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Modification of block Nyström type method (BNTM1) for solving the two point boundary value problems (BVPs) with Robin boundary conditions

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***Abstract**: In this paper, we proposed a modified block Nyström type method (BNTM1) with one-off-step-point for the numerical solution of two points boundary value problems with Robin boundary conditions. The proposed (BNTM1) is formulated from its continuous scheme which is constructed from an appropriate power series via collocation and interpolation techniques. The convergence analysis of the proposed method is presented that shows that the method is consistent, has zero stability and convergence. The proposed method has an advantage of solving the ODEs directly without reducing it into equivalent system of first order.*

The performance of the modified method (BNTM1) is demonstrated by comparing with: Bernoulli polynomial of order eight and ten (BP8, BP10) and diagonal block numerical method (2PDD4), the results shows an encouraging performance of the method.

Keywords: Block method, Collocation method, Nyström method, Robin boundary condition

1. Introduction

In this paper we consider two points boundary value problems associated with Robin boundary conditions which is stated in the form;

$$y''(x) = f(x, y, y'), \text{ for } a \le x \le b \qquad 1.1$$

With

$$\begin{aligned} c_1 y'(a) + c_2 y(a) &= \alpha \\ c_3 y'(b) + c_4 y(b) &= \beta \end{aligned} \qquad 1.2$$

Where $a, b, c_1, c_2, c_3, c_4, \alpha, and \beta$ are all constants and $c_1, c_2, c_3, and c_4$ are nonzero. The Robin boundary condition signifies that both functional values and derivative of the solution are given in (1.2). This boundary conditions arise in areas such as electromagnetic problems, heat transfer problems as impedance [1].

Interestingly, some of the problems e.g (1.2) do not have analytical solution and if they do, it is most times difficult to obtain. Therefore, the development of numerical methods becomes necessary to obtain approximate solutions as an alternative where analytical method fails. [5] suggested that algorithms should have the capacity to generate solutions at several points simultaneously, [6 , 7, 8] implemented the suggestion and obtain solution. [16] developed a diagonal block numerical method for solving two-points boundary value problems with Robin boundary conditions. Method [2] and [16] are linear multi-step methods with high computation time as a result of the number of function evaluations per step

To overcome the challenge stated above, this research is motivated by the speed of computational time associated with Nyström one step method by developing a modified block Nyström type method (BNTM1) suitable for the stated problem (1.1).

The paper is organized as follows; section 2 is derivation of the method, the convergence analysis is discussed in section 3, section 4 we perform the numerical experiments and the conclusion of the paper in section 5.

2. Development of the Block Nyström type Method (BNTM1)

In this chapter, we develop a continuous approximation that is used to obtain the block Nyström type method (BNTM1) for the solution of second order boundary value problems of the form

$$y'' = f(x, y, y'),\ \ y(a) = 0 \text{ and } y(b) = 0,\ x\varepsilon[a,b] \qquad 2.1$$

With the following assumptions;

(i) $f(x, y, y')$ is continuous,

(ii) $\frac{\partial f(x, y, y')}{\partial y}$ exist and is continuous.

Also (2.1) is considered on interval [a, b], for $a = x_0 < x_1 < \cdots < x_{n-1} < x_N = b$ with constant step size $h = \frac{b-a}{N}$, N>0 is an integer.

We initially consider the interval $x_n \le x \le x_{n+1}$ and assumed that the exact solution to (2.1) is approximated by a power series of the form,

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad 2.2$$

With the second derivative of (2.2) given as follow;

$$y''(x) = \sum_{j=0}^{r+s-1} j(j-1)a_j x^{j-2} \qquad 2.3$$

The method (BNTM!) is derived by the introduction of off-step points in the one step scheme following the method of [9, 11, 13, 15] and recently [12] using code in maple 2015 software.

By substituting (2.3) into (2.1) gives;

$$\sum_{j=0}^{r+s-1} j(j-1)a_j x^j = f(x, y, y') \qquad 2.4$$

2.1: Specification of the method (BNTM1)

$s+v$ is the sum of the number of collocation and interpolation points.

To derive a continuous method by considering one-off-step point, $v=\frac{1}{2}$ the following specifications are considered; $r=2, s=3, k=1$ $\alpha_j(x), \beta_j(x)$ and $\beta_v(x)$ are expressed as functions of x to obtain the continuous form;

$$y(x)=\alpha_0 y_n+\alpha_1 hy'_n+h^2[\beta_0 f_n+\beta_1 f_{n+\frac{1}{2}}+\beta_2 f_{n+1}] \qquad 2.5$$

By combining (2.2) and (2.3) with the off-step point, the matrix A of the proposed method with one-off-step point is expressed as follow:

$$A=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 \\ 0 & 0 & 2 & 6x_n+3h & 12(x_n+\frac{1}{2}h)^2 \\ 0 & 0 & 2 & 6x_n+6h & 12(x_n+h)^2 \end{bmatrix} \qquad 2.6$$

Using matrix inversion technique, with the help of maple software we obtain the inverse matrix B, for $B=A^{-1}$. The coefficient of the inverse matrix were obtained and substituted into equation (2.5) to generate the continuous scheme with $y(x)$ and evaluating the equation at point $x=x_{n+1}, x_{n+\frac{1}{2}}$ we obtain the following schemes;

$$y_{n+1}=y_n+hy'_n+\frac{1}{6}h^2 f_n+\frac{1}{3}h^2 f_{n+\frac{1}{2}} \qquad 2.7$$

$$y_{n+\frac{1}{2}}=y_n+\frac{1}{2}hy'_n+\frac{7}{96}h^2 f_n+\frac{1}{16}h^2 f_{n+\frac{1}{2}}-\frac{1}{96}h^2 f_{n+1} \qquad 2.8$$

The following additional schemes are obtained by differentiating equation (2.5) and evaluating at point, $x=x_{n+1}, x_{n+\frac{1}{2}}$

$$y'_{n+1}=y'_n+\frac{1}{6}hf_n+\frac{2}{3}hf_{n+\frac{1}{2}}+\frac{1}{6}hf_{n+1} \qquad 2.9$$

$$y'_{n+\frac{1}{2}}=y'_n+\frac{5}{24}hf_n+\frac{1}{3}hf_{n+\frac{1}{2}}-\frac{1}{24}hf_{n+1} \qquad 2.10$$

3: Analysis of the (BNTM1)

3.1: Order and error constant

We consider the order and error constant, consistency, zero stability and convergence of the method

According to [10] with some modification to incorporate off-step points, define the method by

$$\sum_j \alpha_{ij}^{(\mu)}(x) y_{n+j} = h^2 \sum_j \beta_{ij}^{(\mu)}(x) f_{n+j}, ij = 0, v_1, v_m, k \qquad 3.1$$

Where m is the number of off-step points used, μ is the degree of the derivative and $\alpha_{ij}(x)$ and $\beta_{ij}(x)$ are continuous coefficients.

Definition 3.1: [10]

The block methods is said to have order p if $C_0 = C_1 = ... = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$.

The term C_{p+2} is called the error constant and the local truncation error t_{n+k} of the method is given as

$$t_{n+k} = C_{p+2} h^{p+2} y^{(p+2)}(x) + O(h^{p+3}) \qquad 3.2$$

The error and order constant of the BNTM was generated using a computer code written in maple15 software as follow

$$C_{p+2} = \left(\frac{1}{720}, \frac{1}{1440}, 0, \frac{1}{384} \right)^T \text{ with Order } p = (3,3,3,3)^T$$

3.2: Consistency of the methods

Definition 3.2: [10]

The one step hybrid method is said to be consistent if the following conditions are satisfied

i. The order $p \geq 1$
ii. $\sum_{j=0}^{k} \alpha j = 0$
iii. $p(1) = p'(1) = 0$ and
iv. $p''(1) = 2!\sigma(1)$

According to [14] condition (i) is sufficient to show that the block method is consistent.

By condition (i) of definition (3.2) the block method is consistent, $p = 3 \geq 1$

3.3: Zero Stability of the methods

Definition 3.3: [10]

Hybrid block method is said to be zero stable if the roots $z_s, s = 1,..., n$ of the first characteristic polynomial $p(z)$

$$p(z) = \det[zA - E] \qquad 3.3$$

For $|z_s| \leq 1$ and all roots $|z_s| = 1$ has multiplicity not greater than two in the limit $h \to 0$

$$\rho(z) = \det\left[z\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] = \det\begin{bmatrix} z & -1 & 0 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z-1 \end{bmatrix} = z^3(z-1) \qquad 3.4$$

$z = 0$ or $z = 1$. Therefore the BNTM is stable

3.4: Convergence

We state Dahlquist's theorem for linear multistep methods without proof [10].

Dahlquist's theorem: State that the necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.

Therefore, the proposed (BNTM1) is consistent and zero stable, hence it is convergent.

4: Numerical Experiment

In this section, we present some numerical problems to illustrate the performance of the (BNTM1). The absolute error of the approximate solution is calculated as Error=[y(x)-y(exact)] using codes in maple 2015 software.

Problem 1: [2]

A non-linear differential equation with Robin boundary conditions $y'' = \frac{1}{2}(1 + x + y)^3, \; 0 < x < 1, \quad y'(0) - y(0) = \frac{1}{2}, and, y'(1) + y(1) = 1$

Exact solution: $y(x) = \frac{2}{2-x} - x - 1$

Problem 2: [16]

Nonlinear second-order differential equation with Robin boundary conditions

$y'' = -\exp(-2y), \; 0 \leq x \leq 1, \; y'(0) - y(0) = 1$ and $y'(1) + y(1) = 0.5 + In(2)$

Exact solution: $y(x) = In(1 + x)$.

The following notations are used in the results given as follows:

BP8: Bernoulli Polynomial of order 8 [2]

BP10: Bernoulli Polynomial of order 10 [2]

2PDD4: Direct two-point diagonal block method of order four [16]

Table 4.1 The results for BNTM1 and error of BNTM1, BP8 and BP10 for problem 1 at h=0.1

X	Y[t]	BNTM1	BNTM1 error	BP8 error	BP10 error
0.0	0.0000000000	0.0000000000	0.00E+00	0.00E+00	0.00E+00
0.1	-0.0473684210	-0.0473681985	2.23E-07	3.75E-07	3.80E-09
0.2	-0.0888888890	-0.0888887107	1.78E-07	2.92E-07	6.90E-09
0.3	-0.1235294120	-0.1235292994	1.13E-07	2.27E-07	1.40E-08
0.4	-0.1500000000	-0.1499999880	1.20E-08	4.96E-07	1.16E-08
0.5	-0.1666666670	-0.1666668115	1.45E-07	9.27E-08	2.30E-09
0.6	-0.1714285710	-0.1714289660	3.95E-07	5.53E-07	1.47E-08
0.7	-0.1615384620	-0.1615392628	8.01E-07	9.49E-08	1.31E-08
0.8	-0.1333333330	-0.1333348137	1.48E-06	4.53E-07	4.30E-09
0.9	-0.0818181820	-0.0818208320	2.65E-06	4.86E-07	2.00E-09
1.0	0.0000000000	0.0000000000	0.00E+00	0.00E+00	0.00E+00

From table 4.1, the result shows that at $x = 0.4$, BNTM1 and BP10 has the same error order $E-08$ which is higher than the order of BP8. This shows that the performance of the method BNTM1 is encouraging; also the result converges to the exact solution.

Table 4.2: Result of BNTM1 and error of BNTM1 and 2PDD4 for problem 2 at h=0.1

X	Y[x]	BNTM1	BNTM1 error	2PDD4 error
0.0	0.000000000000	0.000000000000	0.00E+00	0.00E+00
0.1	0.095310179800	0.095310577460	3.98E-07	5.30E-07
0.2	0.182321556800	0.182321820700	2.64E-07	9.48E-07
0.3	0.262364264500	0.262364440800	1.76E-07	1.66E-06
0.4	0.336472236600	0.336472353700	1.17E-07	1.33E-06
0.5	0.405465108100	0.405465183800	7.57E-08	1.53E-06

0.6	0.470003629200	0.470003675400	4.62E-08	1.34E-06
0.7	0.530628251100	0.530628275500	2.44E-08	1.38E-06
0.8	0.587786664900	0.587786673000	8.10E-09	1.25E-06
0.9	0.641853886200	0.641853881700	4.50E-09	1.23E-06
1.0	0.693147180600	0.693147166100	1.45E-08	1.14E-06

The result of table 4.2, shows convergence of BNTM1 to the exact solutions. Also, the comparison between the errors of 2PDD4 and BNTM1 indicate that the BNTM1 is of higher order which implies that the performance is better.

Figure 1.1: **Absolute Error of BNTM1, B8 and B10 for Problem 1, for h=0.1.**

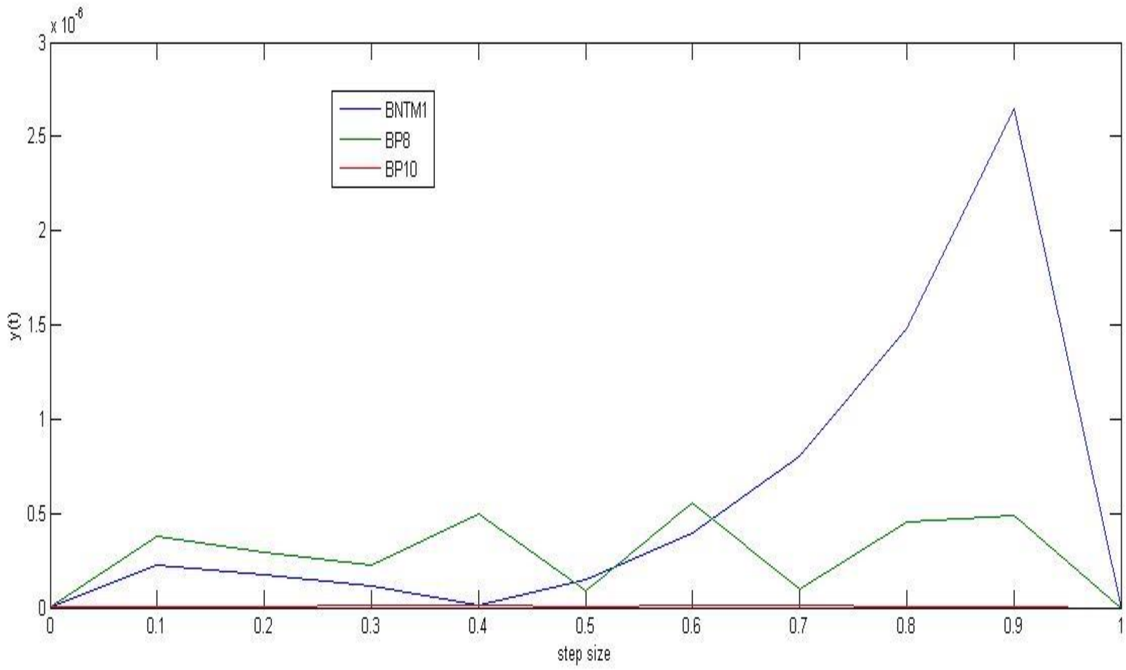
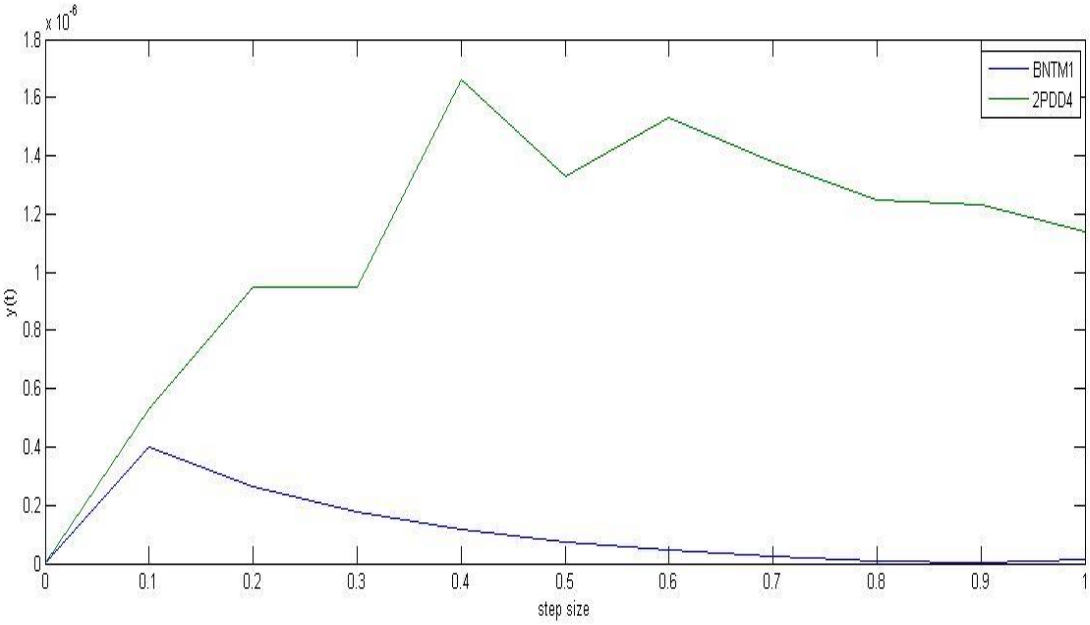


Figure 4.3: The absolute Error of BTNM1 and 2PDD4 for Problem 2, for h=0.1.



5. Conclusion

This study has presented a Block Nyström Type Method (BNTM) with one off-step point that is a modification of Nyström one step method. The analysis of the method has been carried out and shows that the method (BNTM1) has error and order constant $p = 3$, zero stable, consistent and convergence. The method (BNTM1) was used to solve two test problems. All formulations of the method (BNTM1) and numerical computations was carried out using computer code in Maple 2015 software. Also, the computed results were compared with exact solution and show good agreement.

References

[1] T T Akano and O A Fakinlede, 2015. Numerical computation of Sturm-liouville problem with Robin boundary condition, International Journal of mathematical, computational, physical, electrical and computer engineering. 9, 684–688.

[2] M S Islam and A Shirin, 2011. Numerical solutions of a class of second order boundary value problems on using Bernoulli polynomials, Applied Mathematics. 2(9), 1059–1067.

[3] J Duan, R Rach, A Wazwaz, T Chaolu and Z Wang, 2016. Anew modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions, Applied Mathematical Modeling, 37(20–21), 8687–8708.

[4] R Rach, J S Duan and A M Wazwaz, 2016. Solution of higher order, multipoint, nonlinear boundary value problems with high-order robin-type boundary conditions by the adomian decomposition method, Applied Mathematics & Information Sciences, 10(4), 1231–1242.

[5] S O Fatunla, 1995. A class of block methods for second order ivps, International Journal of Computer Mathematics, 55(1-2), 119–133.

[6] P S Phang, Z A Majid, and M Suleiman, 2012. On the solution of two point boundary value problems with two point direct method, International Journal of Modern Physics, 9, pp.566–573.

[7] Z A Majid, M Mughti Hasni, and N Senu, 2013. Solving second order linear Dirichlet and Neumann boundary value problems by block method, IAENG International Journal of Applied Mathematics, 43(2), 71–76.

[8] Z Omar and O Adeyeye, 2016. Solving two-point second order boundary value problems using two-step block method with starting and non-starting values, International Journal of Applied Engineering Research, 11(4), 2407–2410.

[9] C W Gear, 1964. Hybrid Methods for Initial Value Problems in Ordinary Differential Equations, SIAM Journal of Numerical Analysis, vol.2, 69-86.

[10] J P Chollom, J N Ndam and G M Kumleng, 2007. On Some Properties of the Block Linear Multistep Methods, Science World Journal,vol. 2(3), 11-17.

[11] J C Butcher, 2003. Numerical Methods for Ordinary Differential Equations, West Sussex: John Wiley & Sons Ltd.

[12] D O Awoyemi and O M Idowu, 2005. A Class of Hybrid Collocation Methods for Third Order Ordinary Differential Equations. Internationa Journal of Computer Mathematics, 82(10), 1287-1293.

[13] W B Gragg and H J Stetter, 1964. Generalized Multistep Predictor-Corrector Methods, Journal of the ACM, vol.11, 188-209.

[14] S N Jator, 2007. A Sixth Order Linear Multistep Method for the Direct Solution of y’’ = f(x,y,y’). International Journal of Pure and Applied Mathematics, 40(4), 457-472.

[15] J J Kohfield, 1967. Multistep Methods with Modified Predictors and Correctors, Journal of the ACM, 14(1), 155-166.

[16] N M Nasir, Z A Majid, F Ismail and N Bachok, 2018. Diagonal block method for solving two point boundary value problems with Robin boundary conditions, Mathematical problems in engineering, vol.2018

[17] Mathematical Problems in Engineering Volume 2018, Article ID 2056834, 12 pages https://doi.org/10.1155/2018/2056834