ON THE NEGATIVE PELL EQUATION $y^2 = 102x^2 - 18$

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ABSTRACT:
The binary quadratic Diophantine equation represented by the negative Pellian $y^2 = 102x^2 - 18$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.
Introduction

The binary quadratic equations of the form \( y^2 = D x^2 + 1 \) where \( D \) is non-square negative integer has been selected by various mathematicians for its non-trivial integer solutions when \( D \) takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by \( y^2 = 102 x^2 - 18 \) is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of Analysis

The negative Pell equation representing hyperbola under consideration is,

\[
y^2 = 102 x^2 - 18
\]  \hspace{1cm} (1)

The smallest positive integer solutions of (1) are,

\[ x_0 = 3, \ y_0 = 30 \]

Consider the pellian equation

\[
y^2 = 102 x^2 + 1
\]  \hspace{1cm} (2)

The initial solution of (2)

\[ \tilde{x}_0 = 10, \ \tilde{y}_0 = 101 \]

The general solution \((x_n, y_n)\) of (2) is given by,

\[ \tilde{x}_n = \frac{1}{2\sqrt{102}} f_n, \ \tilde{y}_n = \frac{1}{2} f_n \]

where,

\[ f_n = (101 + 10\sqrt{102})^{n+1} + (101 - 10\sqrt{102})^{n+1} \]

\[ g_n = (101 + 10\sqrt{102})^{n+1} - (101 - 10\sqrt{102})^{n+1} \]

Applying Brahmagupta lemma between \((x_0, y_0)\) and \((\tilde{x}_n, \tilde{y}_n)\) the other integer solution of (1) are given by,

\[ x_{n+1} = \frac{3}{2} f_n + \frac{30}{2\sqrt{102}} g_n \]

\[ y_{n+1} = \frac{30}{2} f_n + \frac{306}{2\sqrt{102}} g_n \]

The recurrence relation satisfied by the solution \(x\) and \(y\) are given by,

\[ x_{n+3} - 202 x_{n+2} + x_{n+1} = 0 \]

\[ y_{n+3} - 202 y_{n+2} + y_{n+1} = 0 \]
Some numerical examples of $x$ and $y$ satisfying (1) are given in the Table 1 below.

<table>
<thead>
<tr>
<th>N</th>
<th>$x_n$</th>
<th>$y_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>1</td>
<td>603</td>
<td>6090</td>
</tr>
<tr>
<td>2</td>
<td>121803</td>
<td>1230150</td>
</tr>
<tr>
<td>3</td>
<td>12423906</td>
<td>248484210</td>
</tr>
</tbody>
</table>

From the above table, we observe some interesting relations among the solutions which are presented below.

- $y_n$ values are even.

- Each of the following expression is a nasty number:
  
  - $\frac{6}{9} \left[ 18 + 306x_{2n+2} - 30y_{2n+2} \right]$
  
  - $\frac{6}{90} \left[ 180 + 6090x_{2n+2} - 30x_{2n+3} \right]$
  
  - $\frac{6}{36360} \left[ 72720 + 2460300x_{n+1} - 60x_{n+3} \right]$
  
  - $\frac{6}{1818} \left[ 3636 + 123012x_{2n+2} - 60y_{2n+3} \right]$
  
  - $\frac{6}{183609} \left[ 367218 + 12423906x_{2n+2} - 30y_{2n+4} \right]$
  
  - $\frac{6}{909} \left[ 1818 + 306x_{2n+3} - 6090y_{2n+2} \right]$
  
  - $\frac{6}{183609} \left[ 367218 + 306x_{n+3} - 1230150y_{n+1} \right]$
  
  - $\frac{6}{9180} \left[ 18360 + 306y_{2n+3} - 61506y_{2n+2} \right]$
  
  - $\frac{6}{1854360} \left[ 3708720 + 306y_{2n+4} - 12423906y_{2n+2} \right]$
  
  - $\frac{6}{90} \left[ 180 + 1230150x_{2n+3} - 6090x_{2n+4} \right]$
  
  - $\frac{6}{9} \left[ 18 + 61506x_{2n+3} - 6090y_{2n+3} \right]$
  
  - $\frac{6}{909} \left[ 1818 + 12423906x_{2n+3} - 6090y_{2n+4} \right]$
  
  - $\frac{6}{909} \left[ 1818 + 61506x_{2n+4} - 1230150y_{2n+3} \right]$
  
  - $\frac{6}{9180} \left[ 18360 + 61506y_{2n+4} - 12423906x_{2n+3} \right]
Each of the following expression is a cubical integer.

\[
\frac{1}{9} \left[ 306x_{3n+3} - 30y_{3n+3} + 918x_{n+1} - 90y_{n+1} \right]
\]

\[
\frac{1}{90} \left[ 6090x_{3n+3} - 30x_{3n+4} + 18270x_{n+1} - 90x_{n+2} \right]
\]

\[
\frac{1}{36360} \left[ 2460300x_{3n+3} - 60x_{3n+5} + 7380900x_{n+1} - 180x_{n+3} \right]
\]

\[
\frac{1}{1818} \left[ 123012x_{3n+3} - 60y_{3n+4} + 369036x_{n+1} - 180y_{n+2} \right]
\]

\[
\frac{1}{183609} \left[ 12423906x_{3n+3} - 30y_{3n+5} + 37271718x_{n+1} - 90y_{n+2} \right]
\]

\[
\frac{1}{909} \left[ 306x_{3n+4} - 6090y_{3n+4} + 918x_{n+2} - 18270y_{n+1} \right]
\]

\[
\frac{1}{183609} \left[ 306x_{3n+5} - 1230150y_{3n+3} + 918x_{n+3} - 3690450y_{n+1} \right]
\]

\[
\frac{1}{9180} \left[ 306y_{3n+4} - 61506y_{3n+3} + 918y_{n+2} - 184518y_{n+1} \right]
\]

\[
\frac{1}{1854360} \left[ 306y_{3n+5} - 12423906y_{3n+3} + 918y_{n+3} - 37271718y_{n+1} \right]
\]

\[
\frac{1}{90} \left[ 1230150x_{3n+4} - 6090x_{3n+5} + 3690450x_{n+2} - 18270x_{n+3} \right]
\]

\[
\frac{1}{9} \left[ 61506x_{3n+4} - 6090y_{3n+4} + 184518x_{n+2} - 18270y_{n+2} \right]
\]

\[
\frac{1}{909} \left[ 12423906x_{3n+4} - 6090y_{3n+5} + 37271718x_{n+2} - 18270y_{n+3} \right]
\]

\[
\frac{1}{909} \left[ 61506x_{3n+5} - 1230150y_{3n+4} + 184518x_{n+3} - 3690450y_{n+2} \right]
\]

\[
\frac{1}{9180} \left[ 61506y_{3n+5} - 12423906y_{3n+4} + 184518y_{n+3} - 37271718y_{n+2} \right]
\]

Each of the following expression is a biquadratic integer.

\[
\frac{1}{9} \left[ 306x_{4n+4} - 30y_{4n+4} + 1224x_{2n+2} - 120y_{2n+2} + 54 \right]
\]

\[
\frac{1}{90} \left[ 6090x_{4n+4} - 30x_{4n+5} + 24360x_{2n+2} - 120x_{2n+3} + 540 \right]
\]

\[
\frac{1}{36360} \left[ 2460300x_{4n+4} - 60x_{4n+6} + 9841200x_{2n+2} - 240x_{2n+4} + 218160 \right]
\]

\[
\frac{1}{1818} \left[ 123012x_{4n+4} - 60y_{4n+5} + 492048x_{2n+2} - 240y_{2n+3} + 10908 \right]
\]

\[
\frac{1}{183609} \left[ 12423906x_{4n+4} - 30y_{4n+6} + 49695624x_{n+1} - 120y_{n+3} + 1101654 \right]
\]

\[
\frac{1}{909} \left[ 306x_{4n+5} - 6090y_{4n+4} + 1224x_{2n+3} - 24360y_{2n+2} + 5454 \right]
\]

\[
\frac{1}{183609} \left[ 306x_{4n+6} - 1230150y_{4n+4} + 1224x_{2n+4} - 4920600y_{2n+2} + 1101654 \right]
\]
\[ \frac{1}{9180} \left[ 306y_{4n+5} - 61506y_{4n+4} + 1224y_{2n+3} - 246024y_{2n+2} + 55080 \right] \]
\[ \frac{1}{1854360} \left[ 306y_{4n+6} - 12423906y_{4n+4} + 1224y_{2n+3} - 49695624y_{n+1} + 11126160 \right] \]
\[ \frac{1}{90} \left[ 1230150x_{4n+5} - 6090y_{4n+6} + 4920600x_{2n+3} - 24360x_{2n+4} + 540 \right] \]
\[ \frac{1}{9} \left[ 61506x_{4n+5} - 6090x_{4n+6} + 246024x_{2n+3} - 24360y_{2n+3} + 54 \right] \]
\[ \frac{1}{909} \left[ 12423906x_{4n+6} - 6090y_{4n+6} + 49695624x_{n+2} - 24360y_{n+3} + 5454 \right] \]
\[ \frac{1}{909} \left[ 61506x_{4n+6} - 1230150y_{4n+5} + 246024x_{2n+4} - 4920600y_{2n+3} + 5454 \right] \]
\[ \frac{1}{9180} \left[ 61506y_{4n+6} - 12423906y_{4n+5} + 246024y_{2n+4} - 49695624y_{2n+3} + 55080 \right] \]

Each of the following expression is a quintic integer:

\[ \frac{1}{9} \left[ 306x_{5n+5} - 30y_{5n+5} + 1530x_{3n+3} - 150y_{3n+3} + 3060x_{n+1} - 300y_{n+1} \right] \]
\[ \frac{1}{9} \left[ 6090x_{5n+5} - 30x_{5n+6} + 30450x_{3n+3} - 150x_{3n+4} + 60900x_{n+1} - 300x_{n+2} \right] \]
\[ \frac{1}{36360} \left[ 2460300x_{5n+5} - 60x_{5n+7} + 123035000x_{3n+3} - 300x_{3n+5} + 24603000x_{n+1} - 600x_{n+3} \right] \]
\[ \frac{1}{1818} \left[ 123012x_{5n+5} - 60y_{5n+6} + 615060x_{3n+3} - 300y_{3n+4} + 1230120x_{n+1} - 600y_{n+2} \right] \]
\[ \frac{1}{183609} \left[ 12423906x_{5n+5} - 30y_{5n+7} + 62119530x_{3n+3} - 150y_{3n+5} - 124239060x_{n+1} - 300y_{n+2} \right] \]
\[ \frac{1}{909} \left[ 306x_{5n+6} - 6090y_{5n+5} + 1530x_{3n+4} - 30450y_{3n+3} + 3060x_{n+2} - 60900y_{n+1} \right] \]
\[ \frac{1}{183609} \left[ 306x_{5n+7} - 1230150y_{5n+5} + 1530x_{3n+5} - 7380900y_{3n+3} + 3060x_{n+3} - 11071350y_{n+1} \right] \]
\[ \frac{1}{9180} \left[ 306y_{5n+6} - 61506y_{5n+5} + 1530y_{3n+4} - 307530y_{3n+3} + 3060y_{n+2} - 615060y_{n+1} \right] \]
\[ \frac{1}{1854360} \left[ 306y_{5n+7} - 12423906y_{5n+5} + 1530y_{3n+5} - 62119530y_{3n+3} + 3060y_{n+3} - 124239060y_{n+1} \right] \]
\[ \frac{1}{90} \left[ 1230150x_{5n+6} - 6090x_{5n+7} + 6150750x_{3n+4} - 30450x_{3n+5} + 12301500x_{n+2} - 60900x_{n+3} \right] \]
\[ \frac{1}{9} \left[ 61506x_{5n+6} - 6090y_{5n+6} + 307530x_{3n+4} - 30450y_{3n+5} + 615060x_{n+2} - 60900y_{n+2} \right] \]
\[ \frac{1}{909} \left[ 243906x_{5n+6} - 6090y_{5n+7} + 62119530x_{3n+4} - 30450y_{3n+5} + 124239060x_{n+2} - 60900y_{n+3} \right] \]
\[ \frac{1}{909} \left[ 61506x_{5n+7} - 1230150y_{5n+6} + 307530x_{3n+5} - 6150750y_{3n+4} + 615060x_{n+3} - 12301500y_{n+2} \right] \]
\[ \frac{1}{9180} \left[ 61506y_{5n+7} - 12423906y_{5n+6} + 307530y_{3n+5} - 62119530y_{3n+4} + 615060y_{n+3} - 124239060y_{n+3} \right] \]
Relations among the solutions are given below:

\begin{align*}
&x_{n+2} = 101x_{n+1} + 10y_{n+1} \\
&x_{n+3} = 20401x_{n+1} + 2020y_{n+1} \\
&y_{n+2} = 1020x_{n+1} + 101y_{n+1} \\
&y_{n+3} = 206040x_{n+1} + 20401y_{n+1} \\
&x_{n+3} = 202x_{n+2} - x_{n+1} \\
&10y_{n+2} = 101x_{n+2} - x_{n+1} \\
&10y_{n+3} = 20401x_{n+2} - 101x_{n+1} \\
&20y_{n+2} = x_{n+3} - x_{n+1} \\
&2020y_{n+3} = 20401x_{n+3} - x_{n+1} \\
&101y_{n+3} = 1020x_{n+1} + 20401y_{n+2} \\
&101x_{n+3} = 20401x_{n+2} + 10y_{n+1} \\
&101y_{n+2} = 1020x_{n+2} - y_{n+1} \\
&y_{n+3} = 2040x_{n+2} + y_{n+1} \\
&20401y_{n+2} = 1020x_{n+3} + 101y_{n+1} \\
&20401y_{n+3} = 206040x_{n+3} + y_{n+1} \\
&y_{n+3} = 202y_{n+2} - y_{n+1} \\
&10y_{n+2} = x_{n+3} - 101x_{n+2} \\
&10y_{n+3} = 101x_{n+3} - x_{n+2} \\
&y_{n+3} = 1020x_{n+2} + 101y_{n+2} \\
&101y_{n+3} = 1020x_{n+3} + y_{n+2}
\end{align*}

Remarkable Observations

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below.

<table>
<thead>
<tr>
<th>S.NO</th>
<th>Hyperbola</th>
<th>((X,Y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4Y^2 - 4131X^2 = 1296)</td>
<td>((y_{n+1} - 10x_{n+1}, 306x_{n+1} - 30y_{n+1}))</td>
</tr>
<tr>
<td>2</td>
<td>(Y^2 - 51X^2 = 32400)</td>
<td>((3x_{n+2} - 603x_{n+1}, 6090x_{n+1} - 30x_{n+2}))</td>
</tr>
<tr>
<td>3</td>
<td>(4Y^2 - 408X^2 = 5288198400)</td>
<td>((3x_{n+3} - 121803x_{n+1}, 1230150x_{n+1} - 30x_{n+3}))</td>
</tr>
<tr>
<td>4</td>
<td>(4Y^2 - 408X^2 = 13220496)</td>
<td>((3y_{n+2} - 6090x_{n+1}, 61506x_{n+1} - 30y_{n+2}))</td>
</tr>
<tr>
<td>5</td>
<td>(Y^2 - 102X^2 = 1.348490595 \times 10^{11})</td>
<td>((3y_{n+3} - 1230150x_{n+1}, 12423906x_{n+1} - 30y_{n+3}))</td>
</tr>
</tbody>
</table>
II. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below.

<table>
<thead>
<tr>
<th>S.N</th>
<th>Parabola</th>
<th>((X,Y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(4Y - 459X^2 = 144)</td>
<td>((y_{n+1} - 10x_{n+1}, 306x_{2n+2} - 30y_{2n+2} + 18))</td>
</tr>
<tr>
<td>2</td>
<td>(90Y - 51X^2 = 32400)</td>
<td>((3x_{n+2} - 603x_{n+1}, 6090x_{2n+2} - 30x_{2n+3} + 180))</td>
</tr>
<tr>
<td>3</td>
<td>(36360Y - 408X^2 = 5288198400)</td>
<td>((3x_{n+3} - 121803x_{n+1}, 2460300x_{2n+2} - 60x_{2n+4} + 72720))</td>
</tr>
<tr>
<td>4</td>
<td>(1818Y - 408X^2 = 13220496)</td>
<td>((3y_{n+2} - 6090x_{n+1}, 123012x_{2n+2} - 60y_{2n+3} + 3636))</td>
</tr>
<tr>
<td>5</td>
<td>(Y - 102X^2 = 734436)</td>
<td>((3y_{n+3} - 1230150x_{n+1}, 12423906x_{2n+2} - 30y_{2n+4} + 367218))</td>
</tr>
<tr>
<td>6</td>
<td>(909Y - 102X^2 = 3305124)</td>
<td>((603y_{n+1} - 30x_{n+2}, 306x_{2n+3} - 6090y_{2n+2} + 1818))</td>
</tr>
<tr>
<td>7</td>
<td>(Y - 102X^2 = 734436)</td>
<td>((306x_{n+3} - 1230150y_{n+1}, 306x_{n+3} - 1230150y_{n+1} + 367218))</td>
</tr>
<tr>
<td>8</td>
<td>(9180Y - 102X^2 = 337089600)</td>
<td>((6090y_{n+1} - 30y_{n+2}, 306y_{2n+3} - 61506y_{2n+2} + 18360))</td>
</tr>
<tr>
<td>9</td>
<td>(1854360Y - 102X^2 = 1.375460404 \times 10^{13})</td>
<td>((1230150y_{n+1} - 30y_{n+2}, 306y_{2n+4} - 12423906y_{2n+2} + 3708720))</td>
</tr>
</tbody>
</table>
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\[
10^2 Y^2 - 102X^2 = 32400
\]
\[
(603x_{n+3} - 121803x_{n+2} + 1230150x_{2n+3} - 6090x_{2n+4} + 180)
\]

\[
9Y^2 - 102X^2 = 324
\]
\[
(603y_{n+2} - 6090x_{n+2} + 61506x_{2n+3} - 6090y_{2n+3} + 18)
\]

\[
909Y^2 - 102X^2 = 3305124
\]
\[
(603y_{n+3} - 1230150x_{n+2} + 12423906x_{2n+3} - 6090y_{2n+4} + 1818)
\]

\[
909Y^2 - 102X^2 = 3305124
\]
\[
(121803y_{n+2} - 6090x_{n+3} + 61506x_{2n+4} - 1230150y_{2n+3} + 1818)
\]

\[
9180Y^2 - 102X^2 = 337089600
\]
\[
(1230150y_{n+2} - 6090y_{n+3} + 61506y_{2n+4} - 12423906y_{2n+3} + 18360)
\]

\[
1495Y^2 - 23X^2 = 8940100
\]
\[
(518y_{n+3} - 24852y_{n+2} + 119186y_{2n+3} - 2484y_{2n+4} + 2990)
\]

**SPECIAL CASES:**

\[
P_y^2(t_{3,x+1})^2 = 918P_x^6(t_{3,y})^2 + 18(t_{3,y})^2(t_{3,x+1})^2
\]

\[
9P_y^6(t_{3,x})^2 = 102P_x^1(t_{3,y+1})^2 + 18(t_{3,y+1})^2(t_{3,x})^2
\]

\[
P_y(t_{3,x+2})^2 = 102(6P_y^4)^2(t_{3,y})^2 + 18(t_{3,y})^2(t_{3,x+2})^2
\]

\[
36P_y^8(t_{3,x})^2 = 102P_x^1(t_{3,x+2})^2 + 18(t_{3,x+2})^2(t_{3,y})^2
\]

\[
9P_y(t_{3,x+2})^2 = 102(36P_y^8)^2(t_{3,y+1})^2 + 18(t_{3,y+1})^2(t_{3,x+2})^2
\]

\[
(6P_y^4)^2(t_{3,x+1})^2 = 102(3P_x^1)^2(t_{3,x+2})^2 + 18(t_{3,x+2})^2(t_{3,y+1})^2
\]

**Conclusion**

In this paper, we have presented infinitely many integer solutions for the negative Pell Equation \( y^2 = 102x^2 - 18 \). As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

**REFERENCES**