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ON p-GROUPS OF ORDER 2^{2n+e}, $e \in \{0,1\}$ SATISFYING $k(G) = p^e + (p^2-1)n + (p^2-1)(n-1)t$

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Abstract. Let $k(G)$ be the number of conjugacy classes of a group G. Then, there exists a non-negative integer $t = t(G)$ such that:

$$k(G) = p^e + (p^2-1)n + (p^2-1)(p-1)t \;\; t \geq 0, \;\; n \in Z^+ \text{ and } e \in \{0,1\}$$

Define: even = positive (+) and odd = negative (−).

Then, the following hold

(i) If n and t are of the same sign then, $k(G)$ is negative.
Otherwise, $k(G)$ is positive.

(ii) If n is 0, e and t are of the same sign, then $k(G)$ is negative.
Otherwise $k(G)$ is positive.
And the result is generally in harmony with the Sylow's C, D and E theorems [15], [21], [22], [23].

1. Introduction

The three main Sylow theorems are the Sylow E theorem, D theorem and C theorem sometimes referring to the "existence", "development" and "conjugate" theorem respectively. Another theorem gives the number of Sylow p-subgroups of a group for fixed prime p; this is sometimes referred to as the Sylow "counting theorem [15], [21].

A Sylow p-subgroup (sometimes p-Sylow subgroup) of a group G is a maximal p-subgroup of G, i.e. a subgroup which is a p-group and which is not a proper subgroup of any other p-subgroup of G. The set of all Sylow p-subgroups for a given prime p can be denoted by $Syl_\rho(G)$. Here, all members are actually isomorphic to each other and have the largest possible order: if $|G| = p^n m$, $n > 0$ where p does not divide m, then any Sylow p-subgroup W has order $|W| = p^n$. That is, W is a p-group and $(|G : W|, p) = 1$. This properties can be exploited to further analyze the structure of G [22], [23].

Suppose that $G > \{1\}$ is a p-group and $\chi \in Irr(G)$ = the set of irreducible complex characters of G.
If $\chi \in Irr(G)$, then $\chi(1)^2 \leq |\ G : Z(G)\ |$. Thus , $\chi(1)^2$ is a divisor of $|\ G : Z(G)\ |$. Let $k(G)$ be the class number of the group G. Then ,consider $Irr(G/Q)$ the subset $\{\chi \in Irr(G) | Q \leq ker(\chi)\}$ of $Irr(G)$. Let $Lin(G) = \{\chi \in Irr(F) \mid \chi(1) = 1\}$ be the set of linear characters of G ; then $Irr_1(G) = Irr(G) \backslash Lin(G)$ is the set of nonlinear irreducible characters of G . Clearly , $Irr_1(G) = Irr(G/G')$.
Proposition A :[20] Let G be a group of order p^m and let $D \leq Z(G) \ni |D| = p$. Then , $k(G) \geq p - 1 + k(G \mid D)$. We have that $|\ Irr(G \mid D)\ | \geq p - 1$

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(i) If $k(G) = p - 1 + k(G \mid G)$, then , $Z(G) = D$ and $\chi(1)^2 = p^{m-1}$ $\forall$ $\chi \in Irr(G \mid D)$ .Hence ,m is odd .
(ii) If m is even , then $k(G) \geq p^2 - p + k(G \mid D)$.

Let $cl(G) = \{A_1, A_2, \cdots, A_t\}$ be the set of all G-classes . i. e . t $= k(G)$; $a_i \in A_i$. Then ,the number of commuting ordered pairs $(x, y) \in GxG$ is equal to

$$\sum_{i=1} \mid C_G(a_i) \mid . \mid A_i \mid = \mid G \mid .k(G).$$

$\therefore$ the number of non-commutating ordered pair $(x, y) \in G$xG is equal to $|G|^2$ - $|G|.k(G)$.Let $\alpha_2(G)$denote the number of ordered non-commuting pairs (x, y) $\in G$ x $G \ni G = \langle x, y\rangle$ whence $\alpha_2(G) = 0$ if G is either abelian or not generated by 2 elements . Then , Mann has shown that :

$$\sum_{Q \leq G} \alpha_2(Q) = |G|^2 - |G|.k(G). \cdots (1)$$

The following theorem is due to P.Hall .

Proposition α**:**(see[20])

(a) Suppose that G is a 2-generator p-group then , $\varphi_2(G) = (p^2 - 1)(p^2 - p).|\Phi(G)|^2$
(b) If $G = ES(m, p)$ is an exraspecial group of order p^{2m+1} , then , the number $t = t(m, p)$ of nonabelian subgroup of order p^3 in G is equal to $\frac{p^{2m}-1}{p^2-1}.p^{2m-2}$

Proof :(a) $\varphi_2(G)$ is equal to the number of (ordered) minimal bases of G . Now, by assumption , $G/\Phi(G)$ is abelian of type (p,p) . If $a \in G\backslash\Phi(G)$, $G\backslash\langle x, \Phi(G)\rangle$, then ,$G = \langle a, b\rangle$. This element a , can be chosen in $|G| - |\Phi(G)| = |\Phi(G)|(p^2 - 1)$ ways and b , after the choice of a in $|G| - p|\Phi(G)| = |\Phi(G)|(p^2 - p)$ ways . Applying the combinatorial product rule , the result follows.
(b) Every nonabelian 2-generator subgroup of G has order p^3 .By using (1) and (a) , taking into cognisance the value $k(G) = p^{2m} + p - 1$, we have that $p^3(p-1)(p^2-1)t$ $= p^{2m+1}(p^{2m+1} - p^{2m} - p + 1) = p^{2m+1}(p^{2m} - 1)(p - 1)$ $\square$

Theorem β**:** Let $|G| = p^{2n+e}$, $e \in \{0, 1\}$. Then $\exists$ a non-negative integer $t = t(G)$ $\ni$ k(G) $= p^e + (p^2 - 1)[n + (p - 1)t] = p^e + (p^2 - 1)n + (p^2 - 1)(p - 1)t$.
If $Q \trianglelefteq G$ then $t(G/Q) \leq t(G)$.

Proof : (See [13]) By $\alpha(a)$, and (1) we get $|G|^2 \equiv |G|.k(G) mod (p^2 - 1)(p^2 - p)$. And so, $k(G) \equiv |G|(mod(p^2 - 1)(p - 1))$.
Now , observe that: $p^{2n+e} = p^e + (p^{2n} - 1)p^e$
$= p^e + (p^2 - 1)(p^{2n-2} + p^{2n-4} + \cdots + p^2 + 1)(p^e - 1 + 1)$
$= p^e + (p^2 - 1)(p^{2n-2} + p^{2n-4} + \cdots + p^2 + 1)(p^2 - 1)$
$+ (p^2 - 1)(p^{2n-2} + p^{2n-4} + \cdots + p^2 + 1)$
$\equiv p^e + (p^2 - 1)(p^{2n-2} + \cdots + p^2 + 1) \equiv p^e + (p^2 - 1)n(mod(p^2 - 1)(p - 1))$
(since $p^{2n-2} + \cdots + p^2 + 1 \equiv n(mod(p - 1))$) .
$\therefore k(G) = p^e + (p^2 - 1)n + (p^2 - 1)(p - 1)t = p^e + (p^2 - 1)[n + (p - 1)t]$, for some $t = t(G) \in \mathbb{Z}$. Now, to prove that $t(G)$ is non negative , use induction on $|G|$ .By assuming that $n > 1$. Let $Q \leq Z(G)$ be such that $|Q| = p$. Then ,$k(G) > k(G/Q)$ since $\bigcap_{\chi \in Irr(G)} Ker\chi(1) = Ker(\rho G) = \{1\}$, where ρG is the regular character of G. Now ,let $e = 0$, then $|G| = p^{2n}$ and $|G/Q| = p^{2(n-1)+1}$, so , by the result above , $k(G) = 1 + (P^2 - 1)[n + (p - 1)t(G)]$ and $k(G/Q) = p + (p^2 - 1)[n - 1 + (p - 1)t(G/Q)]$
And so, $0 < k(G)$ - $k(G/Q)$ $= 1 - p + (p^2 - 1)[1 + (t(G) - t(G/Q))(p - 1)]$.
$\therefore t(G) \geq t(G/Q)$ is obtained , since $t(G)$ - $t(G/Q) \in \mathbb{Z}$,

and 1+ $[t(G)-t(G/Q)](p-1) > \frac{p-1}{p^2-1} > 0$.
And so ,$t(G) \geq t(G/Q) \geq 0$, by induction.
Let $e = 1$, then , $|G| = p^{2n+1}$, $|G/Q| = p^{2n}$. Then , by the result, we have that : $k(G) = p + (p^2-1)[n+(p-1)t(G)]$ and $k(G/Q) = 1+ (p^2-1)[n+(p-1)t(G/Q)]$
So that $0 < k(G)$ - $k(G/Q) = p-1+ (p^2-1)(p-1)[t(G)-t(G/Q)]$. This must be that $t(G)-t(G/Q) \geq 0$. Thus , $t(G) \geq t(G/Q) \geq 0$ □

Proposition : Suppose that $Q \lhd G$ and $|G:Q| = p$. Let s be the number of G-invariant characters in $Irr(Q)$. Then ,

(i) $pk(G) = k(Q) + (p^2-1)s$.
(ii) If , in addition , G is a p-group , then $p-1$ divides $s-1$.

Proof : (i) If $\tau \in Irr(Q)$ is G-invariant , then , $\tau^(G) = \chi^1 + \cdots + \chi^p$
where $Irr(\tau^G) = \{\tau^1, \cdots, \tau^p\}$, $\chi(1) = \tau(1)$.(This was given by Clifford). Now , let $\varphi \in Irr(Q)$ be not Ginvariant, then, $\varphi^G = \chi \in Irr(G)$ and $\chi_Q = \varphi_1 + \cdots + \varphi_p$ is called the Clifford decomposition , where $\varphi = \varphi_1$.
Now , since $\bigcup_{\tau \in Irr(Q)} Irr(\tau^G) = Irr(G)$,

$$Irr((\sum_{\chi \in Irr(G)} \chi)Q) \quad = \quad Irr(Q)$$

we thus have that : $k(G) = |Irr(G)| = sp + \frac{1}{p}(|Irr(Q)|-s) = sp + \frac{1}{p}(k(Q)-s)$, and the result follows . By theorem (β) , we have the proof of (ii) . □

Proposition : (See [13])The number $s = s(G)$ of nonabelian subgroups of order p^3 is a p-group G , $|G| = p^m$, $m \geq 5$ is divisible by p^2 .

Proof : (See [20])By proposition $\alpha(a)$ and (1), we have that:
$sp^2(p-1)(p^2-1)$ +

$$\sum_{Q \leq G, |Q| > p^3} \alpha_2(Q) = p^m.[p^m - k(G)] \cdots (ii)$$

If $Q \leq G$ is a 2-generator subgroup of order $\geq p^4$, then $|\Phi(Q)| \geq p^2$. Thus , $\alpha_2(Q)$ is divisible by p^5 ,by $\alpha(a)$.
By (ii) , we have the result . □

It was proved by Mann [in sec. 2 [20]] that

$$k(G) = p^e + (p^2-1)[n+(p-1)t].$$

Now, to exhaust all possibilities,

$$\begin{aligned} k(G) = p^e + t(1-p) - n(\text{mod}p) &= p^e + t - pt - n(\text{mod}p) \\ &= p^e + t - n(\text{mod}p) \end{aligned}$$

Now, when $e = 0$, $k(G) = 1 + t - n(\text{mod})$
$|G| = 2^{2n}$ implies $n \geq 1$, $p = 2$.
If n is even, (then $n = 2u$ say) implies that $k(G) = 1+t-2u(\text{ mod } 2) = 1+t(\text{ mod } 2)$.
Further, if t is even, $k(G) = 1(\text{mod}2)$. But if t is odd, $t = 2d+1$, say, thus, $k(G) = 0$ (mod2) implies that $k(G)$ is even.
If n is odd, (then, $n = 2w+1$ say), implies that $k(G) = 1+t-2w-1 = t(\text{mod}2)$.
So, if t is odd, $k(G) = 1(\text{mod}2)$ and even t implies $k(G)$ is even.
By Sylow's, if $B = \{B_i\}$ is a family of all subsets of G, each of order p^k.
Then, $|B| =$ the number of conjugacy classes of order p^k.
Hence, it implies that t is odd for $p = 2$ in harmony with the result

$k(G) = 1(\mathrm{mod}2)$.
Summarily, we have the generalized result given in tabular form as follows:

	n	t	$k(G)$	
0	even	even	odd	
0	even	odd	even	
0	odd	odd	odd	
0	odd	even	even	
1	0	even	even	
1	0	odd	odd	
1	1	even	even	
1	1	odd	odd	
1	even	even	odd	
1	odd	odd	odd	

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