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MODELLING OF MULTIPHASE POROUS MEDIA FLOW EQUATIONS

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Abstract
We modelled simgle phase, two phase and three phase flow equations in a porous medium from first principle. The model was developed using the principle of conservation of mass, Darcy law, saturation and capillary pressure relations. Also we provided a modeling process for the pressure gradient in a hypothetical water flooding experiment. Our modeled equations have the potentials to capture inherent flows scnarios in porous media.

Keywords: Porous media, Modelling, Saturation, Porousity and Permeability

1 INTRODUCTION

Fluid transport modeling through porous media is essential to numerous environmental, biological and industrial systems. Processes such as the movement of contaminants in the subsurface and their remediation, geologic nuclear waste disposal, medical application such as brain and liver cancer treatment and most notably in oil recovery from petroleum reservoirs Arezou *et al.* (2019) are some examples of porous media transport. In petroleum reservoirs, the inherent heterogeneity of subsurface porous media, as well as the complexity involved in the multiphase physics, highlights some of the most important technological challenges of our time (Komal *et.at*, 2023; Vincent *et.at* 2022; Pan and Miller 2003; Nagi, 2009). In order to understand the dynamics of porous media transport , we must have sufficient knowledge of the constitutive relationships between the macroscopic properties of the system such as relative permeabilities, capillary pressures and fluid saturations which are essential in the modeling of the flow transport (Mohammed and Pramod, 2015). The determination of these constitutive relationships are however not without challenge as they are dependent on the fluid properties, the pore space as well as the saturation history. The inherent complexity of pore-scale displacement through the irregular geometry of natural porous media makes the prediction of multiphase flow mechanism in geological processes a very difficult task. Therefore, any scientific approach to this problem does not only requires a detailed understanding of the multiphase displacement mechanisms at the micro scale level but must also understand the structure of the porous medium (Pereira *et.al*. 1996; Corey 1994; Helmig 1997). Our current civilized world, will very likely continue to depend on petroleum products either as energy resource or as vital materials for consumer products in the near future. The complexity in the understanding of the pore scale displacement mechanism in the petroleum reservoir, has resulted to a decline in the production of conventional petroleum products Tore and Eyvind (2008), thereby mounting pressure on the discoveries of new oil wells as well as oil exploration in vulnerable areas such as the arctic regions. In the petroleum industry, the economic value of a reservoir is determined by the amount of oil which can be produced from the reservoir, which is affected by either field-scale fluid flow behavior within the porous media as well as pore-scale behaviour of the flow. The pore-scale behaviour of the flow dictates the macroscopic (core-scale) properties of porous media, such as capillary pressure as well as the relative permeability. In reality, due to the complicated transport phenomena involved, the multiphase flow and heat transfer remain poorly understood and analytically intractable (Starikovicius, 2003, Zuonaki and Orukari 2021). This can only be achieved if there is a robust mathematical model for multiphase flow phenomena; which this is the motivation behind this research

2 MATHEMATICAL FORMULATION

Our inability to predict accurately multiphase flow phenomena in porous media is simply because existing models fail to accurately capture the inherent transport processes in the medium. This is because either there were over simplifying assumptions made during the modeling process or some parameters which play significant role in the flow mechanisms are not captured in the model. We acknowledged that there are inherent challenges in subsurface flow modeling. For example, in petroleum reservoir modeling, it is almost impossible to accurately predict the porosity and the permeability of the rock properties. Another challenge is how the different fluid phases such as oil, gas and water interact as well as the rate of mass transfer. Thus we develop an enhanced mathematical model which is not based on over simplifying assumptions but which captures sufficient parameters relevant to flow processes within the continuum scale. The mathematical model of this physical system is set by differential equations and some special boundary conditions. To this end, we will apply the fundamental rule of conservation of mass and Darcy equation for each phase as well as constitutive relations. We start with the modeling of single phase mass conservation equation in a porous medium from the first principle.

2.1 Development Of Flow Equations In Porous Media

2.1.1 Mass conservation

The principle of conservation of mass discusses the balance between the rate of mass change in an arbitrary volume and the inflow of mass through the boundary surface area. In integral form, this can be expressed as follows:

$$\frac{\partial}{\partial t}\iiint \rho\phi dV + \iint \rho\mathbf{u}\cdot\mathbf{n}dS = \iiint q dV \tag{2.1}$$

The double and triple integrals in (2.1) are taken over the surface and volume respectively while the parameters $\rho, \phi, \mathbf{u}, \mathbf{n},$ and q represent the fluid density, the porosity the medium, the velocity vector, the unit outward normal vector and the external mass flow rate respectively. The second term on the right hand side of equation (2.1) can be converted into a volume integral form by using the Gauss' divergence theorem such as:

$$\iint \rho\mathbf{u}\cdot\mathbf{n}dS = \iiint \nabla\cdot(\rho\mathbf{u})dV \tag{2.2}$$

Using equation (2.2) in (2.1) and for a fixed control volume, the integral form of the conservation law results to

$$\iiint \left[\frac{\partial(\rho\phi)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - q \right] dV = 0 \qquad (2.3)$$

since $dV \neq 0$ (i.e the control volume), it implies that

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - q = 0 \text{ or}$$

$$\frac{\partial(\rho\phi)}{\partial t} = -\nabla \cdot (\rho \mathbf{u}) + q \qquad (2.4)$$

where ∇ is the del operator defined as

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

Equation (2.4) is known as the mass conservation equation

We remark that q by convention is negative for sinks and positive for sources. Introducing the formation volume factor B defined as $B = \frac{\rho_s}{\rho} \Rightarrow \rho = \frac{\rho_s}{B}$ where ρ_s is the fluid density at standard conditions. Substituting $\rho = \frac{\rho_s}{B}$ into equation (2.4), we have

$$\frac{\partial}{\partial t}\left(\frac{\phi}{B} \right) = -\nabla \cdot \left(\frac{1}{B} \mathbf{u} \right) + \frac{q}{\rho_s} \qquad (2.5)$$

Equations (2.4) and (2.5) are equivalent mass conservation equations

2.1.2 Darcy's law

This is an empirically observed law (Darcy; 1856) which states that the flow rate of a single phase fluid through a horizontal homogeneous porous medium is proportional to the pressure gradient across the medium and inversely proportional to the viscosity of the fluid. It is mathematically expressed as:

$$Q = \frac{KA}{\mu}\frac{\nabla P}{L} \tag{2.6}$$

where Q and K are the volumetric flow rate and permeability of the fluid respectively, μ is the viscosity of the fluid, ∇P is the pressure gradient across the medium while A and L are the cross sectional area and length of the system respectively while K is the permeability of the porous. A porous material will produce a permeability of one darcy if a pressure gradient of 1atmosphere applied across a sample of the material with a cross-sectional area of 1square cm and a length of 1 cm will produce a flow rate of 1 cubic cm per second when the fluid viscosity is 1 centipoise.

The differential form of Darcy law (2.6) is given as:

$$\mathbf{u} = \frac{Q}{A} = -\frac{K}{\mu}\frac{\partial P}{\partial x} \tag{2.6*}$$

where $\mathbf{u}$ is the superficial Darcy velocity and the negative sign signifies that the fluid flows in the direction of decreasing pressure. For multidimensional flow, Darcy law is given as:

$$\mathbf{u} = -\frac{K}{\mu}(\nabla P - \rho g \nabla D) \tag{2.7}$$

where $\mathbf{u}$ is the fluid flow velocity, P, the fluid pressure is the unknown function to be determined by the flow model, K is the absolute permeability tensor and a parameter of the solid matrix only and may depend on position. μ is the dynamic viscosity of the given fluid and is taken either as a constant or as a function of pressure. g is the gravitational vector, ρ is the fluid density and D, the physical depth. Darcy's law is valid for slow flow of a Newtonian fluid through porous medium with rigid solid matrix (Zhangxin *et al.* 2006; Zhangxin C. 2007).

2.1.3 Single phase flow equation

By substituting equation (2.7) into equation (2.4) results to

$$\frac{\partial(\phi\rho)}{\partial t} = \nabla\cdot(\frac{\rho K}{\mu}(\nabla P - \rho g\nabla D)) + q \quad (2.8)$$

where $\phi, \rho, \mu, \mathrm{K}, P, g, D, q$ as earlier defined represent the porosity, density, viscosity, permeability, pressure, gravity, physical depth and external mass flow rate respectively. Equation (3.8) is a single-phase flow equation in porous media. In most practical applications, substituting equation (3.7) into equation (3.5) we have an alternative form of the single phase flow equation as

$$\frac{\partial}{\partial t}\left(\frac{\phi}{B}\right) = \nabla\cdot\left(\frac{K}{\mu B}(\nabla P - \rho g\nabla D)\right) + \frac{q}{\rho_s} \quad (2.9)$$

The new parameter in equation (2.9) is the formation volume factor B

2.1.4 Two -phase immiscible flow equation

In any petroleum reservoir, there exists at least two different fluid phases. The single phase scenario seldom occurs. Here, we develop the model for the displacement of oil by either water or gas. The challenge is that this happens in a simultaneous flow and not with a sharp edge. To circumvent this challenge, we assume that there is no mass transfer between the two fluids. We consider two-phase flow where the fluids are immiscible and one fluid phase is considered a wetting phase (the phase which wets the porous medium more) while the other is considered non-wetting. In a water – oil system, water is considered the wetting phase while oil is regarded as the non-wetting phase but in an oil – gas system, oil is referred to as the wetting phase while the gas is the non-wetting phase. We refer to the wetting phase by the subscript w and to the non-wetting phase by the subscript n. Thus we have

$$s_w + s_n = 1 \quad (2.10)$$

where s_w, s_n are the saturations of the wetting and non-wetting phase respectively. Also, due to the curvature and surface tension of the interface between the two phases, the pressure in the wetting fluid is less than that in the non-wetting fluid as mention by Held and Celia (2001). The pressure difference is given by the capillary pressure. As an empirical fact, the capillary pressure is a function of the saturation and the wetting phase Mohammad and Pramod (2015) and is defined by

$$p_{cnw}(\mathrm{s_w}) = p_n - p_w \tag{2.11}$$

At this point, we extend Darcy's law from single phase flow to two-phase flow by assuming that the phase pressure forces for each phase to flow. Thus equation (2.8) can be written as

$$\mathbf{u}_n = -\frac{Kk_{rn}}{\mu_n}(\nabla P_n - \rho_n G) \tag{2.12}$$

$$\mathbf{u}_w = -\frac{Kk_{rw}}{\mu_w}(\nabla P_w - \rho_w G) \tag{2.13}$$

Where α represents the phase (wetting and non-wetting), $Kk_{r\alpha}, P_\alpha, \mu_\alpha$ are the phase permeability, phase pressure and phase viscosity respectively and $G = g\nabla D$. Except for the accumulation term, the same derivation that led to (2.4) also applies to the mass conservation equation for each fluid phase. To obtain the rate of accumulation, we multiply the differential volume by the phase saturation $s_\alpha, (\alpha = w, n)$ Zhangxin et.al (2006). Thus the mass accumulation in a differential volume per unit time is represented as

$$\frac{\partial(\phi\rho_\alpha s_\alpha)}{\partial t}$$

Considering this and with the assumption that there is no mass transfer between phases in the immiscible flow, mass is conserved within each phase. Thus we obtain:

$$-\nabla\cdot(\rho_n\mathbf{u}_n) + q_n = \frac{\partial(\phi\rho_n s_n)}{\partial t} \tag{2.14}$$

$$-\nabla\cdot(\rho_w\mathbf{u}_w) + q_w = \frac{\partial(\phi\rho_w s_w)}{\partial t} \tag{2.15}$$

for the non-wetting phase and wetting phase respectively. Again, applying Darcy's law in equations (2.14) and (2.15) results to

$$\nabla \cdot \left[\frac{\rho_n K k_{rn}}{\mu_n} (\nabla p_n - \rho_n G) \right] + q_n = \frac{\partial(\phi \rho_n s_n)}{\partial t} \quad (2.16)$$

$$\nabla \cdot \left[\frac{\rho_w K k_{rw}}{\mu_w} (\nabla p_w - \rho_w G) \right] + q_w = \frac{\partial(\phi \rho_w s_w)}{\partial t} \quad (2.17)$$

Equations (2.16) and (2.17) represent the mathematical model describing the flow of two phase immiscible fluids in porous media.

2.1.5 Three-phase immiscible flow equation

Consider a system which involves three immiscible fluids such as gas, oil and water $(g, o\ w)$. We assume that no mass transfer between the three fluids. The derivation of three phase flow equation is analogous to that of two phase flow equations with slight modifications of the saturation and capillary pressure relations.

$$s_w + s_o + s_g = 1$$

(2.18)

In this case we have three capillary pressures in which two are independent and defined as follows:

$$p_{cow} = p_o - p_w \quad (2.19)$$

$$p_{cgo} = p_g - p_o \quad (2.20)$$

$$p_{cgw} = p_g - p_w = p_{cgo} + p_{cow} \quad (2.21)$$

Now the conservation equation for each of the three phases results to the following:

$$-\nabla \cdot (\rho_g \mathbf{u}_g) + \mathrm{q_g} = \frac{\partial(\phi \rho_g s_g)}{\partial t} \quad (2.22)$$

$$-\nabla \cdot (\rho_w \mathbf{u}_w) + \mathrm{q}_w = \frac{\partial(\phi \rho_w s_w)}{\partial t} \quad (2.23)$$

$$-\nabla\cdot(\rho_o\mathbf{u}_o)+\mathrm{q}_o=\frac{\partial(\phi\rho_o s_o)}{\partial t} \quad (2.24)$$

Next, we apply Darcy's law to equations (2.22) – (2.24) which results to the following set of equations:

$$\nabla\cdot\left[\frac{\rho_g K k_{rg}}{\mu_g}(\nabla p_g-\rho_g G)\right]+q_g=\frac{\partial(\phi\rho_g s_g)}{\partial t} \quad (2.25)$$

$$\nabla\cdot\left[\frac{\rho_w K k_{rw}}{\mu_w}(\nabla p_w-\rho_w G)\right]+q_w=\frac{\partial(\phi\rho_w s_w)}{\partial t} \quad (2.26)$$

$$\nabla\cdot\left[\frac{\rho_o K k_{ro}}{\mu_o}(\nabla p_o-\rho_o G)\right]+q_o=\frac{\partial(\phi\rho_o s_{go})}{\partial t} \quad (2.27)$$

As in the case of two phase flow equations, equations (2.25) - (2.27) represent the mathematical model describing the flow of three phase immiscible fluids in porous media. In order to solve equations (2.16), (2.17), (2.25), (2.26) and (2.27) for the transient pressure and saturation of each phase, the following additional information are required:

(i) appropriate boundary and initial conditions

(ii) capillary pressure and relative permeabilities as functions of saturation and

(iii) the porosity and fluid properties such as phase densities and viscosities as functions of pressure.

2.1.6 Initial and boundary conditions

Since our flow equations describe the changes of the function values in space and in time; to get their values at any given time and location, an initial condition as well as boundary conditions have to be defined. In practical applications, the processes to be investigated take place in a concrete geometry (e.g., in turbines, car engines, heat exchangers, chemical reactors, soil etc.) during a finite interval of time. The choice of the domain and of the time interval to be considered is dictated by the nature of the problem at hand, the objectives of the analytical or numerical study, and by the available resources. Furthermore, the choice of initial and/or boundary conditions that lead to a well-posed problem is of great importance.

2.1.6.1 Definition: Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $(0,\mathrm{T})$ be a time interval of interest. In general, the boundary Γ of Ω may consists of an inflow part $\Gamma_- = \{x \in \Gamma \,|\, \mathbf{u}\cdot\mathbf{n} < 0\}$, the outflow part $\Gamma_+ = \{x \in \Gamma \,|\, \mathbf{u}\cdot\mathbf{n} > 0\}$ and a solid wall $\Gamma_0 = \{x \in \Gamma \,|\, \mathbf{u}\cdot\mathbf{n} = 0\}$ where $\mathbf{n}$ denotes the unit outward normal to the boundary at the point $x \in \Gamma$.

Remarks: Since our flow equations contain time derivative, we must therefore include an initial condition that defines the distribution of mass at $t = 0$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega \tag{2.28}$$

Furthermore, since the fluid inside Ω interacts with the surrounding medium, it is therefore, necessary to prescribe suitable boundary conditions on Γ. If the values of $\mathbf{u}$ are known on $\Gamma_D \subset \Gamma$, they can be imposed as Dirichlet boundary conditions (boundary condition which specifies the value(s) of the unknown along the boundary of the domain)

$$\mathbf{u}(\mathbf{x},\mathrm{t}) = \mathbf{u}_D(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_D, \quad \forall t \in (0,T) \tag{2.29}$$

As a rule, this boundary condition is used at the inlet Γ_- and/or on the solid wall Γ_0.

Alternatively, a given normal flux may be prescribed on the complementary boundary part $\Gamma_N = \Gamma \setminus \Gamma_D$. This is referred to as Neumann boundary condition defined as

$$\mathbf{f}\cdot\mathbf{n} = g(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_N, \quad \forall t \in (0,T) \tag{2.30}$$

The flux $\mathbf{f}$ may consist of a convective and/or a diffusive part, depending on the information available. If $\mathbf{f} = \mathbf{f}_C$ (the convective flux) or the diffusive flux $\mathbf{f}_D$ is required to vanish, then the right-hand side of (2.30) is given by

$$g = (\mathbf{u}\cdot\mathbf{n})\mathrm{u} \text{ on } \Gamma_\pm \text{ and } g = 0 \text{ on } \Gamma_0 \tag{2.31}$$

2.1.7 Example of physical modeling approach: Two phase water flooding problem

Consider a petroleum reservoir of two dimensions. This is not out of place since the areal dimensions in petroleum reservoirs are usually much greater than the thickness, and since petroleum reservoirs are usually more permeable in the horizontal direction than the vertical direction, (ie $\mathrm{K}_H > \mathrm{K}_V$) many reservoirs are modeled as two-dimensional reservoirs. The injection and production wells would be considered as point sources or sinks in the continuity equation. The two fluid phases oil and water would be denoted by the subscripts o and w respectively. For the present illustration, we would neglect the effect of capillary pressure. Now from the above information, the equations describing the system are as follows:

$$\frac{\partial(\phi\rho_w s_w)}{\partial t} = -\nabla\cdot(\rho_w \mathbf{u}_w) + \sum_{m=1}^{NW} \mathrm{q}_{w_m}\,\delta(x_m - x)\delta(y_m - y) \tag{2.32}$$

$$\frac{\partial(\phi\rho_o s_o)}{\partial t} = -\nabla\cdot(\rho_o \mathbf{u}_o) + \sum_{m=1}^{NW} \mathrm{q}_{o_m}\,\delta(x_m - x)\delta(y_m - y) \tag{2.33}$$

$$\mathbf{u}_w = -\frac{\mathrm{K}k_{rw}}{\mu_w}\nabla P_w \tag{2.34}$$

$$\mathbf{u}_o = -\frac{\mathrm{K}k_{ro}}{\mu_o}\nabla P_o \tag{2.35}$$

$$s_o + s_w = 1 \tag{2.36}$$

where $\delta(x_m - x)$ is the Dirac delta function, NW is the number of wells while x_m and y_m denote the location of a single well. A positive value of q_{w_m} or q_{o_m} implies injection of water or oil while a negative value stands for production. For impermeable boundary, the appropriate boundary and initial conditions are follows:

$$\frac{\partial P}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \tag{2.37}$$

$$P(t=0) = P_{in} \tag{2.38}$$

$$s_w(t=0) = s_{\mathrm{in}} \tag{3.39}$$

Where **n** is the outward unit normal vector on the reservoir boundary $\partial\Omega$ while P_{in} and s_{in} represent the specified initial pressure and saturation. Rather than no-flux boundary condition on pressure, values of pressure can be specified over some or entire reservoir boundary. The following relations are adopted to represent the fluid densities and the reservoir porosity as functions of pressure:

$$c_w = \frac{1}{\rho_w}\frac{d\rho_w}{dP} \qquad (2.40)$$

$$c_o = \frac{1}{\rho_o}\frac{d\rho_o}{dP} \qquad (2.41)$$

$$c_r = \frac{1}{\phi}\frac{d\phi}{dP} \qquad (2.42)$$

where c_w, c_o and c_r are the compressibilities of water, oil and the porous space respectively. Equations (3.32-3.42) constitute the reservoir model which can be solved for the transient pressure and saturation when the reservoir properties such as ϕ, K, k_{rw} and k_{ro} and the well rates q_w and q_o are specified.

2.1.8 Generalized model for multiphase immiscible flow equations

In this section, we investigate further on the structure of the coupled partial differential equations: (2.16), (2.17), (2.25) – (2.27) by deriving a single partial differential equation involving pressure and saturation only for both the two phase and three phase flows respectively. This formulation is amenable to practical applications.

2.1.9 Pressure equation for two phase immiscible flow

By expanding the time derivatives of equations (2.14) and (2.15); we have:

$$-\nabla\cdot(\rho_n\mathbf{u_n}) + q_n = \left[\rho_n s_n\frac{\partial\phi}{\partial t} + \phi s_n\frac{d\rho_n}{dp_n}\frac{\partial p_n}{\partial t} + \phi\rho_n\frac{\partial s_n}{\partial t}\right] \qquad (2.43)$$

$$-\nabla\cdot(\rho_w\mathbf{u_w})+q_w=\left[\rho_w s_w\frac{\partial\phi}{\partial t}+\phi s_w\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}+\phi\rho_w\frac{\partial s_w}{\partial t}\right] \quad (2.44)$$

Now we divide (2.43) by ρ_n and (2.44) by ρ_w result to

$$\frac{-\nabla\cdot(\rho_n u_n)}{\rho_n}+\frac{q_n}{\rho_n}=\left[s_n\frac{\partial\phi}{\partial t}+\frac{\phi s_n}{\rho_n}\frac{d\rho_n}{dp_n}\frac{\partial p_n}{\partial t}+\phi\frac{\partial s_n}{\partial t}\right] \quad (2.45)$$

$$\frac{-\nabla\cdot(\rho_w u_w)}{\rho_w}+\frac{q_w}{\rho_w}=\left[s_w\frac{\partial\phi}{\partial t}+\frac{\phi s_w}{\rho_w}\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}+\phi\frac{\partial s_w}{\partial t}\right] \quad (2.46)$$

adding equations (2.45) and (2.46) gives

$$-\frac{\nabla\cdot(\rho_n u_n)}{\rho_n}-\frac{\nabla\cdot(\rho_w u_w)}{\rho_w}+\left(\frac{q_n}{\rho_n}+\frac{q_w}{\rho_w}\right)=(s_n+s_w)\frac{\partial\phi}{\partial t}+\frac{\phi s_n}{\rho_n}\frac{d\rho_n}{dp_n}\frac{\partial p_n}{\partial t}+\frac{\phi s_w}{\rho_w}\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}+\phi\frac{\partial}{\partial t}(s_n+s_w)$$

(2.47)

using the two phase saturation relation; equation (2.10), phase compressibility $c_i=\frac{1}{\rho_i}\frac{d\rho_i}{dp_i}$ and letting $Q=\frac{q_n}{\rho_n}+\frac{q_w}{\rho_w}$ equation (2.47) becomes

$$-\frac{\nabla\cdot(\rho_n u_n)}{\rho_n}-\frac{\nabla\cdot(\rho_w u_w)}{\rho_w}+Q=\frac{\partial\phi}{\partial t}+\phi s_n c_n\frac{\partial p_n}{\partial t}+\phi s_w c_w\frac{\partial p_w}{\partial t} \quad (2.48)$$

Equation (2.48) is the pressure equation for two phase immiscible flow in a porous medium; where Q is the total volumetric injection rate, c_n and c_w are the respective phase compresibilities, ρ_n and ρ_w are the respective phase densities while P_n and P_w are the respective phase pressures. μ_n , μ_w, s_n, s_w, represent phase viscosities and phase saturations respectively while ϕ is the rock porosity.

2.2.3 Pressure equation for three phase immiscible flow

Similar to the two phase immiscible flow equations; we expand the time derivatives of equations (2.22, 2.23 and 2.24). This gives:

$$-\nabla\cdot(\rho_g\mathbf{u}_g)+\mathrm{q_g}=\left[\rho_g s_g\frac{\partial\phi}{\partial t}+\phi s_g\frac{d\rho_g}{dp_g}\frac{\partial p_g}{\partial t}+\phi\rho_g\frac{\partial s_g}{\partial t}\right] \tag{2.49}$$

$$-\nabla\cdot(\rho_w\mathbf{u}_w)+\mathrm{q}_w=\left[\rho_w s_w\frac{\partial\phi}{\partial t}+\phi s_w\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}+\phi\rho_w\frac{\partial s_w}{\partial t}\right] \tag{2.50}$$

$$-\nabla\cdot(\rho_o\mathbf{u}_o)+\mathrm{q}_o=\left[\rho_o s_o\frac{\partial\phi}{\partial t}+\phi s_o\frac{d\rho_o}{dp_o}\frac{\partial p_o}{\partial t}+\phi\rho_o\frac{\partial s_o}{\partial t}\right] \tag{2.51}$$

Now divide through equation (2.49) by ρ_g, equation (2.50) by ρ_w and equation (2.51) by ρ_o we have,

$$\frac{-\nabla\cdot(\rho_g\mathbf{u}_g)}{\rho_g}+\frac{\mathrm{q_g}}{\rho_g}=\left[s_g\frac{\partial\phi}{\partial t}+\phi\frac{s_g}{\rho_g}\frac{d\rho_g}{dp_g}\frac{\partial p_g}{\partial t}+\phi\frac{\partial s_g}{\partial t}\right] \tag{2.52}$$

$$\frac{-\nabla\cdot(\rho_w\mathbf{u}_w)}{\rho_w}+\frac{\mathrm{q}_w}{\rho_w}=\left[s_w\frac{\partial\phi}{\partial t}+\phi\frac{s_w}{\rho_w}\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}+\phi\frac{\partial s_w}{\partial t}\right] \tag{2.53}$$

$$\frac{-\nabla\cdot(\rho_o\mathbf{u}_o)}{\rho_o}+\frac{\mathrm{q}_o}{\rho_o}=\left[s_o\frac{\partial\phi}{\partial t}+\phi\frac{s_o}{\rho_o}\frac{d\rho_o}{dp_o}\frac{\partial p_o}{\partial t}+\phi\frac{\partial s_o}{\partial t}\right] \tag{2.54}$$

Adding equations (2.52) – (2.54) results to

$$\frac{-\nabla\cdot(\rho_g\mathbf{u}_g)}{\rho_g}\frac{-\nabla\cdot(\rho_w\mathbf{u}_w)}{\rho_w}\frac{-\nabla\cdot(\rho_o\mathbf{u}_o)}{\rho_o}+\frac{\mathrm{q_g}}{\rho_g}+\frac{\mathrm{q_w}}{\rho_w}+\frac{\mathrm{q_o}}{\rho_o}=(s_g+s_w+s_o)\frac{\partial\phi}{\partial t}+\phi\frac{s_g}{\rho_g}\frac{d\rho_g}{dp_g}\frac{\partial p_g}{\partial t}+\phi\frac{s_w}{\rho_w}\frac{d\rho_w}{dp_w}\frac{\partial p_w}{\partial t}$$
$$+\phi\frac{s_o}{\rho_o}\frac{d\rho_o}{dp_o}\frac{\partial p_o}{\partial t}+\phi\frac{\partial}{\partial t}(s_g+s_w+s_o) \tag{2.55}$$

now using the three phase saturation relation in equation (2.18) and the phase compressibility $c_i=\frac{1}{\rho_i}\frac{d\rho_i}{dp_i}$ and letting $Q=\frac{q_g}{\rho_g}+\frac{q_w}{\rho_w}+\frac{q_o}{\rho_o}$ equation (2.55) becomes

$$\frac{-\nabla\cdot(\rho_g\mathbf{u}_g)}{\rho_g}\frac{-\nabla\cdot(\rho_w\mathbf{u}_w)}{\rho_w}\frac{-\nabla\cdot(\rho_o\mathbf{u}_o)}{\rho_o}+Q=\frac{\partial\phi}{\partial t}+\phi s_g c_g\frac{\partial p_g}{\partial t}+\phi s_w c_w\frac{\partial p_w}{\partial t}+\phi s_o c_o\frac{\partial p_o}{\partial t} \tag{2.56}$$

In the above equation. Q_i is the total volumetric injection rate, ϕ is the rock porosity while c_g, c_w and c_o are the phase compressibilities of gas, water and oil respectively. Others parameters are as defined above.

3. Results

In this section, we present the model equations of two phase and three phase equations respectively as well as their pressure formulations. For easy identification, we maintained the equation numbers as formulated in section two

(i) Two phase flow equation

$$\nabla \cdot \left[\frac{\rho_n K k_{rn}}{\mu_n} (\nabla p_n - \rho_n G) \right] + q_n = \frac{\partial(\phi \rho_n s_n)}{\partial t} \tag{2.16}$$

$$\nabla \cdot \left[\frac{\rho_w K k_{rw}}{\mu_w} (\nabla p_w - \rho_w G) \right] + q_w = \frac{\partial(\phi \rho_w s_w)}{\partial t} \tag{2.17}$$

$$-\frac{\nabla \cdot (\rho_n u_n)}{\rho_n} - \frac{\nabla \cdot (\rho_w u_w)}{\rho_w} + Q = \frac{\partial \phi}{\partial t} + \phi s_n c_n \frac{\partial p_n}{\partial t} + \phi s_w c_w \frac{\partial p_w}{\partial t} \tag{2.48}$$

Equations (2.16) and (2.17) are two phase equation while equation (2.48) is the pressure equation for two phase immiscible flow in a porous medium.

(ii) Three phase flow equation

$$\nabla \cdot \left[\frac{\rho_g K k_{rg}}{\mu_g} (\nabla p_g - \rho_g G) \right] + q_g = \frac{\partial(\phi \rho_g s_g)}{\partial t} \tag{2.25}$$

$$\nabla \cdot \left[\frac{\rho_w K k_{rw}}{\mu_w} (\nabla p_w - \rho_w G) \right] + q_w = \frac{\partial(\phi \rho_w s_w)}{\partial t} \tag{2.26}$$

$$\nabla \cdot \left[\frac{\rho_o K k_{ro}}{\mu_o} (\nabla p_o - \rho_o G) \right] + q_o = \frac{\partial(\phi \rho_o s_{go})}{\partial t} \tag{2.27}$$

$$\frac{-\nabla \cdot (\rho_g \mathbf{u}_g)}{\rho_g} \frac{-\nabla \cdot (\rho_w \mathbf{u}_w)}{\rho_w} \frac{-\nabla \cdot (\rho_o \mathbf{u}_o)}{\rho_o} + Q = \frac{\partial \phi}{\partial t} + \phi s_g c_g \frac{\partial p_g}{\partial t} + \phi s_w c_w \frac{\partial p_w}{\partial t} + \phi s_o c_o \frac{\partial p_o}{\partial t} \tag{2.56}$$

Equations (2.25) - (2.27) represent the mathematical model describing the flow of three phase immiscible fluids in porous media while equation (2.56) is the pressure equation for three phase immiscible flow equation in a poruous medium.

4. Conclusion

In this research, we have developed single phase, two phase and three phase flow equations in a porous medium. The mass balance equation for each fluid phase, darcy's law was modified to accommodate the different fluid phases as well as the constitutive relations for pore fluids and solid skeleton formed the basis of the multiphase formulation. Our flow equations were transform into pressure and saturation formulations and by rigorous mathematical applications, we are able to develop reservoir flow equations for two phase and three phase flows which have more essential parameters ever reported in literature. We hope that simulation results of these equations would capture the inherent flow scenarios observed in the laboratory and in the field.

5. References

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