for the relation in equation (25) to converge to α with convergence order $\rho = 6$, the first and second term of equation (28) must vanish. It follows that $\theta = 1$, which leads to the asymptotic error equation

$$e_{k+1} = \left(2c_2^5 - \frac{9c_2^3c_3}{2} + \frac{9c_2c_3^2}{2}\right)e_k^6 + O(\|e_k^7\|) \tag{29}$$

This completes the proof. ■

For the parameter $\theta = 1$ in equation (25), a new quadrature based iterative method of convergence order $\rho = 6$ is proposed:

Method 3

Given an initial guess x_0, approximate the solution x_{k+1} by the iterative scheme:

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$z_k = y_k - \frac{f(y_k)}{2f'\left(\frac{x_k + y_k}{2}\right) - f'(x_k)},$$

$$z_k = z_k - \frac{f(z_k)}{2f'\left(\frac{x_k + y_k}{2}\right) - f'(x_k)}, \qquad k = 1,2,3,\ldots \tag{30}$$

with asymptotic error equation

$$e_{k+1} = \left(2c_2^5 - \frac{9}{2}c_2^3c_3 + \frac{9}{2}c_2c_3^2\right)e_k^6 + O(\|e_k^7\|) \tag{31}$$

and efficiency index $EEF = 1.4310$.

4. Numerical Experimentation

In this section, the proposed iterative methods (Method 1 ($M_{1,3}$), Method 2 ($M_{2,4}$) and Method 3 ($M_{3,6}$), where $M_{i,\rho}$ is Method i of convergence order ρ) are implemented on four problems (Examples 1-4) in literature in other to illustrate their efficiency. Numerical results obtained by the proposed methods ($M_{1,3}$, $M_{2,4}$, and $M_{3,6}$), are compared with methods from which they have been derived (Algorithm 2.6 ($N_{1,3}$) and Algorithm 2.12 ($N_{2,4}$) in [10]) and Newton method (NM). All numerical computations presented in Tables 4.1-4.4, are carried out in a PYTHON 2.7.12 environment with 25 digits floating arithmetic. Intel Celeron(R) CPU 1.6 GHz with 2 GB of RAM processor was used to execute all programs. The stopping criterion used for all programs is $|f(x_k)| < 10^{-15}$.

The measurements used for comparison are number of iteration required by method to achieve convergence (IT), number of functional evaluations required by method to achieve convergence (NFE), norm of difference of last two consecutive iterations ($|x_{k+1} - x_k|$), function of the last iteration ($|f(x_{k+1})|$), and Computer execution time in seconds (CPU).

The following problems used for implementation of the methods are taken from Noori [16].

Example 1

$f_1(x) = x^5 - 10, \qquad x_0 = 2.5$

Example 2

$f_2(x) = e^{x^2+7x-30} - 1, \quad x_0 = 3.5$

Example 3

$f(x) = \sin(x) - \frac{x}{2}, \quad x_0 = 1.6$

Example 4

$f(x) = e^x \sin(x) + \ln(x^2 + 1), \quad x_0 = 2$

Table 3.1 Computational results for Example 1

Method	**IT**	**NFE**	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	**CPU-time**
NM	6	12	1.4298e-06	8.1392e-11	0.0298
$N1$	4	16	9.8660e-06	4.8244e-14	0.0174
$M1$	3	12	1.5313e-03	3.6690e-08	0.0127
$M2$	2	8	1.0503e-01	1.7761e-03	0.0085
$N2$	3	18	9.1883e-04	4.5068e-11	0.0136
$M3$	2	10	5.3127e-02	6.4939e-07	0.0078

Table 3.2 Computational results for Example 2

Method	**IT**	**NFE**	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	**CPU-time**
NM	11	22	1.9615e-07	3.2898e-12	0.0534
$N1$	7	28	5.0831e-06	7.3853e-14	0.0371
$M1$	5	20	2.2128e-04	6.0862e-10	0.0224
$M2$	4	16	3.7500e-03	2.3416e-09	0.0182
$N2$	5	30	3.2548e-03	4.0080e-07	0.0268
$M3$	3	15	7.2798e-02	1.4826e-03	0.0168

Table 3.3 Computational results for Example 3

Method	**IT**	**NFE**	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	**CPU-time**
NM	54	108	2.9212e-15	1.4606e-15	0.2807
$N1$	63	253	1.1102e-15	1.3015e-15	0.3720
$M1$	39	156	2.6645e-15	1.7196e-15	0.2002
$M2$	30	120	1.0261e-14	1.7101e-15	0.1509
$N2$	42	852	1.5543e-15	1.0150e-15	0.2442
$M3$	19	95	1.8228e-14	1.3020e-15	0.0969

Table 3.4 Computational results for Example 4

Method	**IT**	**NFE**	$\lvert x_{k+1} - x_k \rvert$	$\lvert f(x_{k+1}) \rvert$	**CPU-time**
NM	5	10	1.4773e-06	4.3649e-12	0.0220
$N1$	3	12	2.0423e-03	3.3697e-08	0.0181
$M1$	4	16	1.3526e-03	4.4471e-09	0.0234
$M2$	3	12	3.3152e-04	8.4364e-14	0.0171
$N2$	3	18	1.2360e-04	1.8654e-15	0.0191
$M3$	2	10	7.5436e-02	4.5395e-06	0.0152

From Tables 3.1-3.4, the computational results indicates that the proposed iterative methods converges faster and required less execution time compared with the iterative methods from which they have been derived. This agrees with the theoretical results obtained on convergence order and efficiency index of the proposed methods in Section 3.

5. Conclusion

In this paper, a technique utilized to improve convergence order and efficiency i of a two-step and three-step quadrature based iterative methods (Algorithm 2.6 and Algorithm 2.12) in [10] for the approximation of solution of nonlinear equations is developed. The convergence analysis of the developed methods are established via Taylor's series expansion technique. The implementation results suggest that, the methods proposed outperformed methods from which they have been derived on account of convergence order and efficiency index.

Competing Interests

The authors do not have any competing interest in the manuscripts.

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