_{(1+\varepsilon)_1} v^j - w^j\right) \in T^j_{(1+\varepsilon)_1}(V) + U' \subset U,$$ and

$$\left|\sum_j \left(\rho((1+\varepsilon)_1 + (1+\varepsilon)_2) - \lambda_j \mu^j\right)\right| = \left|\sum_j \left(\rho(1+\varepsilon)_1 \rho(1+\varepsilon)_2 - \lambda_j \mu^j\right)\right| \leq$$

$$\left|\sum_j \left(\rho(1+\varepsilon)_1 - \mu^j\right)\right| . \left|\rho(1+\varepsilon)_2\right| + \sum_j \left|\mu^j\right| . \left|\sum_j \left(\rho(1+\varepsilon)_2 - \lambda_j\right)\right| < 2\varepsilon .$$

Hence $\lambda_j \mu^j \in F^j_{u^j,w^j}$.

Fix now $x \in HC(\mathcal{T}^j)$. By Steps 1, 2 and 3, $F^j_{x,x}$ is a nonempty closed subsemigroup of $\mathbb{T}$. Firstly, suppose that $F^j_{x,x} = \mathbb{T}$. Then, given $y \in X$, by Steps 1 and 3 we get $F^j_{x,y} = \mathbb{T}$. In particular $1 \in F^j_{x,y}$, which yields the existence of a sequence $((1+\varepsilon)_n)_n \subset \mathbb{R}_+$ tending to infinity such that $lim_n T^j_{(1+\varepsilon)_n} x = y$ and $lim_n \rho(1+\varepsilon)_n = 1$. Write $(1+\varepsilon)_n$ as $(1+\varepsilon)_n = k_n + \varepsilon_n$ with $k_n \in \mathbb{N}$ and $\varepsilon_n \in [-1/2, 1/2]$. Then $lim_n \varepsilon_n = 0$. Let $U \in \mathcal{U}_0(X)$. We fix $U', V \in \mathcal{U}_0(X)$ with $U' + U' \subset U$ and $T^j_s(V) \subset U'$, $0 \leq s \leq 2$. Let $n \in \mathbb{N}$ be large enough such that $T^j_{(1+\varepsilon)_n}x - y \in V$ and $T^j_{1-\varepsilon_n}y - T^j_1 y \in U'$. Then we have

$$T^j_{k_n+1}x - T^j_1 y = T^j_{1-\varepsilon_n}\left(T^j_{(1+\varepsilon)_n}\, x - y\right) + \left(T^j_{1-\varepsilon_n} - T^j_1\right)y$$
$$\in T^j_{1-\varepsilon_n}(V) + U' \subset U' + U' \subset U.$$

Hence $T^j_1 y \in \overline{Orb\left(T^j_1, x\right)}$. Since T^j_1 has dense range and $y \in X$ is arbitrary, then x is hypercyclic for T^j_1.

For the rest of the proof we assume that $F^j_{x,x} \neq \mathbb{T}$, and we will show that it leads to a contradiction.

Step 4. There exists some $k \in \mathbb{N}$ such that, for each $y \in HC(\mathcal{T}^j)$, there is $\lambda_j \in \mathbb{T}$ satisfying $F^j_{x,y} = \lambda_j z{:}\, z^k = 1$. It turns out that there is $k \in \mathbb{N}$ such that $F^j_{x,x} = \{z \in \mathbb{T}{:}\, z^k = 1\}$. Indeed, given $z \in F^j_{x,x}$, the set $\{z^n{:}\, n \in \mathbb{N}\}$ is either dense in $\mathbb{T}$ or finite. Since it is contained in the closed semigroup $F^j_{x,x} \neq \mathbb{T}$, it should be finite. Now, given $y \in HC(\mathcal{T}^j)$, $\lambda_j \in F^j_{x,y}$, and $\mu^j \in F^j_{x,y}$, by Step 3, $\lambda_j F^j_{x,x} \subset F^j_{x,y}$, and $\mu^j F^j_{x,y} \subset F^j_{x,x}$, then $\#\left(F^j_{x,y}\right) = \#\left(F^j_{x,x}\right)$. This implies that $F^j_{x,y} = \lambda_j F^j_{x,x}$.

Step 5. There is a continuous function $h_j{:}\, \mathbb{D} \to \mathbb{T}$, whose restriction to the unit circle is homotopically nontrivial. A contradiction.

Let us recall that two maps $f_j, g_j : X \to Y$ are homotopic if there is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f_j(x)$and $H(x,1) = g_j(x), x \in X$. f_j is homotopically trivial if it is homotopic to a constant map. If f_j is homotopically trivial, then so are all its restrictions. Any continuous map $f_j : \mathbb{D} \to Y$ is homotopically trivial. We say that a continuous map $g_j : \mathbb{T} \to \mathbb{T}$ has index n ($n \in \mathbb{Z}$), if it is homotopic to the map $z \to z^n$. Any continuous map $g_j : \mathbb{T} \to \mathbb{T}$ has some index, and it is homotopically trivial if and only if it has index 0. We refer the reader to, e.g., Aubry (1995).

Consider the function $f_j{:}\, HC(\mathcal{T}^j) \to \mathbb{T}$ as $f_j(y) \coloneqq \lambda_j^k$, where $\lambda_j \in F^j_{x,y}$. Clearly, by Steps 2 and 4, f_j is well defined and continuous. Besides, $f_j(x) = 1$ and, since $x \in HC(\mathcal{T}^j)$, then $T^j_{(1+\varepsilon)}x \in HC(\mathcal{T}^j)$ for every $\varepsilon \geq -1$ by Corollary 2.2. Therefore it easily follows that $e^{2\pi i(1+\varepsilon)} \in F^j_{x, T^j_{(1+\varepsilon)}x}$ and $f_j\left(T^j_{(1+\varepsilon)}x\right) = e^{2\pi i(1+\varepsilon)k}$ for every $\varepsilon \geq -1$.

We will find $g_j : \mathbb{D} \to HC(\mathcal{T}^j)$ such that $h_j := f_j \circ g_j$ is the desired function which will give the contradiction. We first define $g_j : \mathbb{T} \to HC(\mathcal{T}^j)$, and then extend it to $\mathbb{D}$. To do this, since f_j is continuous at x, we find $U \in \mathcal{U}_0(X)$ such that $|f_j(y) - 1| < 1$ if $y \in HC(\mathcal{T}^j)$ and $y - x \in U$. We now fix $(1+\varepsilon)_0 > 1$ satisfying $T^j_{(1+\varepsilon)_0}x - x \in U$. Let us define $g_j : \mathbb{T} \to HC(\mathcal{T}^j)$ by

Clearly, g_j is well defined and continuous. By Corollary 2.2, we have $g_j(\mathbb{T}) \subset HC(\mathcal{T}^j)$. Since U is balanced, $g_j\left(e^{2\pi i(1+\varepsilon)}\right) - x \in U$, for $-1/2 \le \varepsilon < 0$. This implies

$$\left|f_j\left(g_j\left(e^{2\pi i(1+\varepsilon)}\right)\right) - 1\right| < 1, -1/2 \le \varepsilon < 0. \quad \text{Moreover}$$

$f_j\left(g_j\left(e^{2\pi i(1+\varepsilon)}\right)\right) = e^{4\pi i(1+\varepsilon)(1+\varepsilon)_0 k}, -1 \le \varepsilon < -1/2$, which yields that the index of $f_j \circ g_j$ at 0 is between $[(1+\varepsilon)_0]k$ and $([(1+\varepsilon)_0]+1)k$ (depending on the difference $(1+\varepsilon)_0 - [(1+\varepsilon)_0]$. -

We extend the function g_j to $\mathbb{D}$ by defining $g_j(z) := (1-|z|)x + |z|g_j(z/|z|)$ for each $z \neq 0$, and $g_j(0) = x$. Clearly, this extension is also continuous on $\mathbb{D}$, and $g_j(z) \in HC(\mathcal{T}^j)$ for every $z \in \mathbb{D}$ since $g_j(z)$ is a non-zero linear combination of x and $T^j_{(1+\varepsilon)}x$, for some $0 < (1+\varepsilon) \le (1+\varepsilon)_0$ (Corollary 2.2).

To sum up, we have a continuous function $h_j := f_j \circ g_j : \mathbb{D} \to \mathbb{T}$, such that its restriction to the unit circle is homotopically nontrivial, a contradiction.

3. Frequently hypercyclic operators and semigroups

We prove the analogous result for the stronger concept of frequent hypercyclicity. We first need a technical lemma concerning the frequently hypercyclic vectors of a C_0-semigroup $\mathcal{T}^j$.

Lemma 3.1. Let $\sum_j \mathcal{T}^j = \sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{\varepsilon \ge -1}$ be a frequently hypercyclic semigroup in $L(X)$, and let $x \in FHC(\mathcal{T}^j)$. For every $k \in \mathbb{N}$, $y \in X$, and $U \in \mathcal{U}_0(X)$

$\underline{Dens}\left(\left\{(1+\varepsilon) \in \bigcup_{n\in\mathbb{N}}\left[n-1/k,n\right): T^j_{(1+\varepsilon)}x - y \in U\right\}\right) > 0.$

Proof. Clearly, $T^j_{r/k}x \in HC(\mathcal{T}^j)$ for every $r = 0,\dots,k-1$, and even more, $T^j_{r/k}x \in HC(T^j_1)$ by Theorem 2.3. Fix $U, U' \in \mathcal{U}_0(X)$ such that $U' + U' \subset U$, and $y \in X$. Then there are some $n_r \in \mathbb{N}$ such that $T^j_{n_r+r/k}x - y \in U'$, $r = 0,\dots,k-1$. Besides, there is some $V \in \mathcal{U}_0(X)$ such that $T^j_s(V) \subset U'$ if $s \le N_0 := \max\{n_r : r = 0,\dots,k-1\} + 1$. Since $x \in FHC(\mathcal{T}^j)$, we have $\underline{Dens}(\{(1+\varepsilon) \in \mathbb{R}^+ : T^j_{(1+\varepsilon)}x - x \in V\}) > 0$. So there are $C > 0$ and $N_1 \in \mathbb{N}$ such that $\mu(\{\varepsilon \le N-1 : T^j_{(1+\varepsilon)}x - x \in V\}) \ge CN$ for every $N \ge N_1$.

For every $N \in \mathbb{N}$, let us define $L := \left\{\varepsilon \le N-1 : T^j_{(1+\varepsilon)}x - x \in V\right\}$. In addition, for every $j = 0,\dots,k-1$, we define the sets $I_j := n[n + j/k, n + (j+1)/k)$, $L_j := L \cap I_j$, and the mapping

$f_j : \mathbb{R}_+ \to \mathbb{R}_+$ as $f_j(1+\varepsilon) := (1+\varepsilon) + n_{k-j-1} + (k-j-1)/k$. These mappings satisfy that $f_j(1+\varepsilon) \in I_{k-1}$ for every $\varepsilon \in L_j - 1$, and

$$
\begin{aligned}
T^j_{f_j(1+\varepsilon)}x - y &= T^j_{n_{k-j-1+(k-j-1)/k}}\left(T^j_{(1+\varepsilon)} - x\right) + \left(T^j_{n_{k-j-1+(k-j-1)/k}}x - y\right) \\
&\in T^j_{n_{k-j-1+(k-j-1)/k}}(V) + U' \subset U.
\end{aligned}
$$

Finally, for $N \ge N_0 + N_1$ we have

$$\mu(\{\varepsilon \le 2N-1 : T_{(1+\varepsilon)}x - y \in U \text{ and } \varepsilon \in I_{k-1} - 1\})$$

$$\ge \mu\left(\bigcup_{j=0}^{k-1} f_j(L_j)\right) \ge \sum_{j=0}^{k-1} \mu f_j(L_j)/k$$

$$= \sum_{j=0}^{k-1} \mu(L_j)/k = \mu(L)/k \ge CN/k$$

Hence $Dens\left(\left\{\varepsilon \in I_{k-1} - 1 : T^j_{(1+\varepsilon)}x - y \in U\right\}\right) > 0$, and we are done.

Theorem 3.2. Let $\sum_j \mathcal{T}^j = \sum_j \left\{T^j_{(1+\varepsilon)}\right\}_{\varepsilon \ge -1}$ be a frequently hypercyclic semigroup in $L(X)$, and let $x \in FHC(\mathcal{T}^j)$. Then $x \in FHC(T^j_{(1+\varepsilon)_0})$ for every $(1+\varepsilon)_0 > 0$.

Proof. Without loss of generality, we may again assume that $(1+\varepsilon)_0 = 1$ as in the proof of Theorem 2.3. Fix $y \in X, U \in \mathcal{U}_0(X)$, and select $k \in \mathbb{N}, U' \in \mathcal{U}_0(X)$, such that $U' + U' \subset \mathrm{U}$ and $T^j_{(1+\varepsilon)}y - y \in U'$ for every $-1 \leq \varepsilon \leq 1/k$. Since $\mathcal{T}^j$ is strongly continuous there is some $V \in \mathcal{U}_0(X)$ such that $T^j_{(1+\varepsilon)}(V) \subset U'$ for every $-1 \leq \varepsilon \leq 1/k$. By the previous lemma, we know that $\underline{Dens}\left(\left\{(1+\varepsilon) \in \bigcup_{n\in\mathbb{N}}\left[n-\frac{1}{k},n\right) : T^j_{(1+\varepsilon)}x - y \in V\right\}\right) > 0$.

If $(1+\varepsilon) \in [n - 1/k, n)$ for some $n \in \mathbb{N}$ and $T^j_{(1+\varepsilon)}x - y \in V$, then we define $\eta_{(1+\varepsilon)} \coloneqq [1+\varepsilon] - \varepsilon$. Each $\eta_{(1+\varepsilon)}$ satisfies $0 < \eta_{(1+\varepsilon)} \leq 1/k$, and $(1+\varepsilon) + \eta_{(1+\varepsilon)} \in \mathbb{N}$. So

$T^j_{(1+\varepsilon)+\eta_{(1+\varepsilon)}}x - y = T^j_{\eta_{(1+\varepsilon)}}\left(T^j_{(1+\varepsilon)}x - y\right) + \left(T^j_{\eta_{(1+\varepsilon)}}y - y\right) \in T^j_{\eta_{(1+\varepsilon)}}(V) + U' \subset U$. Hence

$\underline{dens}\left(\left\{n \in \mathbb{N}: T^j_n x - y \in U\right\}\right) \geq \underline{Dens}\left(\left\{(1+\varepsilon) \in \bigcup_{n\in\mathbb{N}}\left[n-\frac{1}{k},n\right) : T^j_{(1+\varepsilon)}x - y \in V\right\}\right) > 0$.

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