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# ON A MODIFIED HIGHER ORDER FORM OF CONJUGATE GRADIENT METHOD FOR SOLVING SOME OPTIMAL CONTROL PROBLEMS

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ABSTRACT:

In this paper, we embed a control operator in a modified conjugate gradient method. The embedded control operator in the modified conjugate gradient method was used to solve the Lagrange form of optimal control problems. The solutions obtained in all the problems converged appropriately in just few iterations by considering the repetition of the functional values and gradient norm as the basis of convergence.



KEYWORDS: Control operator, Embed, Lagrange Form, Modified Conjugate Gradient Method, Optimal control problem.

## 1.0 INTRODUCTION

Optimal control theory is a branch of applied mathematics that deals with finding a control law for a dynamical system over a period of time such that an objective function is optimized. It has numerous applications in the fields of science, engineering, medicine and social sciences. Several methods are available for solving optimal control problems. Ibiejugba and Onumanyi [4] constructed a control operator, the explicit knowledge of which was used to devise an Extended Conjugate Gradient Method (ECGM) algorithm for solving the Lagrange form of optimal control problems without delay in the state equations. Aderibigbe [2] constructed a control operator which enabled the ECGM to be employed in solving the Lagrange form of optimal control problems but with delay parameter in the state equations. The work of [2] was subsequently extended to the Bolza form of optimal control problems by [1]. As a result of the successes recorded by [4] and [2], Olorunsola and Olotu [6] came up with the idea of discretizing the component of the control operators to solve continuous optimal control problems constrained by evolution equation. The work of these researchers gave us the desire to embed the control operator constructed by [4] in a Modified Conjugate Gradient Method Algorithm developed in [5] to solve the Lagrange form of optimal control problems and the results obtained converges appropriately in just few iterations.

## 2.0 MATERIAL AND METHODS

### 2.1 Modified Conjugate Gradient Methods (MCGM)

The Modified Conjugate Gradient Method (MCGM) developed in [5] was designed to solve equations of the form:

Minimize $f(x) = \frac{1}{2}x^T Ax - b^T x$ (1)

The MCGM is a variant of the gradient method. In its simplest form, the gradient method uses the iterative scheme:

$x_{i+1} = x_i - \alpha \nabla f(x)$ (2)

to generate a sequence $\{x_i\}_{i=1}^{n}$ of vectors which converges to the minimum of $f(x)$, [3]. The parameter $\alpha$ appearing in (2) denotes the step length of the descent direction. In particular, if F is a function on a Hibert space $\mathcal{H}$, F admits a Taylor series expansion

$f(x) = f_0 + \langle a, x\rangle_{\mathcal{H}} + \frac{1}{2}\langle x, Qx\rangle_{\mathcal{H}}$ (3)

where $a, x \in \mathcal{H}$ and Q is a positive definite, symmetric, linear operator, then it can be shown that f possesses a unique minimum $x^* \in \mathcal{H}$ and that $\nabla f(x) = 0$, [3]. The MCGM algorithm for iteratively locating the minimum $x^*$ of $f(x)$ in $\mathcal{H}$ as described by [5] is given as follows:

Step 1: Guess the initial element, $x_0$

Step 2: Compute the gradient of the function at initial guess, $g_0$

Step 3: Compute the descent direction, $p_0 = -g_0$

Step 4: Set $x_{i+1} = x_i + \alpha_i p_i$, $\forall\, i = 0, 1, 2, \ldots, n$

where $\alpha_i = \frac{{g_i}^T g_i}{{p_i}^T Q p_i}$, $\forall\, i = 0, 1, 2, \ldots, n$

Step 5: Compute $g_{i+1} = g_i + \alpha_i Q p_i$, $\forall\, i = 0, 1, 2, \ldots, n$

Step 6: Update the descent direction, $p_{i+1} = -g_{i+1} + \beta_i p_i$,

where $\beta_i = \frac{-(g_{i+1} + g_i)^T g_{i+1}}{{p_i}^T g_i}$, $\forall\, i = 0, 1, 2, \ldots, n$

Step 7: $If\ g_i = 0$ for some *i* then, terminate the sequence; else set

$i = i + 1$ and go to Step 4

In the iterative steps 2 through 7 above, $g_i$ denotes the gradient of the function f at $x_i$, $p_i$ denotes the descent direction at i-th step of the algorithm and $\alpha_i$ denotes the step length of the descent sequence $\{x_i\}$. Step 3, 4, 5, and 6 of the algorithm reveal the crucial role of the linear operator Q in determining the step length of the descent sequence and also in generating a conjugate direction of search.

### 2.2 The Extended Modified Conjugate Gradient Method (EMCGM)

According to [2], an extended modified conjugate gradient method (EMCGM) adopts the MCGM to obtain the solution of Lagrange form of optimal control problem of the form:

$$\min_{(x,\,u)} J = \int_{t_0}^{t_f} \{x^T(t)Ax(t) + u^T(t)Bu(t)\}\, dt$$

(4)

subject to: $\dot{x}(t) = Cx(t) + Du(t);\ 0 \le t \le T$ (5)

$x(0) = x_0$

where A, B, C and D are specified constants such that $A > 0,\ B > 0$; $x_0$ and $t_f$ are given, $\dot{x}(t)$ denotes the derivative of the state vector, x(.), with respect to time and u(.) is the control vector. As conventional with penalty function techniques, (4) and (5) may equivalently be written in the form:

$$\min_{(x,\,u)} J = \int_0^T \left\{Ax^2(t) + Bu^2(t) + \mu||\dot{x}(t) - Cx(t) - Du(t)||^2\right\} dt,\ \mu > 0 \quad (6)$$

where $\mu$ is the penalty parameter and $\mu||\dot{x}(t) - Cx(t) - Du(t)||^2$ is the penalty term. The control operator, Q, is related to the problem in the sense that:

$$\langle y,\ \hat{Q}y\rangle_H \cong \int_0^T \left\{Ax^2(t) + Bu^2(t) + \mu\left\|\dot{x}(t) - Cx(t) + Du(t)\right\|^2\right\} dt \quad (7)$$

where $y^T(t) = \big(x(t),\ u(t)\big)$ and $\mathcal{H}$ is a suitably chosen Hilbert space. According to [4], the operator $\hat{Q}$ is then utilized in the iterative procedure of the MCGM in order to arrive at a solution. Generally, for discrete type of Optimization problems that satisfies the hypotheses in (8) below, the linear operator Q is readily determined in [4].

$$f(x) = f_0 + \langle a, x\rangle_{\mathcal{H}} + \frac{1}{2}\langle x, Qx\rangle_{\mathcal{H}}, \quad (8)$$

Such problems enjoyed the beauty of the MCGM as a computational scheme since MCGM exhibit quadratic convergence and require only a little more computation per iteration. According to [4], the operator Q is such that:

$$(\hat{Q}z)(t) \equiv \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} (Q_{11}x)(t) + (Q_{12}u)(t) \\ (Q_{21}x)(t) + (Q_{22}u)(t) \end{bmatrix} \quad (9)$$

with the composite operators $Q_{11},\ Q_{12,}\ Q_{21}\ and\ Q_{22}$ explicitly given by

$$(Q_{11}x)(t) = -\mu\big[\dot{x}(0) - Cx(0)\big]Sinh + \mu\int_0^t \big[\dot{x}(s) - Cx(s)\big]Cosh(t-s)ds - \int_0^t [(A +$$

$$\mu C^2)x(s) - \mu C\dot{x}(s)]\, Sinh(t-s)ds + [(a + \mu C^2)x(0) - \mu C\dot{x}(0)]Cosh(t) + \frac{Sinh(t)}{Sinh(T)}\Big\{(a +$$

$$\mu C^2)x(T) - \mu C\dot{x}(T) + \mu Sinh(T)\big[\dot{x}(0) - Cx(0)\big] - \mu\int_0^T \big[\dot{x}(s) - Cx(s)\big]Cosh\,(T-s)ds +$$

$\int_0^T [(A+\mu C^2)x(s) - \mu C\dot{x}(s)]\, Sinh(T-s)ds - \ [(A+\mu C^2)x(0) - \mu C\dot{x}(0)]Cosh(T)\Big\}, 0 \leq$

$t \ \leq T$ (10)

$(Q_{21}x)(t) = \mu CDx(t) - \ \mu D\dot{x}(\mathrm{t}), 0 \leq t \ \leq T$ (11)

$(Q_{12}u)(t) = \mu Du(0)Sinh(t) - \ \mu \int_0^t Du(s)Cosh(t-s)ds + \mu \int_0^t CDu(s)Sinh(t-s)ds +$

$\mu CDu(0)Cosh(t) + \frac{Sinh(t)}{Sinh(T)}\Big\{\mu CDu(T) - \ \mu Du(0)Sinh(T) + \mu \int_0^T Du(s)Cosh(T-s)ds \ +$

$\mu \int_0^T CDu(s)Sinh(T-s)ds + \ \mu CDu(0)Cosh(T)\Big\}, 0 \leq t \ \leq T$ (12)

$(Q_{22}u)(t) = Bu(t) + \mu D^2 u(t), 0 \leq t \ \leq T$ (13)

### 2.3 Extended Modified Conjugate Gradient Method (EMCGM) Algorithm

By embedding the control operator Q in the Modified Conjugate Gradient Method, we have the algorithm as follows:

Step 1: Guess the initial element, $x_0, u_0 \in \mathcal{H}$

Step 2: Compute the descent direction, $p_{x,0} = -g_{x,0}$ and $p_{u,0} = -g_{u,0}$

Step 3: Set $x_{i+1} = x_i + \alpha_{x,i} p_{x,i}$ ,

where $\alpha_{x,i} = \frac{{g_{x,i}}^T g_{x,i}}{{p_{x,i}}^T Q p_{x,i}}$ and

Set $u_{i+1} = u_i + \alpha_{u,i} p_{u,i}$ ,

where $\alpha_{u,i} = \frac{{g_{u,i}}^T g_{u,i}}{{p_{u,i}}^T Q p_{u,i}}$,

Step 4: Compute $g_{x,i+1} = g_{x,i} + \alpha_{x,i} Q p_{x,i}, \ \ \forall\, i = 0,\ 1,\ 2, \ldots, n$

and Compute $g_{u,i+1} = g_{u,i} + \alpha_{u,i} Q p_{u,i}, \ \ \forall\, i = 0,\ 1,\ 2, \ldots, n$

Step 5: Set $p_{x,i+1} = -g_{x,i+1} + \beta_{x,i} p_{x,i}$,

where $\beta_{x,i} = \frac{-(g_{x,i+1} + g_{x,i})^T g_{x,i+1}}{{p_{x,i}}^T g_{x,i}}$

and Set $p_{u,i+1} = -g_{u,i+1} + \beta_{u,i} p_{u,i}$,

where $\beta_{u,i} = \frac{-(g_{u,i+1} + g_{u,i})^T g_{u,i+1}}{{p_{u,i}}^T g_{u,i}}$

Step 6: If $g_i = 0$ for some i then, terminate the sequence;

else Set $i = i + 1$ and go to Step 3

In the iterative steps 2 through 6 above, $g_i$ denotes the gradient of the function f at $x_i$, $p_i$ denotes the descent direction at i-th step of the algorithm and $\alpha_i$ denotes the step length of the descent sequence $\{x_i\}$. Step 3, 4, 5, and 6 of the algorithm reveal the crucial role of the linear operator Q in determining the step length of the descent sequence and also in generating a conjugate direction of search.

## 3.0 COMPUTATIONAL RESULTS

### Problem 1

$$\min_{(x,u)} \int_0^1 \{x^2(t) + u^2(t)\}dt; 0 \leq t \leq 1$$

$$subject\ to: \dot{x}(t) = 3x(t) + 2u(t);\ 0 \leq t \leq 1$$

$$x(0) = 1,\ \mathrm{u}(0) = 1$$

**Table 1: Solution to Problem 1**

| ITR | X | U | FV | NORM |
|---|---|---|---|---|
| 0 | 1.00000000 | 1.00000000 | 127.000000 | 2.82842712 |
| 1 | 0.94459164 | 0.94459164 | 113.316178 | 0.3843021100 |
| 2 | 0.99207674 | 0.89744997 | 115.608048 | 0.2511465700 |
| 3 | 0.99702840 | 0.90252829 | 116.823502 | 0.25336093e-1 |
| 4 | 0.99346609 | 0.90596876 | 116.640165 | 0.1575323e-1 |
| 5 | 0.99316228 | 0.90564622 | 116.564396 | 0.13720856e-2 |
| 6 | 0.99335280 | 0.90546863 | 116.574820 | 0.6408406e-3 |

| | | | | |
|---|---|---|---|---|
| 7 | 0.99336482 | 0.90548199 | 116.577876 | 0.42015591e-4 |
| 8 | 0.99335895 | 0.90548720 | 116.577530 | 0.11481269e-4 |
| 9 | 0.99335875 | 0.90548695 | 116.577476 | 0.38605869e-6 |
| 10 | 0.99335881 | 0.90548690 | 116.577479 | 0.51135172e-7 |

**Problem 2**

$$\min_{(x,u)} \int_0^1 \{2x^2(t) + 2u^2(t)\}dt; 0 \leq t \leq 1$$

$$subject\ to:\ \dot{x}(t) = 3x(t) + u(t);\ 0 \leq t \leq 1$$

$$x(0) = 1,\ \mathrm{u}(0) = 1$$



**Table 2: Solution to Problem 2**

| ITR | X | U | FV | NORM |
|---|---|---|---|---|
| 0 | 1.00000000 | 1.00000000 | 84.0000000 | 5.65685425000 |
| 1 | 0.65660979 | 0.65660979 | 36.2154592 | 0.17067082000 |
| 2 | 0.80765882 | 0.540223 | 45.7910629 | 1.10796240000 |
| 3 | 0.75173267 | 0.49197846 | 39.3491829 | 0.33574668000 |
| 4 | 0.65980865 | 0.59925232 | 34.8368095 | 0.27399550000 |
| 5 | 0.67294755 | 0.60584264 | 36.0846718 | 0.20026079e-1 |
| 6 | 0.67694711 | 0.59743355 | 36.1695126 | 0.32464018e-1 |

| | | | | |
|---|---|---|---|---|
| 7 | 0.67854124 | 0.59830155 | 36.3245773 | 0.61276952e-2 |
| 8 | 0.67735273 | 0.60040097 | 36.2877823 | 0.50764396e-2 |
| 9 | 0.67711125 | 0.60025604 | 36.2638985 | 0.56479051e-3 |
| 10 | 0.67722327 | 0.60007912 | 36.2679647 | 0.23378155e-3 |

**Problem 3**

$$\min_{(x,u)} \int_0^1 \{-x^2(t) - u^2(t)\}dt; 0 \leq t \leq 1$$

$$subject\ to: \dot{x}(t) = 3x(t) + 2u(t);\ 0 \leq t \leq 1$$

$$x(0) = 1, \mathrm{u}(0) = 1$$



**Table 3: Solution to Problem 3**

| ITR | X | U | FV | NORM |
|---|---|---|---|---|
| 0 | 1.00000000 | 1.00000000 | 123.000000 | 2.8284271200 |
| 1 | 1.05775675 | 1.05775675 | 137.618468 | 0.3571074700 |
| 2 | 0.98757777 | 1.12735212 | 133.862040 | 0.5199856400 |
| 3 | 0.97776469 | 1.11705675 | 131.306666 | 0.74244322e-1 |
| 4 | 0.98854321 | 1.10685321 | 131.925267 | 0.06200040000 |
| 5 | 0.9896995 | 1.10810198 | 132.229413 | 0.70059025e-2 |

| | | | | |
|---|---|---|---|---|
| 6 | 0.98865632 | 1.10906026 | 132.166464 | 0.50916197e-2 |
| 7 | 0.98856343 | 1.10895652 | 132.141675 | 0.48872354e-3 |
| 8 | 0.98863549 | 1.10889256 | 132.146250 | 0.27767104e-3 |
| 9 | 0.98864043 | 1.1088983 | 132.147590 | 0.21025203e-4 |
| 10 | 0.98863733 | 1.10890093 | 132.147382 | 0.80774159e-5 |

## 4.0 DISCUSSION OF RESULTS AND CONCLUSION

From table 1 above, we can easily see that the functional value of problem 1 converges to 116.577479 as the gradient norm tends to zero in just few iterations. The same is also true for problem 2 that converges to 36.2679647, in table 2, as the gradient norm tends to zero in just few iterations. The result of problem 3 is shown in table 3 where the functional value converges to 132.147382 in the tenth iteration.

Therefore, with the penalty parameter $\mu = 5$, the results obtained in each of the problems performs very well as we can see from the results in tables 1 to 3. Hence, all the problems considered converge appropriately in just few iterations by considering the repetition of the functional values and gradient norm as the basis of convergence.

## 5.0 REFERENCES

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