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Interpretation of the Physical Significance of the Fourier Analysis of Signals using Complex Exponentials

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Abstract

Fourier analysis amount to finding the Fourier coefficient of the signal in question and then substituting these coefficients into the general complex Fourier series $\left(C_o + C_n \sum_{-\infty}^{\infty} e^{jnx} \right)$ *expression, from which the properties of the signal been analyzed can be deduced. Through this analysis, it was found that the saw-tooth wave form consists of a continuous sine series:* $y(t) = {}^{p}\!/_{2} + {}^{p}\!/_{\pi}\left[Sin\,\omega t + \frac{1}{2} Sin2\omega t + \frac{1}{3} Sin3\omega t + ... \right]$ *and in the same manner the triangular wave form was analyzed and found to consist of discrete sine series:* $y(t) = {}^{p}\!/_{2} + {}^{2p}\!/_{\pi}\left[Sinx + \frac{1}{3} Sin3x + \frac{1}{5} Sin5x + ... \right]$*. Through this work I have used the complex exponential* $\left(e^{-jnx}\right)$ *method in finding the Fourier coefficients* $\left(C_n = {}^{1}\!/_{2\pi} \int_{-\pi}^{\pi} f(x) e^{-jnx} dx \right)$ *and e then used trigonometric approach to authenticate my results. At the end of this work I have also used the Fourier analysis to analyze how well do the half and the full wave rectifier diode valve converts an A.C to a D.C. And for the half-wave rectifier diode the expression was:* $I(t) = {}^{I_o}\!/_{\pi} - {}^{2I_o}\!/_{\pi}\left[\frac{1}{3} Cos2\omega t + \frac{1}{15} Cos4\omega t + ... \right] + \frac{1}{2} I_o Sin\,\omega t$*. While that of a full-wave rectifier the expression was:* $I(t) = {}^{2}\!/_{\pi} - {}^{4}\!/_{\pi}\left[\frac{1}{3} Cos2\omega t + \frac{1}{5} Cos4\omega t + ... \right]$*. Hopefully by the end of this work, the reader can be able to analyze a signal or an electronic gadgets such as the diode-valve and the like, by the use of Fourier analysis and predict with precision their behavior by just looking at the final expression, also called the design equation.*

Keywords:

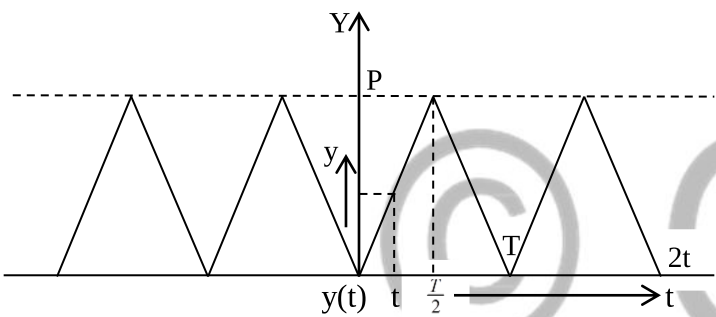
Fourier analysis, Fourier coefficients, complex exponentials, discrete sine series, discrete cosine series, full and half wave rectifier, diode valve, square wave, triangular wave, and the saw-tooth wave

DISCUSSIONS

Any periodic function or signal can be expressed as a sum of sinusoids e.g $f(x) = {}^{a_o}\!/_{2} + a_n Cosnx + b_n Sinnx$. To explain this expression in a layman's language. Suppose you have a full loaf of bread and then divide this loaf of bread into two equal halves. Each half of the bread can be thought of as the average value of the signal, D.C, $a_o/2$, value (zero-frequency). Now keep one half of the bread as it is, but divide the other half into unequal pieces. Now if you continue to add the divided pieces of the bread to the other half of the bread, you will eventually get back full loaf of bread. And if you want to modify the shape of the bread, you will simply refuse to add the some divided pieces of the bread to the other half. Obviously now, you can modify the shape of this bread into an infinite number of different shapes. Now you can think of the sine and cosine as the divided pieces of the bread. The same procedure is applied during audio compression, take or record a voice as a signal e.g an audio speech, decompose these into the average and the sine terms. You can now drop or neglect some of these sine terms and them recombined the remaining terms to give back the original signal. The argument is that, the human ear cannot be able to notice any difference between the original and the modified (sound). All these procedures are a function of the Fourier coefficients therefore all the reason why the concepts of finding the Fourier coefficients are important.

FOURIER ANALYSIS OF SIGNALS:

1. THE TRIANGULAR WAVE



(Fig. 1)

The triangular wave is defined by:

Y = 0	at	t = 0
Y = p	at	t = T/2
Y = 0	at	t = T

And this repeats every period T, as shown in fig. (1) from which we can deduce;

$$y = \frac{2pt}{T}\left(0 < t < T/2\right)$$

And

$$y = \frac{2p(T-t)}{T}\left(T/2 < t < T\right)$$

Recall that: the Fourier series in complex form is:

$$f(x) = C_o + \sum_{n=\infty}^{\infty} C_n e^{jnx}$$

Where C_o and C_n are the coefficients to be calculated.

Using: $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}dx$

So that,

$$C_n = \frac{1}{\pi}\left[\int_0^{T/2}\frac{2pt}{T} + \int_{T/2}^{T}\frac{2p(T-t)}{T}\right]e^{-j\cdot 0.\omega t}dt$$

On integrating we get,

$$C_o = \frac{2p}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2} + \left[T(t) - \frac{t^2}{2}\right]_{T/2}^{T}$$

Substituting the limits we get,

$$C_o = \frac{2p}{T^2}\left[\frac{T^2}{8}\right] + \frac{2p}{T^2}\left[T^2 - \frac{T^2}{2}\right] - \frac{2p}{T^2}\left[\frac{T^2}{2} - \frac{T^2}{8}\right]$$

$$C_o = \frac{p}{4} + 2p - p - p + \frac{p}{4} = \frac{2p}{4}$$

$$\boxed{\therefore C_o = \frac{p}{2}}$$

For C_n using

$$C_n = \underbrace{\frac{2p}{T^2}\int_0^{T/2} t \cdot e^{-jn\omega t}dt}_{C_{n_1}} + \underbrace{\frac{2p}{T^2}\int_{T/2}^{T}(T-t)\cdot e^{-jn\omega t}dt}_{C_{n_2}}$$

So that $C_n = C_{n_1} + C_{n_2}$

From,

$C_{n_1} = \frac{2p}{T^2}\int_0^{T/2} t \cdot e^{-jn\omega t}dt$ (using integration by parts)

That is $I = U\int V - \int\left(U^1\right)\int V$

$$C_{n_1} = \frac{2p}{T^2}\left[\frac{t}{-jn\omega}\cdot e^{-jn\omega t} - \frac{1}{j^2n^2\omega^2}\cdot e^{-jn\omega t}\right]_0^{T/2}$$

Substituting the limit we get,

$$= \frac{2p}{T^2}\left[\frac{T}{-2jn\omega}\cdot e^{-jn\omega T/2} - e^{-jn\omega T/2} - \left(0 - \frac{1}{j^2n^2\omega^2}\right)\right]$$

$$= \frac{2p}{T^2}\left[\frac{T\cdot(-1)}{-2jn\omega} + \frac{(-1)^n}{n^2\omega^2} - \frac{1}{n^2\omega^2}\right] \text{ Where } j^2 = -1$$

If n = odd then,

$$C_{n_1} = \frac{2p}{T^2}\left[\frac{T\cdot(-1)}{-2jn\omega} + \frac{(-2)}{n^2\omega^2}\right]$$

For C_{n_2} we get

$$C_{n_2} = \frac{2p}{T^2}\int_{T/2}^{T}(T-t)\cdot e^{-jn\omega t}dt$$

After integrating, the first term and second term by part

$$C_{n_2} = \left[\frac{T}{-jn\omega}\cdot e^{-jn\omega t}\right]_{\frac{T}{2}}^{T} - \left[\frac{t}{-jn\omega}\cdot e^{-jn\omega t} - \left(\frac{1}{j^2n^2\omega^2}\cdot e^{-jn\omega t}\right)\right]_{\frac{T}{2}}^{T} \times \frac{2p}{T^2}$$

Substituting the limits we get,

$$C_{n_2} = \frac{2p}{T^2}\left[\frac{T\cdot e^{-jn\omega T}}{-jn\omega} - \frac{T\cdot e^{-jn\omega T/2}}{-jn\omega}\right] - \left[\frac{T\cdot e^{-jn\omega T}}{-jn\omega} - \frac{1\cdot e^{-jn\omega T}}{j^2n^2\omega^2} - \left(\frac{T\cdot e^{-jn\omega T/2}}{-2jn\omega} - \frac{1\cdot e^{-jn\omega t}}{j^2n^2v^2}\right)\right]$$

For n = even

$$C_{n_2} = \frac{2p}{T^2}\left\{\left[\frac{2T}{-jn\omega}\right] - \left[\frac{T(1)}{jn\omega}\right] - \left[\frac{T(-1)^n}{-jn\omega} - \frac{1}{j^2n^2\omega^2} - \frac{(-1)^n}{j^2n^2\omega^2}\right]\right\}$$

For n = odd (1, 3, 5)

$$C_{n_2} = \frac{2p}{T^2}\left[\frac{-2T}{jn\omega} + \frac{3T}{2jn\omega} - \frac{2}{n^2\omega^2}\right] \text{ Where } j^2 = -1$$

$$\therefore C_{n_2} = \frac{2p}{T^2}\left[\frac{-T}{2jn} - \frac{2}{n^2\omega^2}\right]$$

So that $C_n = C_{n_1} + C_{n_2}$

$$\therefore C_n = \left[\frac{2(-1)^n}{-2jn\omega} + \frac{(-2)}{n^2\omega^2} + \frac{(-2)}{n^2\omega^2} - \frac{T}{-2jn\omega}\right] \times \frac{-2p}{T^2}$$

$$\Rightarrow C_n = \frac{2p}{T}\left[0 - \frac{4}{n^2\omega^2}\right] = \frac{-8p}{T^2n^2\omega^2} = \frac{-8p}{4\pi^2n^2} = \frac{-2p}{n^2\pi^2}$$

$$\therefore C_n = \frac{-2p}{n^2\pi^2}$$

Now substituting these coefficients into the Fourier series expression we get,

$$f(t) = C_o + \sum_{-\infty}^{\infty} C_n e^{jn\omega t} = C_o + C_n\sum_{-\infty}^{\infty} e^{jn\omega t}$$

But $C_o = \frac{p}{2}$ and $C_n = \frac{-2p}{n^2\pi^2}$ therefore,

$$f(t) = p/2 + \left(\frac{-2p}{\pi^2}\right)\left[\ldots + \frac{e^{-j\omega t}}{-1\times n^2} + \frac{e^{-j3\omega t}}{-3\times n^2} + \frac{e^{-j5\omega t}}{-5\times n^2} + \ldots + \frac{e^{j\omega t}}{1\times n^2} + \frac{e^{j3\omega t}}{3\times n^2} + \frac{e^{j5\omega t}}{5\times n^2}\right]$$

$$\therefore f(t) = \frac{p}{2} - \frac{4p}{\pi^2}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2\times 1^2} + \frac{e^{j3\omega t} - e^{-j3\omega t}}{2\times 3^2} + \frac{e^{j5\omega t} - e^{-j5\omega t}}{2\times 5^2} + \ldots\right]$$

But $\frac{e^{j\omega t} - e^{-j\omega t}}{2} = \cos\omega t$ (Euler's formula)

$$\therefore f(t) = \frac{p}{2} - \frac{4p}{\pi^2}\left[\cos\omega t + \frac{1}{3^2}\cos 3\omega t + \frac{1}{5^2}\cos 5\omega t + \ldots\right]$$

Interpretation of the physical significance of the triangular wave expression

(i). It consist of all cosine terms, apart from P/2 (average value)

(ii). The convergence is fast $\left(\frac{1}{n^2}\right)$ meaning by just adding a few terms you will get back the 'original' signal or function.

(iii). It is a discrete cosine series (ideal for image processing)

PROOF: <u>USING TRIGONOMETRIC APPROACH FOR THE TRIANGULAR WAVE:</u>

From $f(t) = a_o + \sum_{n=1}^{\infty} a_n Cosn\omega t + \sum_{n=1}^{\infty} b_n Sin\omega t$

$$a_o = \frac{1}{T}\int_0^{T/2} \frac{2pt}{T}dt + \frac{1}{T}\int_{T/2}^{T} \frac{2p(T-t)}{T}dt$$

Integrating w.r.t t, we get

$$a_o = \frac{2p}{T^2}\left[\frac{t^2}{2}\right]_0^{\frac{T}{2}} + \frac{2p}{T^2}[t]_{\frac{T}{2}}^{T} - \frac{2p}{T^2}\left[\frac{t^2}{2}\right]_{\frac{T}{2}}^{T}$$

Substituting the limits we get

$$a_o = \frac{2p}{T^2}\times\frac{T^2}{8} + \frac{2p}{T^2}\times\left[T^2 - \frac{T^2}{2}\right] - \frac{2p}{T^2}\left[\frac{T^2}{2} - \frac{T^2}{8}\right]$$

$$= \frac{p}{4} + p - p + \frac{p}{4} = \frac{p}{4} + \frac{p}{4} = \frac{2p}{4} = \frac{p}{2}$$

$C_o = \frac{p}{2}$ $\quad \boxed{\therefore a_o = \frac{p}{2}}$ $\quad$ Q.E.D

For the coefficient a_n using:

$$a_n = \frac{2}{T}\int_0^{T/2}\left(\frac{2at}{T}\right)Cosn\omega tdt + \int_{T/2}^{T}\frac{2a(T-t)}{T}Cosn\omega tdt$$

Integration of the first term by parts and substituting the limits we get

$$a_{n_1} = \frac{4p}{T^2 n^2 \omega^2}\left[Cos\frac{n\omega t}{2} - 1\right] \qquad \text{Part (1)}$$

Integrating the second term by parts and substituting the limit we get

$$a_{n_2} = \frac{4p}{T}\left[\frac{1}{n\omega}Sinn\omega t\Big|_{\frac{T}{2}}^{T} - \frac{4p}{T^2n^2\omega^2}(Cosn\omega t)\Big|_{\frac{T}{2}}^{T}\right.$$

But $Sinn\omega t = 0$ for any value of n

$$\Rightarrow 0 - \frac{4p}{T^2n^2\omega^2}\left[Cos\frac{n\omega t}{2} - 1 + \left(\frac{4p}{T^2n^2\omega^2}\left[Cosn\omega t - Cos\frac{n\omega t}{2}\right]\right.\right.$$

But $\omega t = 2\pi$

$$\therefore a_n = \frac{p}{n^2\pi^2}[2Cosn\pi - Cos2n\pi - 1]$$

For n = even, 0, 2, 4 we get

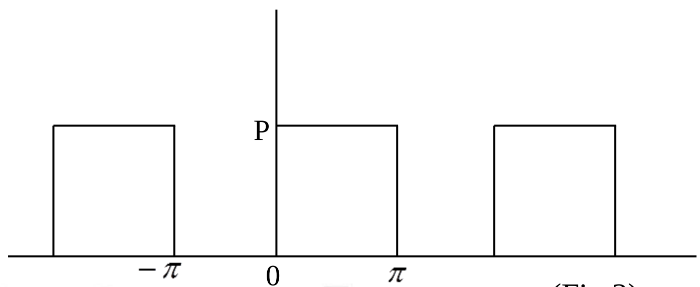
$a_n = 0$

$$a_n = \frac{p}{n^2\pi^2}[2Cos\pi - Cos2\pi - 1]$$

$$\therefore a_n = \frac{p}{n^2\pi^2}[-2-1-1] = \boxed{\frac{-4p}{n^2\pi^2}} = C_n$$

Q.E.D

(2). <u>SQUARE WAVEFORM (High Frequencies)</u>



(Fig 2)

The above square wave can be defined by the function f(x) such that

$f(x) = 0$ from $-\pi < x < 0$

and

$f(x) = p$ from $0 < x < \pi$

To analyze the properties of the square wave all we need is the Fourier coefficients given by;

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}dx$$

So that,

$$C_n = \frac{1}{2\pi}\int_0^{\pi} p\cdot e^{-j\cdot 0x}dx = \frac{p}{2\pi}[x]_0^{\pi} = \frac{p}{2\pi}\times\pi = \frac{p}{2}$$

$$\therefore \boxed{C_o = \frac{p}{2}}$$

For C_n using;

$$C_n = \frac{1}{2\pi}\int_0^{\pi} p \cdot e^{-jnx} dx = \frac{p}{2\pi}\left[\frac{1}{-jn} \times e^{-jnx}\right]_0^{\pi}$$

Substituting the limit we get,

$$\therefore C_n = \frac{p}{2\pi}\left[\frac{e^{-jn\pi}}{-jn} - \frac{e^{0}}{-jn}\right] = \frac{p}{2\pi}\left[\frac{1}{-jn}\left(e^{-jn\pi} - 1\right)\right]$$

$\therefore C_n = \frac{p}{-2\pi jn}\left[e^{-jn\pi} - 1\right] = \frac{p}{-jn2\pi}\left((-1)^n - 1\right) = 0$ for n = even

But if n = odd 1, 3, 5 we get

$$C_n = \frac{p}{-jn2\pi}\left((-1)^n - 1\right) = \frac{-2p}{-jn2\pi} = \frac{p}{jn\pi}$$

$$\therefore \boxed{C_n = \frac{p}{jn\pi}}$$

Writing the Fourier series we get

$$f(x) = C_o + \sum_{-\infty}^{\infty} C_n e^{jnx}$$

But $C_o = \frac{p}{2}$ and $C_n = \frac{p}{jnx}$

$$\therefore f(x) = \frac{p}{2} + \frac{p}{\pi}\sum_{-\infty}^{\infty}\frac{e^{jnx}}{jn}$$

$$\Rightarrow f(x) = \frac{p}{2} + \frac{p}{\pi}\left[\frac{e^{-jx}}{-1\times j} + \frac{e^{-3jx}}{-3\times j} + \frac{e^{-5jx}}{-5\times j} + \ldots + \frac{e^{jx}}{1\times j} + \frac{e^{3jx}}{3\times j} + \frac{e^{5jx}}{5\times j} + \ldots\right]$$

$$\Rightarrow f(x) = \frac{p}{2} + \frac{2p}{\pi}\left[\frac{e^{jx} - e^{-jx}}{2\cdot j} + \frac{e^{3jx} - e^{-3jx}}{2\times 3\times j} + \frac{e^{5jx} - e^{-5jx}}{2\times 5\times j} + \ldots\right]$$

But $\frac{e^{jx} - e^{-jx}}{2j} = Sinx$ (Euler's formula)

$$\therefore f(x) = \frac{p}{2} + \frac{2p}{\pi}\left[Sinx + \frac{1}{3}Sin3x + \frac{1}{5}Sin5x + \ldots\right]$$

(i). All the cosine have vanished showing that it may be used for audio compression.

(ii). The convergence is slow $\left(\frac{1}{n}\right)$ meaning a square wave have a lot of high frequencies. This implies that a lot of terms should be added together before obtaining the 'original' signal.

PROOF: <u>USING TRIGONOMETRIC APPROACH FOR THE SQUARE WAVE</u>

From $y(t) = a_0 + \sum_{n=1}^{\infty} a_n Cosn\omega t + \sum_{n=1}^{\infty} b_n Sinn\omega t$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} pdt = \frac{p}{2\pi}[t]_0^{\pi} = \frac{p\pi}{2\pi} = \frac{p}{2}$$

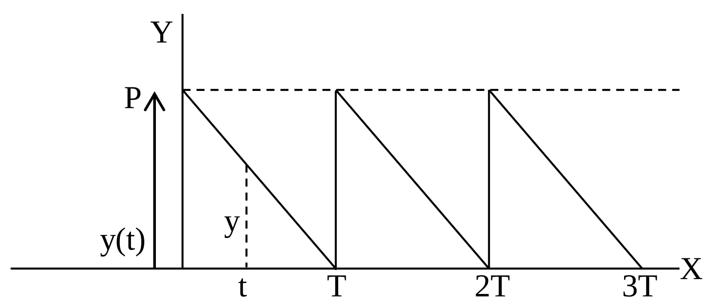
$\therefore \boxed{a_0 = \frac{p}{2}}$ Q.E.D

$$a_0 = 0$$

$b_n = \frac{1}{\pi}\int_0^{\pi} pSinntdt = \frac{p}{n\pi}[1 - Cosn\pi] = \frac{2p}{n\pi}$ for n = odd and 0 for n = even

$\therefore \boxed{b_n = \frac{2p}{\pi} \quad n = 1, 3, 5}$ Q.E.D

(3) <u>THE SAW TOOTH WAVEFORM:</u>



For linear displacement w.r.t time then; y = P at t = 0 to y = 0 at t = T and more. We can then safely write;

$\frac{y}{p} = \frac{T-t}{T}$ or $y = p\left(\frac{T-t}{T}\right) = p\left(1 - \frac{t}{T}\right)$

Recall that: by using the complex analysis we have, $f(x)=\sum_{-\infty}^{\infty} C_n e^{jnx}$ and in order to analyze this signal we need its Fourier coefficients by using;

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}dx$$

If $C_o = \frac{p}{T}\int_0^T\left(1-\frac{t}{T}\right)e^{-jn\omega 0} = \frac{p}{T}\int_0^T\left(1-\frac{t}{T}\right)\cdot e^0$

$$\therefore C_o = \frac{p}{T}\left[t-\frac{t^2}{2}\right]_0^T = \frac{p}{T}\left[T-\frac{T^2}{2T}\right] = \frac{p}{T}\times\frac{T}{2}=\frac{p}{2}$$

$$\Rightarrow \boxed{C_o=\frac{p}{2}}$$

For C_n, using;

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-jnx}dx$$

$$\Rightarrow C_n = \frac{1}{T}\int_0^T p\left(1-\frac{t}{T}\right)e^{-jn\omega t}dt = \frac{p}{T}\int_0^T e^{-jn\omega t}dt - \frac{p}{T^2}\int_0^T te^{-jn\omega t}dt$$

Let $C_{n_1} = \frac{p}{T}\int_0^T e^{-jn\omega t}dt = \frac{p}{T}\left[\frac{e^{-jn\omega t}}{-jn}\right]_0^T$

Substituting the limit we get,

$$C_{n_1} = \frac{p}{T}\left[\frac{e^{-jn\omega t}}{-jn}-\frac{1}{-jn}\right] = \frac{p}{-jnT}\left(e^{-jn\omega t}-1\right) = \frac{p}{-Tjn}[1-1]=0$$

Let

$C_{n_2} = \frac{-p}{T^2}\int_0^T t\cdot e^{-jn\omega t}dt$ Integrating by parts we get,

$= \frac{-p}{T^2}\left[\frac{t\cdot e^{-jn\omega t}}{-jn} - \frac{1\cdot e^{-jn\omega t}}{j^2n^2}\right]_0^T$ Substituting the limit we get

$$C_{n_2} = \frac{-p}{T^2}\left[\frac{T\cdot e^{-jn\omega T}}{-jn} - \frac{e^{-jn\omega T}}{j^2n^2} - \left(0-\frac{1}{j^2n^2}\right)\right]$$

$$C_{n_2} = \frac{-p}{T^2}\left[\frac{T\cdot 1}{-jn}-\frac{1}{j^2n^2}+\frac{1}{j^2n^2}\right] = \frac{-p}{T^2}\left[\frac{T}{-jn}+0\right]$$

$$\therefore \boxed{C_{n_2}=\frac{p}{2\pi jn}}$$

$$C_n = C_{n_1}+C_{n_2} = 0+\frac{p}{2\pi jn}$$

$$\Rightarrow \boxed{C_n = \frac{p}{2\pi jn}}$$

Writing the Fourier series we get,

$y(t)=\sum_{n=1}^{\infty} C_n e^{jnx}$ but $C_o=\frac{p}{2}$ and $C_n=\frac{p}{2\pi jn}$

$$\therefore y(t) = C_o + C_n\sum_{n=1}^{\infty} e^{-jn\omega t} = \frac{p}{2}+\frac{p}{2\pi}\sum_{-\infty}^{\infty}\frac{e^{+jn\omega t}}{nj}$$

$$\Rightarrow y(t)=\frac{p}{2}+\frac{p}{2\pi}\left[...+\frac{e^{-j\omega t}}{-1\times j}+\frac{e^{-2j\omega t}}{-2\times j}+\frac{e^{-3j\omega t}}{-3\times j}+...+\frac{e^{j\omega t}}{1\times j}+\frac{e^{2j\omega t}}{2\times j}+\frac{e^{3j\omega t}}{3\times j}+...\right]$$

$$\Rightarrow y(t) = \frac{p}{2}+\frac{2p}{2\pi}\left[\frac{e^{j\omega t}-e^{-j\omega t}}{2j}+\frac{e^{2j\omega t}-e^{-2j\omega t}}{2\times 2j}+\frac{e^{3j\omega t}-e^{-3j\omega t}}{3\times 2j}+...\right]$$

But $\frac{e^{j\omega t}-e^{-j\omega t}}{2j}=Sin\omega t$ (Euler's formula)

$$\therefore y(t)=\frac{p}{2}+\frac{p}{\pi}\left[Sin\omega t+\frac{1}{2}Sin2\omega t+\frac{1}{3}Sin3\omega t+...\right]$$

INTERPRETATION OF THE PHYSICAL SIGNIFICANCE OF THE ABOVE SAW-TOOTH ANALYSIS EXPRESSION

i. It contains Sine terms only
ii. It is a continuous Sine series (Ideal for audio compression)
iii. It shows a slow convergence $\left(\frac{1}{n}\right)$ which means it may require a lot of terms to add up to the 'original' signal.
iv. It also indicates that while amplitude decreases the frequency increases, which

implies that the convergence to the original signal is assured.

PROOF: <u>USING THE TRIGONOMETRIC APPROACH FOR THE SAW-TOOTH WAVE</u>

Using; $y(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n Cosn\omega t + \sum_{n=1}^{\infty} b_n Sin\omega t$

To analyze the properties of this signal we need their Fourier coefficients, a_o, a_n and b_n

$$a_o = \frac{1}{T}\int_0^T y(t)dt = \frac{p}{T}\int_0^T \left(1 - \frac{t}{T}\right)dt$$

By integrating w.r.t t we get,

$$a_o = \frac{p}{T}\left[t - \frac{t^2}{2T}\right]_0^T = \frac{p}{T}\left[T - \frac{T^2}{2T}\right] = \frac{p}{T}\left[\frac{T}{2}\right] = \frac{p}{2}$$

$$\Rightarrow a_o = \frac{p}{2} = C_o \qquad \text{Q.E.D}$$

$a_n = \frac{2p}{T}\int_0^T \left(1 - \frac{t}{T}\right) Cosn\omega t dt$ by integrating we get,

$$a_n = \frac{2p}{T}\left[\frac{1}{n\omega} Sinn\omega t\Big|_0^T - \frac{1}{T}\left(\frac{t}{n\omega} Sinn\omega t\Big|_0^T + \frac{1}{T}\int_0^T 1\cdot\frac{1}{n\omega} Sinn\omega t dt\right)\right.$$

$Sinn\omega t = 0$ for any value of n

$$\therefore a_n = \frac{2p}{T}\left[0 - 0 + \frac{1}{T}\int_0^T \frac{1}{n\omega} Sinn\omega t dt = \frac{2p}{Tn\omega}\int_0^T Sinn\omega t dt\right.$$

$$= \frac{2p}{Tn^2\omega^2}(1 - Cosn\omega t) \quad \text{But} \quad \omega T = 2\pi$$

$$\therefore a_n = \frac{2p}{Tn^2\omega^2}(1 - Cosn2\pi) \quad \Rightarrow a_n = 0$$

For coefficient b_n

$$b_n = \frac{2p}{T}\int_0^T \left(1 - \frac{t}{T}\right) Sinn\omega t dt = \frac{2p}{T}\left[\int_0^T Sinn\omega t dt - \int_0^T \frac{t}{T} Sinn\omega t dt\right]$$

By integrating w.r.t to t we get,

$$b_n = \frac{2p}{T}\left[\frac{-1}{n\omega} Cosn\omega t + \frac{1}{n\omega} + \frac{1}{n\omega} Cosn\omega T\right]$$

$$\therefore b_n = \frac{2p}{Tn\omega} = \frac{2p}{Tn \times \frac{2\pi}{T}} = \frac{p}{n\pi}$$

$$\therefore \boxed{b_n = \frac{p}{n\pi}} = C_n \quad \text{Q.E.D}$$

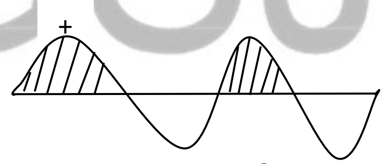
<u>FOURIER ANALYSIS OF ELECTRONIC COMPONENTS (DIODE-VALVE)</u>

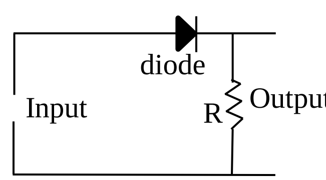
<u>HALF-WAVE RECTIFIER:</u>

After passing through the diode (i.e a half wave rectifier) the sinusoidal current, $I(t) = I_m Sin\omega t$ will be rectified. The rectified current function can be written as;

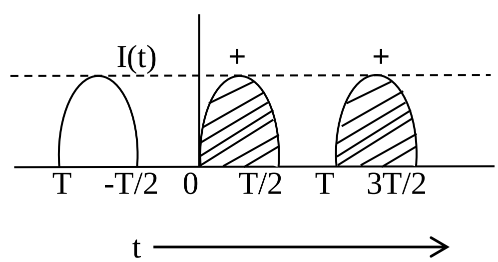
$$I(t) = \begin{cases} I_m Sin\omega t & 0 < t < T/2 \\ 0 & -T/2 \le t < 0 \end{cases}$$

As shown below;





- Input Voltage



With time period T. its Fourier expansion in complex form is:

$$I(t) = C_o + \sum_{-\infty}^{\infty} C_n e^{jn\omega t}$$

Hence the Fourier analysis of this signal involves finding the Fourier coefficients and then substituting this coefficient into the general Fourier expansion like this:

$$C_o = \frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} I_m Sin\omega t \cdot e^{-j\cdot 0\cdot \omega t} = \frac{I_m}{\pi}\left[-Cos\omega t\right|_{-\frac{T}{2}}^{\frac{T}{2}}$$

Substituting the limits we get,

$$\Rightarrow C_o = \frac{I_m}{2\pi}\left[-Cosw\tfrac{T}{2} - (Cos0)\right] = \left[-Cos\pi + Cos0\right]$$

$$= \frac{I_m}{2\pi}\left[-(-1)+1\right] = \frac{I_m}{2\pi}\times 2 = \frac{I_m}{\pi}$$

$$\therefore \boxed{C_o = \frac{I_m}{\pi}}$$

For the coefficient C_n we have

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} I_m Sin\omega t dt = \frac{1}{2\pi}\int_{-\pi}^{0} 0\cdot e^{-jn\omega_0 t}dt + \frac{1}{2\pi}\int_{0}^{\pi} I_m Sin\omega t\cdot e^{-jn\omega_0 t}dt$$

$$\therefore C_n = \frac{I_m}{2\pi}\int_0^{\pi} Sin\omega t\cdot e^{-jn\omega_0 t}dt$$

But by this identity;

$$\int e^{2x} Sin(bx+c) = \frac{e^{2x}}{a^2+b^2}\left(aSin(bx+c) - bCos(bx+c)\right)$$

Where a = 2, b = 1 and c = 0

Similarly,

$$C_n = \frac{I_m}{2\pi}\int_0^{\pi} Sin\omega t\cdot e^{-jn\omega t}dt = \frac{I_m}{2\pi}\cdot\frac{e^{-jn\omega t}}{a^2+b^2}\left(-jnSin\omega t - 1\cdot Cos\omega t\right)$$

So that substituting the limits we get,

$$C_n = \left[\frac{e^{-jn\pi}}{(-jn)^2+1^2}\left(-jnSin\pi - Cos\pi\right) - \left[\frac{e^{-jn\omega 0}}{(-jn)^2+1^2}\left(-jSin0 - Cos0\right)\right]\times\frac{I_m}{2\pi}\right.$$

$$\Rightarrow C_n = \left[\frac{e^{-n\pi}}{1-n^2}\left(0-(-1)\right) - \left[\frac{1}{1-n^2}\left(0-1\right)\right]\times\frac{I_m}{2\pi}\right.$$

But $e^{-n\pi} = Cosn\pi - jSinn\pi$ but $Sinn\pi = 0$ and $Cosn\pi = (-1)^n$

$$\therefore C_n = \left[\frac{(-1)^n}{1-n^2} - \frac{(-1)}{1-n^2}\right]\times\frac{I_m}{2\pi} = \left[\frac{(-1)^n}{1-n^2}\right]\times\frac{I_m}{2\pi}$$

$$\Rightarrow C_n = \frac{(-1)^n+1}{1-n^2} = 0 \text{ for n = odd 1, 3, 5}$$

But

$$C_n = \frac{(-1)^n+1}{1-n^2} = \frac{2}{1-n^2} \text{ for n = even 0, 2, 4}$$

$$C_n = \frac{I_m}{2\pi}\times\frac{2}{1-n^2} = \frac{I_m}{\pi(1-n^2)}$$

$$\therefore \boxed{C_n = \frac{I_m}{\pi(1-n^2)}}$$

But for n = 1 then,

$$\therefore C_1 = \frac{I_m}{\pi(1-1)} = \frac{I_m}{0} \text{ Undefined}$$

Therefore, since this is unacceptable for n = 1 we get

From $\omega = \frac{2\pi}{T}$ but $T = \frac{T}{2}$ $\therefore T = \frac{2\pi}{2} = \pi$ $\therefore T = \pi$

$$\therefore \omega = \frac{2\pi}{T} = 2 \;\therefore T = 2$$

Using and after substituting the limits $(0,\pi)$ we get

$$C_1 = \frac{1}{2}\left[\frac{e^{-j\pi}}{-j^2+1^2}\left(-jnSin\pi - Cos\pi\right) - \left(\frac{e^0}{-j^2+1^2}\left(-jSin0 - Cos0\right)\right]\times I_m\right.$$

$$\Rightarrow C_1 = \frac{1}{2}\left[\frac{(-1)(0-1)}{1^2+1^2} - \left(\frac{1}{1^2+1^2}\left(0-1\right)\right]\times I_m\right.$$

$$\therefore C_1 = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right]\times I_m = \frac{I_m}{2}$$

$$\therefore \boxed{C_1 = \frac{I_m}{2}}$$

Therefore,

$$C_0 = \frac{I_m}{\pi},\quad C_n = \frac{I_m}{\pi(1-n^2)} \text{ and } C_1 = \frac{I_m}{2}$$

Therefore, the general Fourier expression is;

$$I(t) = C_0 + C_n \sum_{n=\infty}^{\infty} e^{-jn\omega t} + C_1$$

$$I(t) = \frac{I_m}{\pi} + \frac{I_m}{\pi}\left[\sum_{n=\infty}^{\infty} \frac{e^{-jn\omega t}}{1-n^2}\right] + C_1$$

$$\Rightarrow I(t) = \frac{I_m}{\pi} + \frac{2I_m}{\pi}\left[\ldots \frac{e^{-j2\omega t}}{2(1-2^2)} + \frac{e^{-j4\omega t}}{2(1-4^2)} + \ldots + \frac{e^{j2\omega t}}{-2(1-2^2)} + \frac{e^{j4\omega t}}{-2(1-4^2)} + \ldots\right] + \frac{I_m}{2} Sin\omega t$$

$$\Rightarrow I(t) = \frac{I_m}{\pi} + \frac{2I_m}{\pi}\left[\frac{e^{j2\omega t} + e^{-j2\omega t}}{2\times 3} + \frac{e^{j4\omega t} + e^{-j4\omega t}}{2\times 15} + \ldots\right] + \frac{I_m}{2} Sin\omega t$$

$C_n = \frac{I_m}{\pi(1-n^2)}$ but $\frac{e^{j2\omega t} + e^{-j2\omega t}}{2} = Coswt$ (Euler's formula)

$$\therefore I(t) = \frac{I_m}{\pi} + \frac{2I_m}{\pi}\left[\frac{1}{3}Cos2\omega t + \frac{1}{15}Cos4\omega t + \ldots\right] + \frac{1}{2} I_0 Sin\omega t$$

INTERPRETATION OF THE PHYSICAL SIGNIFICANCE OF THE ABOVE HALF-WAVE RECTIFIER ANALYSIS EXPRESSION:

- The expression indicates discrete (even) cosine series.
- The convergence is fast $\left(\frac{1}{1-n^2}\right)$. Implies by just adding few terms we can get the original signal.
- The input sinusoidal current is $I_m Sinwt$ but the output sinusoidal current is $\frac{1}{2} I_m Sinwt$. This means only half of this sinusoidal current was converted into D.C. But this properly is useful especially where a smooth d.c. is not required. For example this circuit arrangement can be used in the construction of battery charger, which charges batteries which uses electrolyte.

PROOF: USING THE TRIGONOMETRIC APPROACH FOR THE HALF-WAVE RECTIFIER ANALYSIS:

From $I(t) = a_0 + \sum_{n=1}^{\infty} a_n Cos(n\omega t) + \sum_{n=1}^{\infty} b_n Sin(n\omega t)$

For the coefficient a_0 we have,

$$a_0 = \frac{I_m}{T}\int_0^{T/2} Sin\omega t dt = \frac{I_m}{T}\left[\frac{-Cos\omega t}{\omega}\right]_0^{T/2} = \frac{I_m}{T}\left[-Cos\omega T/2 - (-Cos\omega 0)\right]$$

But $T\omega = 2\pi$

$$a_0 = \frac{I_m}{2\pi}\left[-Cos\pi + 1\right] = \frac{I_m}{2\pi}\left[1+1\right] = \frac{I_m}{2\pi}\times 2 = \frac{I_m}{\pi}$$

$$\therefore \boxed{a_0 = \frac{I_m}{\pi}} = C_0 \text{ (Q.E.D)}$$

For the coefficient a_n we have,

$$a_n = \frac{2I_m}{T}\int_0^{T/2} Sin\omega t \cdot Cosn\omega t dt$$

Using trig. Identities;

$$SinuCosv = \frac{1}{2}\left[Sin(u+v) + Sin(u-v)\right]$$

$$\therefore a_n = \frac{2I_m}{T}\int_0^{T/2}\left[Sin(\omega t + n\omega t) + Sin(\omega t - n\omega t)\right]$$

Integrating w.r.t t, and substituting the limit, we get

$$a_n = \frac{2I_m}{2T\omega}\left\{\frac{1}{2}\left[\frac{-Cos(1+n)\omega t}{\omega(1+n)} - \frac{-Cos(1-n)\omega t}{2(1-n)} + \frac{Cos0}{(1-n)}\right]\right.$$

But $\omega T = 2\pi$

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$$\therefore a_n = \frac{I_m}{2\pi}\left[\frac{1-Cos(\pi+n\pi)}{1+n}+\frac{1-Cos(\pi-n)\pi}{1-n}\right]$$

But from trigonometric identities we get,

$$Cos(A+B) = CosACosB - SinASinB$$

$$\therefore a_n = \left[\frac{1-Cos\pi Cosn\pi - Sin\pi Sinn\pi}{1+n}+\frac{1-Cos\pi Cosn\pi}{1-n}\right]\times\frac{I_m}{2\pi}$$

But $Sinn\pi Sin\pi = 0$ for any value of n

$$\therefore a_n = \frac{I_m}{2\pi}\left[\frac{2\cos n\pi+2}{(1-n^2)}\right] = \frac{I_m}{\pi}\left[\frac{Cosn\pi+1}{(1-n^2)}\right] = \frac{I_m}{\pi}\left[\frac{(-1)^n+1}{1-n^2}\right]$$

$$\Rightarrow a_n = \left[\frac{I_m}{\pi}\left[\frac{-1+1}{1-n^2}\right]\right] = 0 \text{ for n = odd 1, 3, 5}$$

But

$$a_n = \frac{2I_m}{\pi(1-n^2)} = C_n \quad \text{(Q.E.D)}$$

For b_1 using trigonometric we get

$$b_1 = \frac{2I_m}{T}\int_0^{T/2} Sin\omega t\cdot Sinn\omega tdt \text{ But n = 1}$$

$$\therefore b_1 = \frac{2I_m}{T}\int_0^{T/2} Sin^2\omega tdt \text{ But } Sin^2\omega t = \frac{1}{2}[1-Cos2\omega t]$$

$$\therefore b_1 = \frac{2I_m}{2T}\left[\int_0^{T/2}(1-Cos2\omega t)dt\right]$$

Integrating and substituting limits we get,

But $\omega T = 2\pi$

$$b_1 = \left[\frac{I_m}{T}\times\frac{T}{2}\right] - \frac{I_m}{2\pi\cdot 2}\left[\frac{Sin2\cdot 2\pi}{2}-0\right]$$

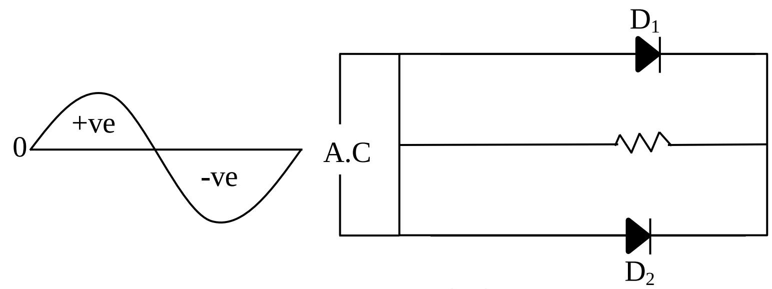
But $Sinn\pi = 0$ for any value of n

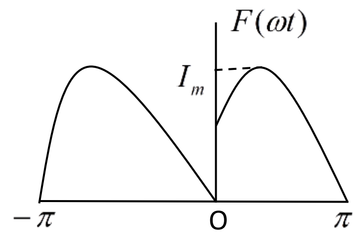
$$\therefore \boxed{b_1 = \frac{I_m}{2}} = 0$$

$$\therefore b_1 = \frac{I_m}{2} = C_1 \text{ (Q.E.D)}$$

FULL-WAVE RECTIER: here I want to use the Fourier analysis to see if the Full-Wave rectifier as it is been called converts A.C into a complete D.C.

The full wave rectifier can thought of as inverting the positive peaks to a negative troughs and vice-versa as shown below:





Output DC signal

This is what the circuit diagram shows, but let us prove it if this true using theory and equation.

To analyze the properties of the full wave rectifier all we need is the Fourier coefficients from this expression;

$$I(t) = C_0 + C_n\sum_{-\infty}^{\infty} e^{+jn\omega t}$$

For a full wave rectifier its defined by this expression;

$$I(t) = Sin\omega t \quad 0<\omega t<\pi$$

$$-Sin\omega t \quad -\pi<\omega t<0$$

For the coefficient C_0, we have,

Since the given function is an even function then,

$$C_0 = \frac{1}{\pi}\int_0^{\pi} I(t)e^{-jn\omega t}dt = \frac{1}{\pi}\int_0^{\pi} Sin\omega t\cdot e^{-j\cdot 0\cdot\omega t}dt$$

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$\Rightarrow C_0 = \frac{1}{\pi}\int_0^{\pi} Sin\omega t dt$ Integrating and substituting the limits we get,

$$C_0 = \frac{1}{\pi}\left[-Cos\omega t\right]_0^{\pi} = \frac{1}{\pi}\left[-(-1)-(-1)\right] = \frac{1}{\pi}\left[1+1\right]$$

$$\therefore \boxed{C_0 = \frac{2}{\pi}}$$

For the coefficient C_n we have, and since the function is an even function then,

$$C_n = \frac{1}{\pi}\int_0^{\pi} Sin\omega t \cdot e^{-jn\omega t}dt$$

Using the identity;

$$\int e^{2x}\cdot Sin(bx+c) = \frac{e^{ax}}{a^2+b^2}\left(aSin(bx+c)-bCos(bx+c)\right)$$

Where a = 2, b = 1 and c = 0

Similarly by using this identity we get,

$$\int e^{-jn\omega t}\cdot Sin\omega t = \frac{e^{-jn\omega t}}{a^2+b^2}\left(aSinbx - bCosbx\right)$$

Where a = -jn, b = 1 and c = 0

$$\Rightarrow C_n = \frac{1}{\pi}\int_0^{\pi} e^{-jn\omega t}\cdot Sin\omega t dt = \frac{e^{-jn\omega t}}{(-jn)^2+1^2}\left(-jnSin0-1.Cos0\right)\Big|_0^{\pi}$$

Substituting the limit we get,

$$C_n = \frac{1}{\pi}\left[\frac{e^{-jn\pi}}{1-n^2}\left(-jnSin\pi - Cos\pi\right) - \left[\frac{1}{1-n^2}\left(-jnSin0-1\cdot Cos0\right)\right]\right]$$

But $e^{-jn\pi} = Cosn\pi - jSinn\pi$

$$C_n = \frac{1}{\pi}\left[\frac{(-1)^n(-(-1))}{1-n^2} - \left[\frac{1}{1-n^2}(0-1)\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n+1}{1-n^2}\right] = \frac{1}{\pi}\left[\frac{2}{1-n^2}\right]$$ if n = even, 0, 2, 4

$$\boxed{C_n = \frac{-2}{\pi(n^2-1)}}$$

Hence the Fourier series of this analysis becomes;

$$I(t) = C_0 + C_n\sum_{n=\infty}^{\infty} e^{+jn\omega t}$$

But $C_0 = \frac{2}{\pi}$ and $C_n = \frac{-2}{\pi(n^2-1)}$

$$\Rightarrow I(t) = \frac{2}{\pi} - \frac{2}{\pi}\left[\ldots + \frac{e^{2j\omega t}}{(2^2-1)} + \frac{e^{4j\omega t}}{(4^2-1)} + \ldots + \frac{e^{-2j\omega t}}{(-2^2-1)} + \frac{e^{-4j\omega t}}{(-4^2-1)} + \ldots\right]$$

$$\Rightarrow I(t) = \frac{2}{\pi} - \frac{2\times 2}{\pi}\left[\frac{e^{2j\omega t}+e^{-2j\omega t}}{2\times 3} + \frac{e^{4j\omega t}+e^{-4j\omega t}}{2\times 15} + \ldots\right]$$

But $\frac{e^{j\omega t}+e^{-j\omega t}}{2} = Cos\omega t$

$$\Rightarrow I(t) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}Cos2\omega t + \frac{1}{15}Cos4\omega t + \ldots\right]$$

INTERPRETATION OF THE PHYSICAL SIGNIFICANCE OF THE FOURIER ANALYSIS OF THE FULL WAVE RECTIFIER EXPRESSION:

- The original frequency (ω) has been eliminated and now the lowest frequency of oscillation is 2ω. Indicating frequency of oscillation is always increasing by (2n) where n is the position of the term in the series.
- The convergence is fairly fast $\left(\frac{1}{n^2-1}\right)$. It means the conversion of the A.C to D.C is instantaneously.
- The expression contains cosine terms only, showing that a full wave rectifier does a good job of approximating an A.C to a D.C.

PROOF: USING TRIGONOMETRIC APPROACH FOR THE FULL-WAVE RECTIFIER.

From $I(t) = a_0 + \sum_{n=1}^{\infty} a_n Cosn\omega t + \sum_{n=1}^{\infty} b_n Sinn\omega t$

Since the function is even function then,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} Sin\omega t d(\omega t) = \frac{1}{\pi}\left[-Cos\omega t\right]_0^{\pi} = \frac{1}{\pi}\left[-(-1)-(-1)\right]$$

$$\Rightarrow \boxed{a_0 = \frac{2}{\pi}} = C_0 \qquad \text{(Q.E.D)}$$

For the coefficient a_n

$$a_n = \frac{2}{\pi}\int_0^{\pi} Sin\omega t \cdot Cosn\omega t d(\omega t)$$

Using trig. Identities;

$$SinuCosv = \frac{1}{2}\left[Sin(u+v) + Sin(u-v)\right]$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left[Sin(\omega t + n\omega t) + Sin(\omega t - n\omega t)\right]$$

Integrating w.r.t t and substituting the limit we get,

$$a_n = \frac{2}{\pi \times 2}\left[\frac{1-Cos(\pi+n\pi)}{(1+n)} + \frac{1-Cos(\pi-n)\pi}{1-n}\right]$$

Using: Cos(A+B) = CosA CosB – SinA SinB

$$\Rightarrow a_n = \left[\frac{1-Cos\pi Cosn\pi - Sin\pi Sinn\pi}{(1+n)} + \frac{1-Cos\pi Cosn\pi}{1-n}\right] \times \frac{2}{2\times\pi}$$

But $Sin\pi Sinn\pi = 0$ for any value of n

$$\therefore a_n = \frac{2}{\pi}\left[\frac{2Cosn\pi + 2}{(1+n^2)}\right] = \frac{2}{\pi}\left[\frac{Cosn\pi + 1}{(1+n^2)}\right] = \frac{2}{\pi}\left[\frac{(-1)^n + 1}{(1+n^2)}\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\frac{(-1)^n + 1}{1+n^2}\right] = \frac{2}{\pi}\left[\frac{2}{1+n^2}\right]$$ if n = even, 0, 2, 4

$$\therefore \boxed{a_n = \frac{-4}{\pi(n^2-1)}} = C_n \quad \text{(Q.E.D)}$$

CONCLUSION

The most important parameter in the Fourier series is the Fourier coefficients. In this work, because of the flexibility of the Fourier series I have used two 'flavours' in finding the Fourier coefficients viz: The complex exponential method and trigonometric approach to find the Fourier coefficients and then use these Fourier coefficients to generate the Fourier series to analyze and interpret the physical significance of the Fourier series generated. Here I have covered both the triangular, square, saw-tooth waves forms as well as the half and full wave rectifiers diodes and went ahead to interpret their physical significance. For example the discrete Cosine series obtained from the analysis of the triangular wave form is good tool for image processing. The continuous Sine series from the analysis of the saw-tooth wave form is also a good tool for audio compression. While the half wave rectifier analysis shows that it can be used for the construction of battery chargers which uses electrolyte and the full wave rectifier analysis is ideal for the construction of battery chargers which do not use electrolyte. As this work is just an 'opening' for the analysis and interpretation of signals using Fourier analysis, I recommend that readers should also strive to look for other uses of this analysis. Above all thank you very much, Jean-Baptiste Fourier, the founder of the Fourier series.

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