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LINDLEY STRETCHED EXPONENTIAL DISTRIBUTION WITH APPLICATIONS

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Abstract

In this paper, we have introduced an innovative generalization of the Stretched Exponential distribution termed as the Lindley Generalized Stretched Exponential distribution. This proposed distribution can model data in form of decreasing and increasing hazard rates. Besides, we have derived some mathematical properties of the distribution covering probability density function, distribution function, survival, failure rate and reversed hazard functions, moments, moment generating function, cumulant generating function, Renyi entropy of introduced distribution. The Maximum Likelihood method has been used for estimation of parameters. We have used real life data sets to demonstrate the worth and significance of introduced distribution. It has been observed that the introduced distribution of three parameters fits better than its specific cases as well as competitive distributions for all data sets.

Keywords. Lindley distribution, Stretched Exponential distribution, Generalized, Transformation, Method of Maximum Likelihood Estimation, Goodness of Fit.

1. Introduction

The Lindley distribution was firstly proposed by Lindley (1958) in the contextual of Bayesian statistics i.e. counter example of Fudicial Statistics that was mixture of Exponential and Gamma distributions. The Lindley distribution is used to explain the lifetime of a device or process. It can be used in extensive areas, including Engineering, Biology, Medicine, Ecology and Finance. Ghitany et. al (2011) stated that it's mainly useful for modelling in mortality studies.

The generalized form of a distribution can be introduced and offered more flexible distribution for modelling real life data. Technique of transformation can be adopted to create Lindley Stretched Exponential distribution for desired purpose. That is Stretched Exponential distribution is transformed into Lindley distribution.

The statistical literature's point of view, the Lindley distribution has generated little attention in excess of the eminent Exponential distribution because these have closed form as well as approach of comparison. In this connection, one improvement of the Lindely distribution is to make a comparison to the exponential distribution. That's why; the Exponential distribution has constant mean residual life function and hazard rate however the Lindley distribution has decreasing mean residual life function and increasing hazard rate.

Some researchers have presented new classes of distributions on basis of modifications of the Lindley distribution along with their properties. The chief awareness is constantly focussed by inserting former and existing distributions to create innovative flexible structures. Many forms of Lindley distribution were described in "A two-parameter form" (Shanker *et al* 2013), A two-parameter weighted form (Ghitany *et al* 2011), An extended (EL) distribution (Bakouch *et al* 2012), An Exponential Geometric distribution (Adamidis and Loukas 1998). The transmuted Lindley-Geometric Distribution (Merovci, and Elbatal). Sankaran (1970) presented the discrete Poisson–Lindley distribution by joining the Poisson and Lindley distributions. Mahmoudi and Zakerzadeh (2010) introduced an extended version of the compound Poisson distribution by combining the Poisson distribution with the generalized Lindley distribution. Louzada, Roman, and Cancho (2011) suggested the complementary exponential geometric distribution by compounding the geometric and the exponential distributions.

The purpose of this paper is to present an extension of the Lindley distribution which offers a more flexible distribution for modelling any real life data. The innovative distribution can assist decreasing and increasing failure rates as well as unimodal. We present the Lindley Generalized Stretched Exponential distribution (**LGSED**) by transformation.

The paper is organized in different sections. Section 2 contributes the density function, distribution function along with shapes as well as special cases of **LGSED**. Section 3 is based on Survival, hazard and Reverse hazard functions for the Lindley Stretched Exponential distribution. We have also discussed the shape of the survival and hazard functions in the same section. In section 4, some mathematical properties of the distribution have been derived including moments, moment generating function, mean, variance, cumulant

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generating function and renyi' entropy. Section 5 deals with the maximum likelihood estimates for the parameters of the distribution. To conclude the distribution, the real life data applications of the newly developed distribution have provided in section 6.

2. Density, Distribution Functions

A random variable X is said to have Lindley distribution (θ), if its probability density function (pdf) and distribution function (cdf) are defined respectively as:

$$\boldsymbol{f(x;\theta) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}} \quad \text{for } \boldsymbol{x,\theta > 0} \tag{1}$$

$$\boldsymbol{F(x;\theta) = 1 - \frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta}} \tag{2}$$

By using transformation, $\boldsymbol{y} = \left\{\left(\frac{x}{a}\right)^{b}\right\}^{c}$, the above pdf and cdf of Lindley distribution are transformed into Lindley Generalized Stretched Exponential (**LGSE**) distribution, then its probability density function and cumulative distribution function will be obtained by.

$$\boldsymbol{f_\theta(x;a,b,c,\theta) = \frac{bc\theta^2}{a(1+\theta)}\left\{\left(\frac{x}{a}\right)^{b}\right\}^{c-1}\left(\frac{x}{a}\right)^{b-1}\left[1+\left\{\left(\frac{x}{a}\right)^{b}\right\}^{c}\right]e^{-\theta\left\{\left(\frac{x}{a}\right)^{b}\right\}^{c}}}$$

for $\boldsymbol{x > 0, a, b, c, \theta > 0}$. (3)
also,

$$\boldsymbol{F_\theta(x;a,b,c,\theta) = 1 - \frac{e^{-\theta\left\{\left(\frac{x}{a}\right)^{b}\right\}^{c}}}{1+\theta}\left[1+\theta+\theta\left\{\left(\frac{x}{a}\right)^{b}\right\}^{c}\right]} \tag{4}$$

For construction of **LGSE** distribution, the baseline distribution is Stretched Exponential distribution.

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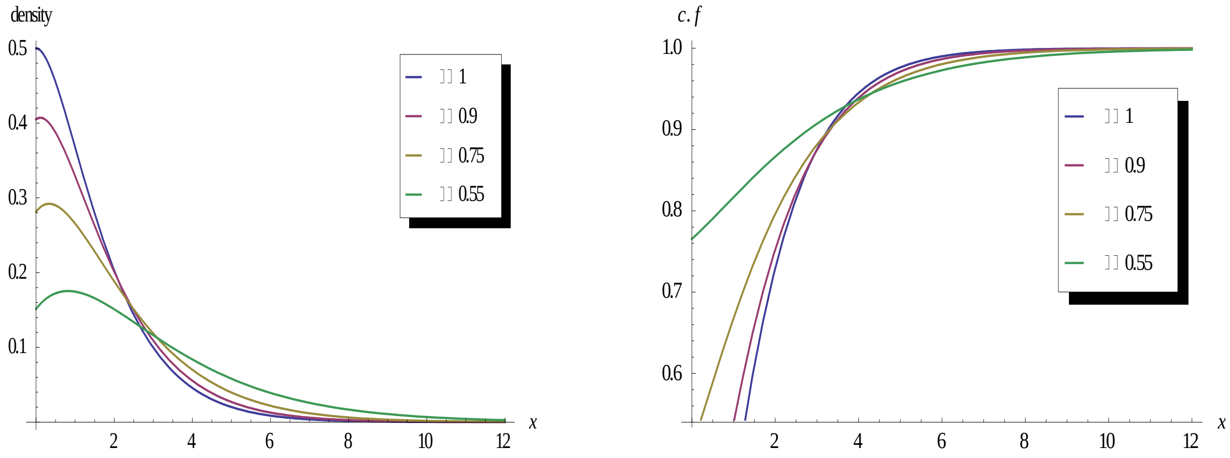

Figure 1: Plots of LGSE density function and distribution function for fixed values of (a, b, c) = 1 with different values of θ.

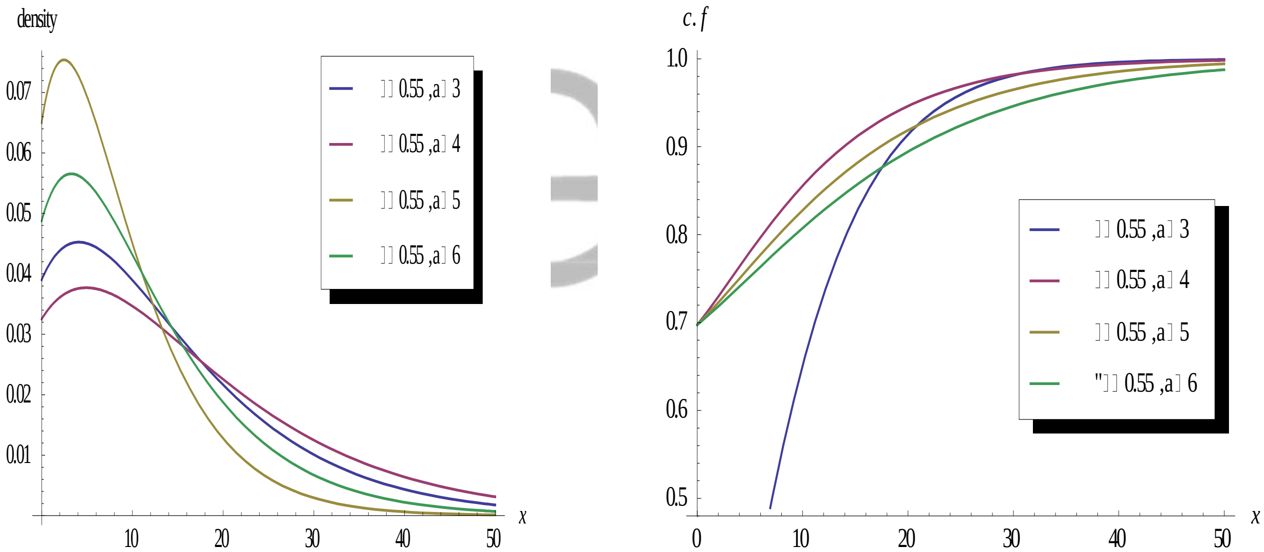

Figure 2: Plots of LGSE density function and distribution function for fixed values of θ = 0.55 (b, c) = 1 with different values of a.

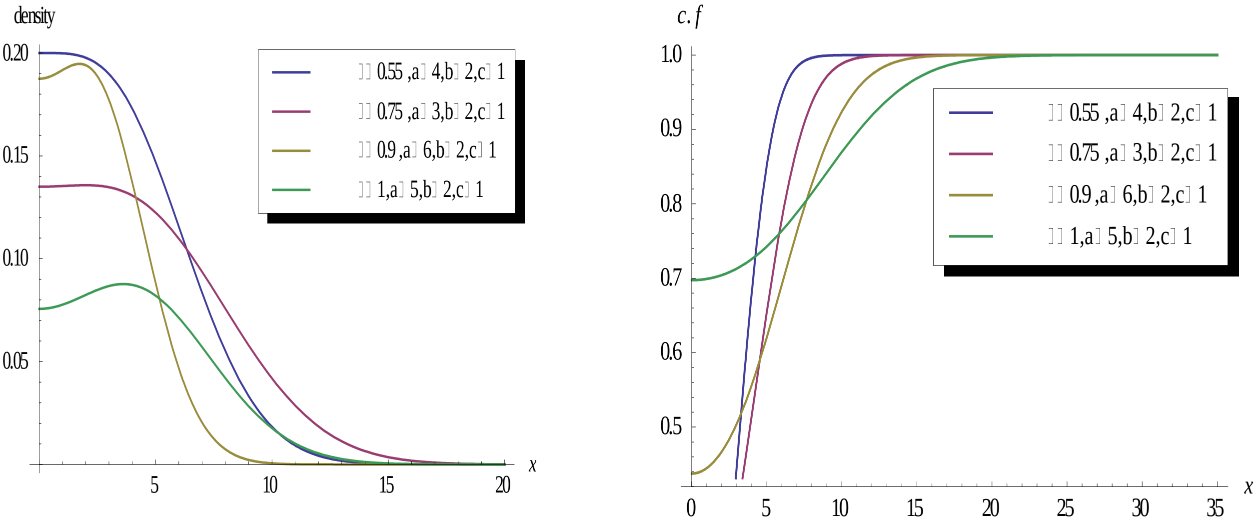

Figure 3: Plots of LGSE density function and distribution function for fixed values of b = 2, c = 1 with different values of a and θ.

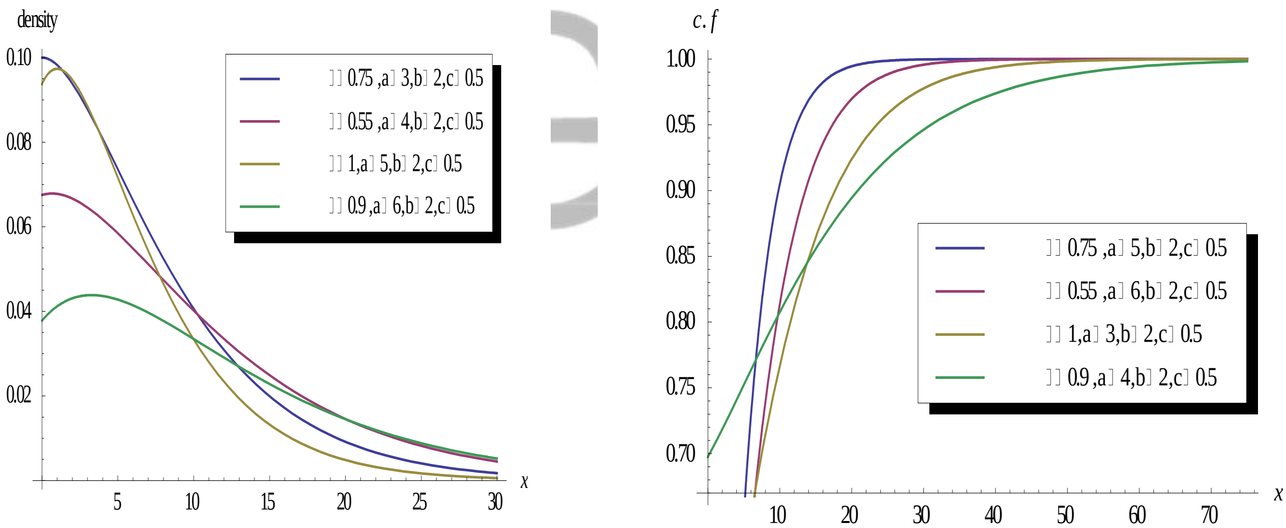

Figure 4: Plots of LGSE density function and distribution function for fixed values of b = 2, c = 0.5 with different values of a and θ.

From all above Figures, it is observed that distribution shows Lindley exponential behaviour and unimodal.

2.1. Special Cases

We can find some existing models through appropriate range of parameters. This particular choice of parameters can fit the observed data too. Consequently, certain Specific cases of **LGSE** distribution are discussed below by replacing different values of a, b, c and θ.

2.1.1. By replacing $a = a,\ b = b,\ c = 1\ and\ \theta = \theta,$ in Equation (**3**), we get three parameter Lindley Stretched Exponential distribution (**LSED**).

$$f_{\theta}(x; a, b, \theta) = \frac{b\theta^2}{a(1+\theta)}\left(\frac{x}{a}\right)^{b-1}\left\{1 + \left(\frac{x}{a}\right)^{b}\right\} e^{-\theta\left(\frac{x}{a}\right)^{b}}$$

for $x > 0, (a, b, \theta) > 0$. **(5)**

2.1.2. By replacing $a = a,\ b = 1, c = 1\ and\ \theta = \theta,$ in Equation (**3**), we get two parameter Lindley Exponential distribution (**LED**).

$$f_{\theta}(x; a, \theta) = \frac{\theta^2}{a(1+\theta)}\left(1 + \frac{x}{a}\right) e^{-\theta\left(\frac{x}{a}\right)}$$

for $x > 0, a, \theta > 0$. **(6)**

2.1.3. By replacing $a = 1,\ b = b, c = 1\ and\ \theta = \theta,$ in Equation (**3**), we get two parameter Lindley Negative Exponential distribution (**LNED**).

$$f_{\theta}(x; b, \theta) = \frac{\theta^2}{(1+\theta)}\left(1 + x^{b}\right) e^{-\theta x^{b}}$$

for $x > 0, b, \theta > 0$.

(7)

2.1.4. By replacing $a = 1,\ b = 1, c = 1\ and\ \theta = \theta,$ in Equation (**3**), we get one parameter Lindley distribution (**LD**).

$$f_{\theta}(x; b, \theta) = \frac{\theta^2}{(1+\theta)}(1 + x) e^{-\theta x}$$

for $x > 0, \theta > 0$. **(8)**

3. Survival, Hazard and Reverse Hazard Functions

A random variable X is said to have Lindley Stretched Exponential distribution with probability density function (*pdf*), $f_{\theta}(x)$, i.e. (**3**) and distribution function (*cdf*), $F_{\theta}(x)$, i.e. (**4**), then the survival, hazard and reverse hazard functions are presented by respectively

$$S_{\theta}(x; a, b, c, \theta) = 1 - F_{\theta}(x; a, b, c, \theta)$$

(9)

By substituting the expression of $F_{\theta}(x; a, b, c, \theta)$ in equation (**9**), we obtain survival function is of the form

$$S_\theta(x;a,b,c,\theta) = \frac{e^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c}\left[1+\theta+\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c\right]}{1+\theta}$$

for $x > 0, (a,b,c,\theta) > 0$ **(10)**

Similarly hazard function is obtained by substituting expressions of pdf and survival functions in the following equation

$$h_\theta(x;a,b,c,\theta) = \frac{f_\theta(x;a,b,c,\theta)}{S_\theta(x;a,b,c,\theta)} = \frac{f_\theta(x;a,b,c,\theta)}{1-F_\theta(x;a,b,c,\theta)}$$

Or

$$h_\alpha(x;a,b,c,\theta) = \frac{bc\theta^2\left\{\left(\frac{x}{a}\right)^b\right\}^{c-1}\left(\frac{x}{a}\right)^{b-1}\left[1+\left\{\left(\frac{x}{a}\right)^b\right\}^c\right]e^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c}}{ae^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c}\left[1+\theta+\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c\right]}$$

for $x > 0, (a,b,c,\theta) > 0$ **(11)**

Also reverse hazard function of (**LGSE**) distribution is attained by substituting expressions of pdf and cdf in the following equation

$$\tau_\theta(x;a,b,c,\theta) = \frac{f_\theta(x;a,b,c,\theta)}{F_\theta(x;a,b,c,\theta)}$$

(12)

$$\tau_\theta(x;a,b,c,\theta) = \frac{bc\theta^2\left\{\left(\frac{x}{a}\right)^b\right\}^{c-1}\left(\frac{x}{a}\right)^{b-1}\left[1+\left\{\left(\frac{x}{a}\right)^b\right\}^c\right]e^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c}}{a\left[1+\theta-e^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c}\left[1+\theta+\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c\right]\right]}$$

for $x > 0, (a,b,c,\theta) > 0$ **(13)**

The behaviour of Survival and Hazard functions can also be checked graphically.

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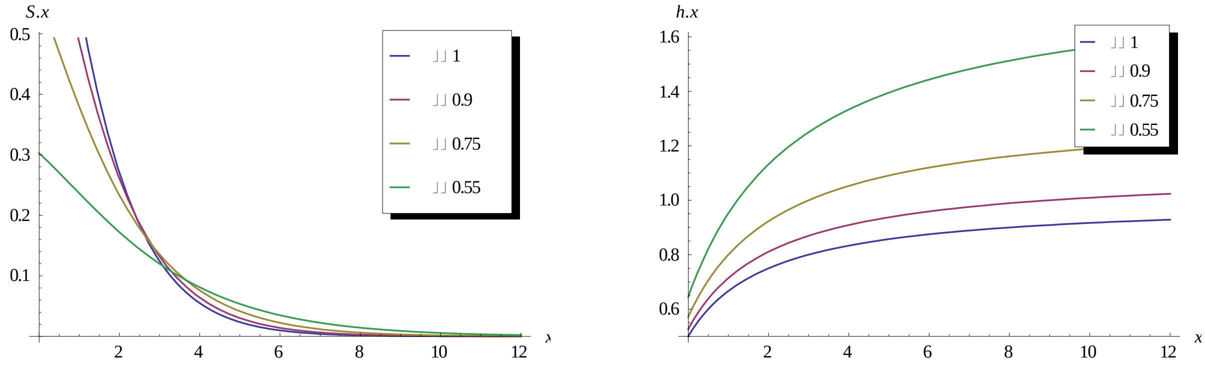

Figure 5: Plots of Survival and Hazard functions for fixed values of (a, b, c) = 1 with different values of and θ.

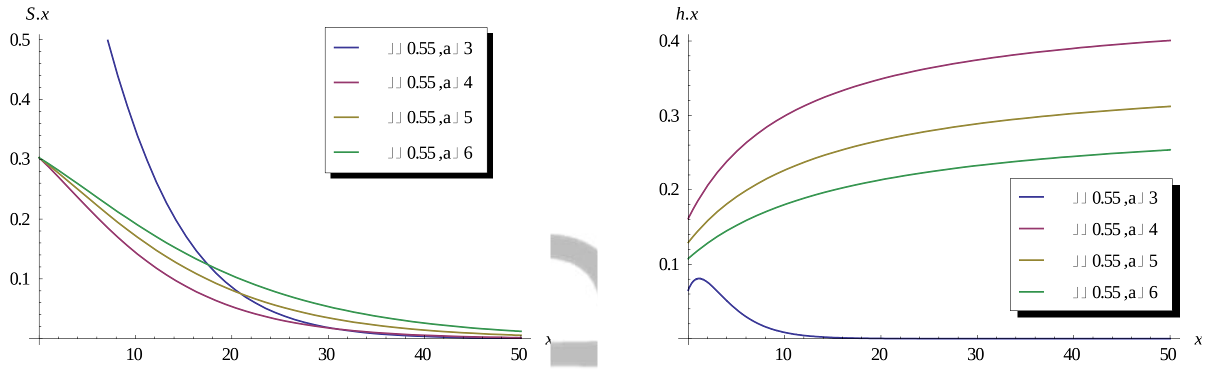

Figure 6: Plots of Survival and Hazard functions for fixed values of (b, c) = 1 and θ = 0.55 with different values of a.

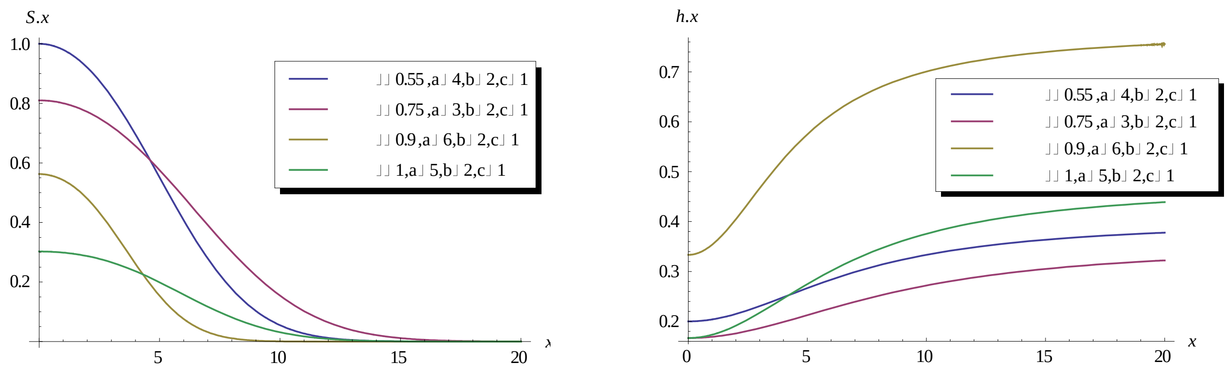

Figure 7: Plots of Survival and Hazard functions for fixed values of $b = 2, c = 1$ with different values of a and θ.

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From above Figures, it is observed that hazard unction is concave increasing for $(a, b, c, \theta) > 1$ except for $a = 3$, at this point, hazard function convex decreases.

4. Properties

In this section, some Mathematical properties of the **LGSE** distribution containing moments, moment generating function, cumulant generating function and information entropy.

4.1. Moments, Moment Generating Function

For the moments of the **LGSE** distribution, the rth moment of Lindley Generalized Stretched Exponential variable $\boldsymbol{X}$ with pdf (**3**) is obtained as:

By definition,

$$\boldsymbol{\mu_r' = E(X^r)}$$

$$= \int_0^\infty x^r \boldsymbol{f_\theta(x; a, b, c, \theta) dx}$$

By substituting right hand side expression of $\boldsymbol{f_\theta(x; a, b, c, \theta)}$ in above expression, we obtain

$$\boldsymbol{\mu_r' = E(X^r) = \frac{bc\theta^2}{a(1+\theta)} \int_0^\infty x^r \left\{\left(\frac{x}{a}\right)^b\right\}^{c-1} \left(\frac{x}{a}\right)^{b-1} \left[1 + \left\{\left(\frac{x}{a}\right)^b\right\}^c\right] e^{-\theta\left\{\left(\frac{x}{a}\right)^b\right\}^c} dx} \quad (14)$$

Now suppose that

$$\boldsymbol{\theta \left\{\left(\frac{x}{a}\right)^b\right\}^c = u}$$

OR **(15)**

$$\boldsymbol{\left\{\left(\frac{x}{a}\right)^b\right\}^c = \frac{u}{\theta}}$$

This implies

$$\boldsymbol{x = a\left\{\left(\frac{u}{\theta}\right)^{1/c}\right\}^{1/b} \Rightarrow dx = \frac{a}{b\theta c}\left\{\left(\frac{u}{\theta}\right)^{1/c}\right\}^{\frac{1}{b}-1} \left(\frac{u}{\theta}\right)^{\frac{1}{c}-1} du} \quad (16)$$

Equation (**14**) can be written as:

$$\boldsymbol{= \frac{bc\theta^2}{a(1+\theta)} \int_0^\infty \left[a\left\{\left(\frac{u}{\theta}\right)^{1/c}\right\}^{1/b}\right]^r \left\{\left(\frac{u}{\theta}\right)^{\frac{1}{c}}\right\}^{c-1} \left[\left\{\left(\frac{u}{\theta}\right)^{1/c}\right\}^{1/b}\right]^{b-1} \left[1 + \frac{u}{\theta}\right] e^{-u} \frac{a}{b\theta c}\left\{\left(\frac{u}{\theta}\right)^{1/c}\right\}^{\frac{1}{b}-1} \left(\frac{u}{\theta}\right)^{\frac{1}{c}-1} du}$$

After simplification and using Gamma function, $\Gamma(z) = \int_0^\infty x^{z-1}\, e^{-x} dx$, we obtain r[th] moment of **LGSE** distribution.

$$\mu_r' = \frac{a^r}{\theta^{r/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{r}{bc}+1\right) + \Gamma\left(\frac{r}{bc}+2\right)\right] \tag{17}$$

For Mean and Variance of X, by putting $r = 1\ and\ 2$ in Equation (**17**), the expressions of $E(X)$ and $E(X^2)$ are given by

$$E(X) = \frac{a}{\theta^{1/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{1}{bc}+1\right) + \Gamma\left(\frac{1}{bc}+2\right)\right] \tag{18}$$

$$E(X^2) = \frac{a^2}{\theta^{2/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{2}{bc}+1\right) + \Gamma\left(\frac{2}{bc}+2\right)\right] \tag{19}$$

Now Variance of X is obtained by

$$V(X) = E(X^2) - [E(X)]^2 \tag{20}$$

By putting expressions of $E(X)$ and $E(X^2)$ in **(20),** we obtain

$$V(X) = \frac{a^2}{\theta^{2/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{2}{bc}+1\right) + \Gamma\left(\frac{2}{bc}+2\right)\right] - \left[\frac{a}{\theta^{1/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{1}{bc}+1\right) + \Gamma\left(\frac{1}{bc}+2\right)\right]\right]^2$$

OR

$$V(X) = \frac{a^2}{\theta^{2/bc}(1+\theta)}\left[\theta\,\Gamma\left(\frac{2}{bc}+1\right) + \Gamma\left(\frac{2}{bc}+2\right) - \frac{1}{(1+\theta)}\left\{\theta\,\Gamma\left(\frac{1}{bc}+1\right) + \Gamma\left(\frac{1}{bc}+2\right)\right\}\right]^2 \tag{21}$$

The moment generating function (***m.g.f***) and cumulant generating function (***c.g.f***) are expressed in the form

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\left[\frac{a^r}{\theta^{r/bc}(1+\theta)}\left\{\theta\,\Gamma\left(\frac{r}{bc}+1\right) + \Gamma\left(\frac{r}{bc}+2\right)\right\}\right], \tag{22}$$

and

$$\boldsymbol{K_X(t)} = \boldsymbol{ln}\left[\sum_{\boldsymbol{r=0}}^{\infty}\frac{\boldsymbol{t^r}}{\boldsymbol{r}!}\left[\frac{\boldsymbol{a^r}}{\boldsymbol{\theta^{r/bc}(1+\theta)}}\left\{\boldsymbol{\theta}\,\Gamma\left(\frac{r}{bc}+1\right)+\Gamma\left(\frac{r}{bc}+2\right)\right\}\right]\right] \qquad \textbf{(23)}$$

Note: We can also obtain mean and variance of **LGSE** distribution with help of ***m.g.f.*** (**22**) by using partial differentiation of ***m.g.f.*** with respect to t, then substituting $t = 0$, we obtain mean (**18**), similarly, again applying partial differentiation of ***m.g.f.*** with respect to t, then substituting $t = 0$, we obtain $E(\boldsymbol{X^2})$ that is used to find variance.

4.2 Information Entropy:

The concept of entropy is significant in various fields of Science, specifically Physics, Theory of communication and Probability.

4.2.1 Renyi' Entropy

The Renyi entropy for the Lindley Stretched Exponential distribution has been obtained as:

Let X be the ***LGSE*** r.v, then the Renyi' entropy can be obtained by using the following relation.

$$I(é) = \frac{1}{1-é}\,log\left\{\int_0^{\infty} f^{é}(x)dx\right\} \qquad \textbf{(24)}$$

Here$f_{\boldsymbol{\theta}}^{é}(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\theta}) = \left[\frac{\boldsymbol{bc\theta^2}}{\boldsymbol{a(1+\theta)}}\left\{\left(\frac{x}{a}\right)^{\boldsymbol{b}}\right\}^{\boldsymbol{c-1}}\left(\frac{x}{a}\right)^{\boldsymbol{b-1}}\left[\boldsymbol{1}+\left\{\left(\frac{x}{a}\right)^{\boldsymbol{b}}\right\}^{\boldsymbol{c}}\right]\boldsymbol{e}^{-\boldsymbol{\theta}\left\{\left(\frac{x}{a}\right)^{\boldsymbol{b}}\right\}^{\boldsymbol{c}}}\right]^{é}$, by substituting the value of $f^{é}(x)$ in equation (**19**), In process of integration of $f^{é}(x)$, we adopt substitution method from Equations (**15**) and (**16**), then after simplification, we obtain required Renyi' entropy as given below:

$$\boldsymbol{I(é)} = \frac{\boldsymbol{1}}{\boldsymbol{1-é}}\boldsymbol{log}\left\{\frac{\boldsymbol{\theta^{2é-1}b^{é-1}c^{é-1}}}{\boldsymbol{a^{é-1}b(1+\theta)^{é}}}\sum_{\boldsymbol{i=0}}^{\infty}\binom{\boldsymbol{é}}{\boldsymbol{i}}\boldsymbol{\Gamma}\left(\boldsymbol{i}+\boldsymbol{é}-\frac{\boldsymbol{é}}{\boldsymbol{bc}}+\frac{\boldsymbol{1}}{\boldsymbol{bc}}\right)\right\} \qquad \textbf{(25)}$$

Note: Another type of entropy i.e. Shannon entropy for **LGSE** distribution can be obtained by using: $E[-ln\boldsymbol{f_\theta}(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\theta})] = -\int_0^{\infty}[ln\boldsymbol{f_\theta}(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\theta})]\boldsymbol{f_\theta}(\boldsymbol{x}; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{\theta})\boldsymbol{dx}$ **(26)**

5. Maximum Likelihood Estimation

Now method of maximum likelihood estimation has been discussed for the purpose of parameters' estimation of ***LGSE*** distribution.

Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from the ***LGSE*** distribution given by equation (**3**). Then

$$L(X;\Omega) = \prod_{i=1}^{n} f_\theta(x_i; a, b, c, \theta)$$

Here, $\Omega = (a, b, c, \theta)$ **(27)**

By substituting right hand expression of $f_\theta(x_i; a, b, c, \theta)$ in (**27**), applying "log" on both sides, then we obtain the following result

$$logL(X;\Omega) = \ell(X;\Omega)$$

$$\ell(X;\Omega) = 2nlog(\theta) + nlog(b) + nlog(c) - nlog(a) - nlog(1+\theta) + (b-1)\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right) + b(c-1)\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right) + \sum_{i=1}^{n} log\left[1+\left\{\left(\frac{x_i}{a}\right)^b\right\}^c\right] - \theta\sum_{i=1}^{n}\left\{\left(\frac{x_i}{a}\right)^b\right\}^c$$

(28)

By partially differentiation to equation (**28**) with respect to "a", "b", "c" and "θ", we obtain the following equations in the form of

$$\frac{\partial\ell(X;\Omega)}{\partial a} = -\frac{n}{a} - \frac{n(b-1)}{a} - \frac{nb(c-1)}{a} + \theta\sum_{i=1}^{n}\frac{bcx_i\left(\frac{x_i}{a}\right)^{-1+b}\left(\left(\frac{x_i}{a}\right)^b\right)^{-1+c}}{a^2} - \sum_{i=1}^{n}\frac{bcx_i\left(\frac{x_i}{a}\right)^{b-1}\left\{\left(\frac{x_i}{a}\right)^b\right\}^{c-1}}{a^2\left[1+\left\{\left(\frac{x_i}{a}\right)^b\right\}^c\right]}$$

(29)

$$\frac{\partial\ell(X;\Omega)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} log\left(\frac{x_i}{a}\right) + (c-1)\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right) - \theta c\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right)\left\{\left(\frac{x_i}{a}\right)^b\right\}^c + \sum_{i=1}^{n}\frac{c\,\log\left(\frac{x_i}{a}\right)\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}{1+\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}$$

(30)

$$\frac{\partial\ell(X;\Omega)}{\partial c} = \frac{n}{c} + b\sum_{i=1}^{n}\log\left(\frac{x_i}{a}\right) - \theta\sum_{i=1}^{n} log\left(\frac{x_i}{a}\right)^b\left\{\left(\frac{x_i}{a}\right)^b\right\}^c + \sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{a}\right)^b\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}{1+\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}$$

(31)

$$\frac{\partial\ell(X;\Omega)}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{a}\right)^b\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}{1+\left\{\left(\frac{x_i}{a}\right)^b\right\}^c}$$

(32)

It have seen that above equations (**29**), (**30**), (**31**) and (**32**) do not give the impression to be resolved directly. Conversely, Fisher's scoring method can be used to solve these equations iteratively.

The MLEs $(\hat{a}, \hat{b}, \hat{c}, \hat{\theta})$ of the parameters of **LGSED** are the solution of the following equations:

$$\begin{bmatrix} \frac{\partial\ell^2(X;\Omega)}{\partial a^2} & \frac{\partial\ell(X;\Omega)}{\partial a\partial b} & \frac{\partial\ell(X;\Omega)}{\partial a\partial c} & \frac{\partial\ell(X;\Omega)}{\partial a\partial\theta} \\ \frac{\partial\ell(X;\Omega)}{\partial a\partial b} & \frac{\partial\ell^2(X;\Omega)}{\partial b^2} & \frac{\partial\ell(X;\Omega)}{\partial b\partial c} & \frac{\partial\ell(X;\Omega)}{\partial b\partial\theta} \\ \frac{\partial\ell(X;\Omega)}{\partial a\partial c} & \frac{\partial\ell(X;\Omega)}{\partial b\partial c} & \frac{\partial\ell^2(X;\Omega)}{\partial c^2} & \frac{\partial\ell(X;\Omega)}{\partial c\partial\theta} \\ \frac{\partial\ell(X;\Omega)}{\partial a\partial\theta} & \frac{\partial\ell(X;\Omega)}{\partial b\partial\theta} & \frac{\partial\ell(X;\Omega)}{\partial c\partial\theta} & \frac{\partial\ell^2(X;\Omega)}{\partial\theta^2} \end{bmatrix}_{\substack{\hat{a}=a_0\\ \hat{b}=b_0\\ \hat{c}=c_0\\ \hat{\theta}=\theta_0}} \begin{bmatrix} \hat{a}=a_0 \\ \hat{b}=b_0 \\ \hat{c}=c_0 \\ \hat{\theta}=\theta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial\ell(X;\Omega)}{\partial a} \\ \frac{\partial\ell(X;\Omega)}{\partial b} \\ \frac{\partial\ell(X;\Omega)}{\partial c} \\ \frac{\partial\ell(X;\Omega)}{\partial\theta} \end{bmatrix}_{\substack{\hat{a}=a_0\\ \hat{b}=b_0\\ \hat{c}=c_0\\ \hat{\theta}=\theta_0}}$$

where a_0, b_0, c_0 and θ_0 are initial values of a, b, c and θ. These equations are solved iteratively till sufficiently close estimates of $\hat{a}, \hat{b}, \hat{c}$ and $\hat{\theta}$ are obtained. In this connection, Mathematica-software can also be used.

6. Applications

The Lindley Generalized Stretched Exponential distribution of three parameters has been applied to some real life data- sets due to make a comparison with other models. Because, In this section, we explore the fitting of three and two-parameter Lindley Stretched Exponential distribution to four real life data-sets and make a comparison of its goodness of fit with other distributions including its Special cases, Weibull, Gamma, Lognormal, Generalized Lindley, Generalized Gamma, LG the one parameter Lindley distributions. In order to compare distributions, $\boldsymbol{-2lnL}$, **AIC** (Akaike Information Criterion), **CAIC** (Consistent Akaike Information Criterion), **BIC** (Bayesian Information Criterion), **HQIC** (Hannan Quinn Information Criteria), K-S Statistics (Kolmogorov-Smirnov Statistics) for real life data sets have been computed by using R software.

Note that: The smaller measures of goodness-of-fit provide better the fit of the data. These measures of goodness-of-fit are defined as:

$$\boldsymbol{AIC = -2lnL + 2k}$$

$$\boldsymbol{BIC = -2lnL + klog(n)}$$

$$\boldsymbol{HQIC = -2lnL + 2klog(log(n))}$$

$$\boldsymbol{CAIC = -2lnL + \frac{2kn}{n-k-1}}$$

where $\boldsymbol{lnL}$ denotes the log-likelihood function estimated at the maximum likelihood estimates, $\boldsymbol{k}$ is the number of parameters, and $\boldsymbol{n}$ is the sample size.

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Data 1:

The data set represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003): The remission times (in months) of bladder cancer patients data is given below:

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	0.52
4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09	0.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	0.62	3.82	5.32
7.32	10.06	14.77	32.15	2.64	3.88	5.32	0.39	10.34	14.83	34.26
0.90	2.69	4.18	5.34	7.59	10.66	0.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	0.19	2.75	4.26	5.41	7.63	17.12
46.12	1.26	2.83	4.33	0.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	0.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46
4.40	5.85	0.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02
2.02	0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49			

Table 1: MLE Estimates of LSED and LD with Measures of Goodness of Fit

Model	MLE Estimates	$-2lnL$	AIC	CAIC	BIC	HQIC	K-S
LSED	$\hat{a} = 223.5189$ $\hat{b} = 1.0473$ $\hat{\theta} = 28.0783$	828.2282	834.2282	834.4218	842.7843	837.7046	0.8696
LD	$\hat{\theta} = 0.1960$	839.0598	841.0598	841.0916	843.9118	842.2186	0.8979

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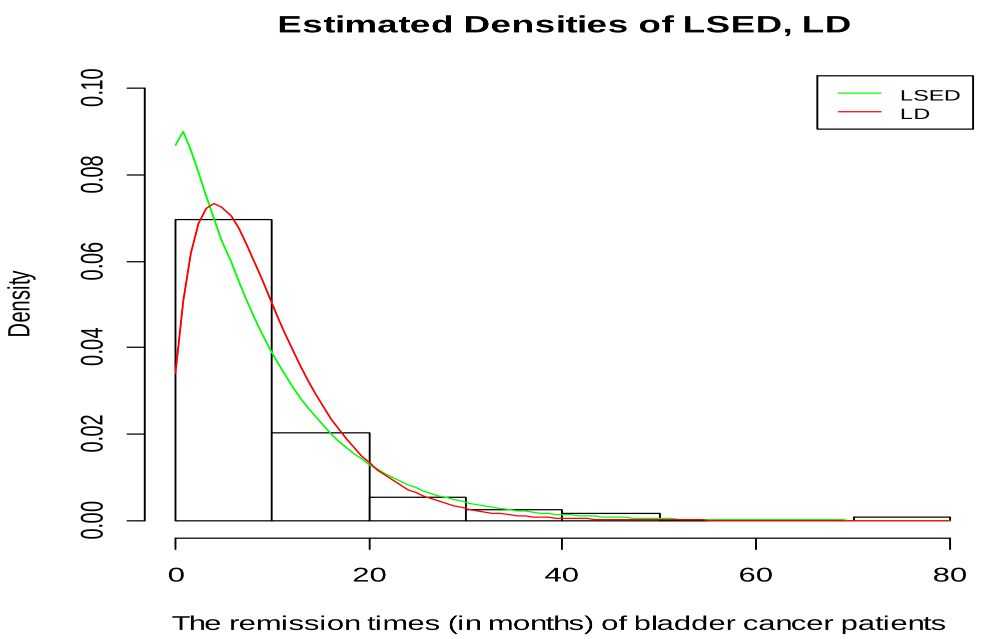

Figure 8 : Fitted Density curves with remission on times of bladder cancer patients data

From Table 2 Figure 9, numerically and graphically, it is observed that, **LSED** gave better performance than **LD** due to its minimum measures of **AIC**, **CAIC**, **BIC** and **HQIC**.

Data 2:

The dataset is taken from Gross and Clark (1975, p. 105) and shows the relief times of 20 patients receiving an analgesic. The data are presented below. We fit the LSED, GD, WD and LG distributions to the real dataset.

1.1	1.4	1.3	1.7	1.9
1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4
3.0	1.7	2.3	1.6	2.0

Table 2: MLE Estimates of LSED, GD, WD and LGD with Measures of Goodness of Fit

Model	MLE Estimates	AIC	BIC
LSED	$\hat{\alpha} = 7.1561$ $\hat{b} = 3.7736$ $\hat{\theta} = 136.9243$	34.03076	37.0180
GD	$\hat{\alpha} = 5.0887$ $\hat{\beta} = 9.6685$	39.6372	41.6287
WD	$\hat{\alpha} = 2.7870$ $\hat{\beta} = 2.1300$	45.1728	47.1643
LG	$\hat{p} = -125.1293$ $\hat{\theta} = 3.1827$	42.6723	44.6638

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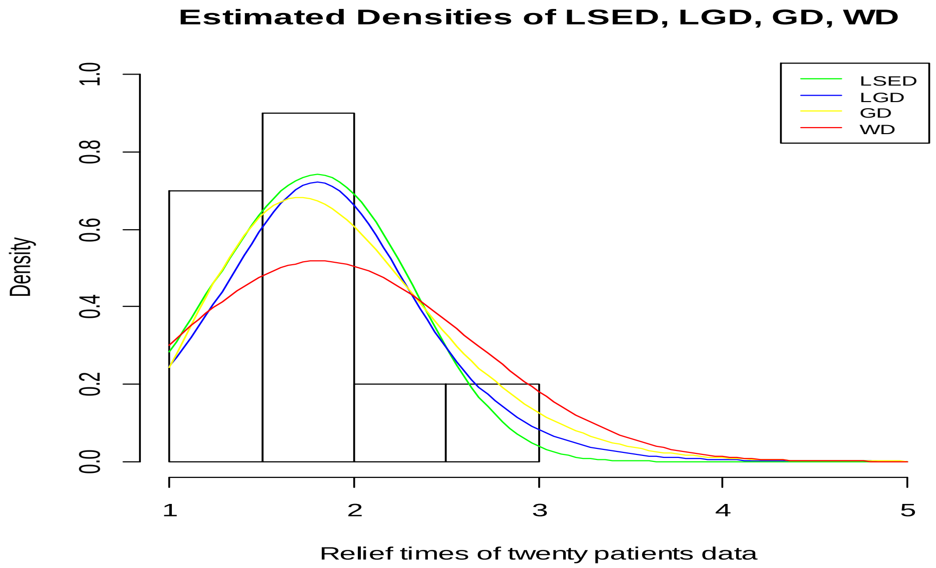

Figure 9 : Fitted Density curves with relief time of twenty patients data

From Table 3 Figure 10, numerically and graphically, it is also observed that, **LSED** gave better performance than **LGD, GD,** and **WD** due to its minimum measures of **AIC** and **BIC**.

Conclusion:

In this paper the Lindley Generalized Stretched Exponential distribution has been introduced. Its mathematical properties have been derived. The method of Maximum Likelihood has been used to estimate its parameters. The usefulness of distribution has also been shown by four real life data sets. It has been observed from both of results, numerically and graphically that Lindley Generalized Stretched Exponential distribution of three parameters proves a better fit for data connected to all real life data as compared to its specific cases as well as other competitive models.

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