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Linear Stability Analysis of Two Immiscible Superpose Homogeneous Fluids

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Abstract

In this research, we considered two immiscible superposed fluids of variable densities. The upper fluid was set in motion with uniform velocity and slide pass the lower stationary fluid. The equations describing the above scenario are mathematically represented and linearized. Thereafter, a normal mode solution was sought and the results obtained were discussed in line with onset of instability. We analysed the case of equal densities and as well equal velocities. The case of variable densities was not discussed in this research due to the fact that the effects of gravity and surface tension were not considered in our model equations, however, readers may consult [12] for details

Keywords: superposed, density, normal mode, gravity

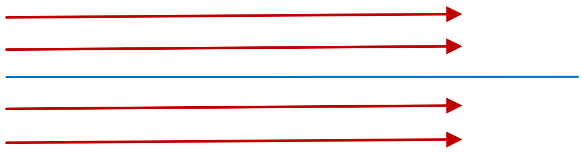
1. Introduction

Superposed immiscible fluids are subject to destabilizing effect of velocity shear and the stabilizing effect caused by density variation. This scenario is often called Kelvin–Helmholtz (KH) instability [1] which was originated and developed by von Helmholtz [2] and Kelvin [3] and had long been considered a standard for fluid mechanics. Also, the basic theory of (KH) can be found in the following text books, such as Lamb [4], Turner [5], Kundu [6] and Scorer [7] to mention a few. [8] observed in their study of Kelvin Helmholtz instability in fan-spine topology that the amplitude and characteristic wavelength of the K–H unstable vortices increased progressively. Critical study of the phenomena of instability, in addition to secondary instabilities had been considered by [9] , while the condition of the fastest-growing K–H mode is given by Miura & Pritchett [10] and Li et al. [11]. In this research, we considered two immiscible superpose fluids and study the dynamics of perturbing one of the fluids as detailed in section two.

2. Problem Formulation

Consider two immiscible fluids of variable densities ρ_1 and ρ_2. The upper fluid with ρ_2 is placed at $(z > 0)$ while the lower fluid with density ρ_1 is located at (z< 0) with $(z = 0)$ signifying the interface. The lower fluid is assume stationary while the upper fluid slide past the lower fluid with uniform velocity and steady velocity $q = Ui$. With introduction of gravity, the above formulation is known as Kelvin Helmholtz instability problem. In the present study, gravity is not considered. The above formulation is as shown in the diagram below.

$\rho = \rho_2$ and $U = U_2 i, \quad z > 0$



$\rho = \rho_1$ and $U = U_1 i, \quad z < 0$

Figure 1: Schematic of the Problem Formulation

Now, by perturbation the interface whose vertical displacement $z = \eta(x, t)$, and correspondingly we assume the velocity and pressure to be q_1 , p_1 and q_2 , p_2 in the regions $z < 0$ and $z > 0$ respectively. Introducing the velocity potential $\phi = \phi(x, t)$ for non-steady state and $\phi = \phi(x)$ for steady state, the equations describing the flow scenario are as follows:

Continuity equation

$$\nabla \cdot q = 0 = \nabla \cdot \nabla\phi = 0 = \nabla^2\phi = 0 \qquad (2.1)$$

$$q_1 = \nabla\phi_1 \quad \text{and} \quad \nabla^2\phi_1 = 0, \quad z < 0 \qquad (2.2)$$

with the Bernoulli's equation given as

$$\frac{\partial\phi_1}{\partial t} + \frac{(\nabla^2\phi_1)^2}{2} = -\frac{p_1}{\rho_1} + C_1 \qquad (2.3)$$

Similarly, in the upper fluid, we have

$$q_2 = \nabla\phi_2 \quad \text{and} \quad \nabla^2\phi_2 = 0, \quad z > 0 \tag{2.4}$$

and the corresponding Bernoulli's equation is given as

$$\frac{\partial\phi_2}{\partial t} + \frac{(\nabla^2\phi_2)^2}{2} = -\frac{p_2}{\rho_2} + C_1 \tag{2.5}$$

The kinematic condition at the interface $G = z = \eta(x,t)$ is given as

$$\frac{\partial G}{\partial t} + \nabla \mathrm{q}_1 \cdot \nabla G = 0 \tag{2.6}$$

and

$$\frac{\partial G}{\partial t} + \nabla \mathrm{q}_2 \cdot \nabla G = 0 \tag{2.7}$$

Or

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi_1}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\phi_1}{\partial z}, \quad z = \eta \tag{2.8}$$

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi_2}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\phi_2}{\partial z}, \quad z = \eta \tag{2.9}$$

The dynamic condition at the interface $z = \eta(x,t)$ is given as $p_1 = p_2$. This therefore implies that the Bernoulli equation becomes

$$\rho_1\left[C_1\left(\frac{\partial\phi_1}{\partial t} + \frac{(\nabla\phi_1)^2}{2}\right)\right] = \rho_1\left[C_2\left(\frac{\partial\phi_2}{\partial t} + \frac{(\nabla\phi_2)^2}{2}\right)\right] \tag{2.10}$$

Note that the distortion at the perturbed interface decreases proportionally away from the interface. Eventually at $z \to \infty$ and $z \to -\infty$, there exists no transient disturbances and thus

$$\nabla\phi_1 = U_1 i \text{ and } \nabla\phi_2 = U_2 i \quad \text{thus} \tag{2.10}$$

$$\rho_1\left(C_1 - \frac{{U_1}^2}{2}\right) = \rho_2\left(C_1 - \frac{{U_2}^2}{2}\right) \tag{2.11}$$

3. Linearization Process

Since the displacement of the interface is very minimal in relation to other scales, the velocity potential is therefore presented as:

$$\phi_1 = U_1 x + \epsilon\phi_1' \qquad z < \epsilon z \tag{3.1}$$

$$\phi_2 = U_2 x + \epsilon\phi_2' \qquad z > \epsilon z \tag{3.2}$$

with $\epsilon \ll 1$. The interface position of the fluids is therefore represented by

$z = \epsilon z(x, y, t)$ as shown in figure 2

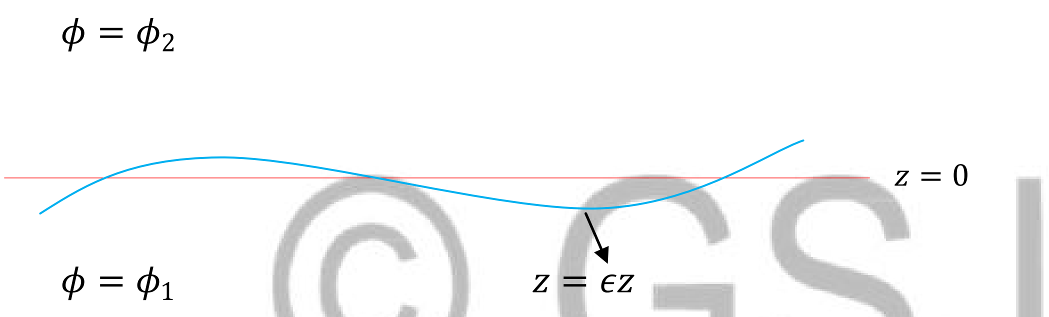


Figure 2: Perturbed interface

Now substituting equations (3.1) and (3.2) into equations (1.1) and (1.3) and neglecting terms $O(\epsilon^2)$ and higher, it results to a linearised set of equations given by

$$\nabla^2\phi_1' = 0, \quad \epsilon z < 0 \tag{3.3}$$

$$\nabla^2\phi_2' = 0, \quad \epsilon z > 0 \tag{3.4}$$

Similarly, using equations (3.1) and (3.2) in the kinematic boundary conditions (1.5) and (1.6), we have the following set of equations

$$\frac{\partial\eta}{\partial t} + U_1\frac{\partial\eta}{\partial x} = \frac{\partial\phi_1'}{\partial z}, \qquad \epsilon z = 0 \tag{3.5}$$

$$\frac{\partial\eta}{\partial t} + U_2\frac{\partial\eta}{\partial x} = \frac{\partial\phi_2'}{\partial z}, \qquad \epsilon z = 0 \tag{3.6}$$

but in the dynamic condition (1.10), the term

$\frac{U_1^2}{2} = \frac{(\nabla\phi_1)^2}{2} = \frac{\left(U_1 i + \epsilon\nabla\phi_1'\right)^2}{2} = \frac{U_1^2}{2} + \epsilon U_1 \frac{\partial \phi_1'}{\partial x}$ similarly,

$\frac{U_2^2}{2} = \frac{(\nabla\phi_2)^2}{2} = \frac{\left(U_2 i + \epsilon\nabla\phi_2'\right)^2}{2} = \frac{U_2^2}{2} + \epsilon U_2 \frac{\partial \phi_2'}{\partial x}$. Thus the linearised dynamic condition results to

$$\rho_1 \left(\frac{\partial \phi_1'}{\partial t} + U_1 \frac{\partial \phi_1'}{\partial x}\right) = \rho_2 \left(\frac{\partial \phi_2'}{\partial t} + U_1 \frac{\partial \phi_2'}{\partial x}\right), \ \epsilon z = 0 \tag{3.7}$$

4. NORMAL MODE ANALYSIS

Let the perturbation at the interface $z = \eta(x, y, t)$ be sinusoidal of the form

$$\eta = \alpha e^{i(kx - \omega t)} \tag{4.1}$$

and

$$\phi_1' = \beta_1 e^{-kz} e^{i(kx - \omega t)} \tag{4.2}$$

$$\phi_2' = \beta_2 e^{-kz} e^{i(kx - \omega t)} \tag{4.3}$$

where α, β_1 and β_2 are unknown constants.

Now substituting equations (4.1) – (4.3) into the kinematic and dynamic boundary conditions (3.5), (3.6) and (3.7), we have the following:

$$(-i\omega + ikU_1)\alpha = -k\beta_1 \tag{4.4}$$

$$(-i\omega + ikU_2)\alpha = -k\beta_2 \tag{4.5}$$

$$\rho_1(-i\omega + ikU_1)\beta_1 = \rho_2(-i\omega + ikU_2)\beta_2 \tag{4.6}$$

Equations (4.4) –(4.6) are rearranged as

$$(-i\omega + ikU_1)\alpha \qquad + \qquad k\beta_1 \qquad\qquad = 0 \tag{4.7}$$

$$(-i\omega + ikU_2)\alpha \qquad - k\beta_2 \qquad = 0 \tag{4.8}$$

$$\rho_1(-i\omega + ikU_1)\beta_1 \qquad - \rho_2(-i\omega + ikU_2)\beta_2 \qquad = 0 \tag{4.9}$$

and presented in matrix form as

$$\begin{bmatrix} -i\omega + ikU_1 & k & 0 \\ -i\omega + ikU_2 & 0 & -k \\ 0 & \rho_1(-i\omega + ikU_1) & -\rho_2(-i\omega + ikU_2) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{bmatrix} = 0 \tag{4.10}$$

A nontrivial solution exists if and only if the determinant of the matrix of coefficients in equation (4.10) vanishes, which results to:

$$\rho_1(-\omega + kU_1)^2 + \rho_2(-\omega + kU_2)^2 = 0 \tag{4.11}$$

$$\rho_1(\omega^2 - 2\omega kU_1 + k^2U_1^2) + \rho_2(\omega^2 - 2\omega kU_2 + k^2U_2^2) = 0$$

$$(\rho_1 + \rho_2)\omega^2 - 2k(\rho_1U_1 + \rho_2U_2)\omega + k^2(\rho_1U_1^2 + \rho_2U_2^2) = 0 \tag{4.12}$$

Observe that equation (4.12) is a quadratic equation in ω. Therefore, solving equation (4.12) by formula method we have

$$\omega = \frac{2k(\rho_1U_1 + \rho_2U_2) \pm \sqrt{4k^2(\rho_1U_1 + \rho_2U_2)^2 - 4k^2(\rho_1 + \rho_2)(\rho_1U_1^2 + \rho_2U_2^2)}}{2(\rho_1 + \rho_2)}$$

Simplification of the terms under the square root symbol, result to the following:

$$4k^2[\rho_1^2U_1^2 + 2\rho_1\rho_2U_1U_2 + \rho_2^2U_2^2] - 4k^2[\rho_1^2U_1^2 + \rho_1\rho_2U_1^2 + \rho_1\rho_2U_2^2 + \rho_2^2U_2^2]$$

$$= 4k^2\rho_1^2U_1^2 + 8k^2\rho_1\rho_2U_1U_2 + 4k^2\rho_2^2U_2^2 - 4k^2\rho_1^2U_1^2 - 4k^2\rho_1\rho_2U_1^2 - 4k^2\rho_1\rho_2U_2^2 - 4k^2\rho_2^2U_2^2$$

$$= 8k^2\rho_1\rho_2U_1U_2 - 4k^2\rho_1\rho_2(U_1^2 + U_2^2)$$

$$= 4k^2\rho_1\rho_2[2U_1U_2] - 4k^2\rho_1\rho_2(U_1^2 + U_2^2)$$

$$= -4k^2\rho_1\rho_2[U_1^2 - 2U_1U_2 + U_2^2]$$

$$= -4k^2\rho_1\rho_2(U_1 - U_2)^2$$

Thus

$$\sqrt{4k^2(\rho_1U_1 + \rho_2U_2)^2 - 4k^2(\rho_1 + \rho_2)(\rho_1U_1^2 + \rho_2U_2^2)} = \sqrt{-4k^2\rho_1\rho_2(U_1 - U_2)^2}$$

$$= 2ik(U_1 - U_2)\sqrt{\rho_1\rho_2}$$

$$\therefore \omega = k\frac{(\rho_1U_1+\rho_2U_2)}{\rho_1+\rho_2} \pm ik\frac{\sqrt{\rho_1\rho_2}(U_1-U_2)}{\rho_1+\rho_2} \quad (4.13)$$

5. RESULTS AND DISCUSSION

Considering equation (4.13), the following can be deduced

(i) if $\rho_1 = \rho_2$ and suppose $U_1 = -U_2 = U > 0$, equation (4.13) it reduces to

$$\omega = \pm ikU \quad (5.1)$$

is purely imaginary. Thus the term $+ikU$ signifies instability and the growth rate is given as $Uk = \frac{2\pi U}{\lambda}$ where λ is the wave length.

(ii) if $U_1 = U_2 = U$, the imaginary part of equation (4.13) vanishes and thus

$$\omega = kU \quad (5.2)$$

In (5.2), the magnitude of the wave front is a function of U and decays linearly. The special cases of $\rho_1 < \rho_2$ and $\rho_1 > \rho_2$ and cannot be sufficiently interpreted using equation (4.13) since surface tension and gravitational effects were not considered in our model. More details can be obtain from the book of [12]

5. **Conclusions**

We considered two immiscible fluids of variable densities ρ_1 and ρ_2. The upper fluid with ρ_2 is placed at $(z > 0)$ while the lower fluid with density ρ_1 is located at (z< 0) with $(z = 0)$ signifying the interface. The upper fluid was set in motion with uniform velocity and slide pass the lower stationary fluid. The equations describing the above scenario are mathematically represented and linearized. Thereafter, a normal mode solution was sought and the results obtained were discussed in line with onset of instability.

6. **Conflict of Interests**

The authors hereby declare that there is no conflict of interests during and at the completion of this research.

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