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ON THE NEGATIVE PELL EQUATION $y^2 = 102x^2 - 18$

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ABSTRACT:

The binary quadratic Diophantine equation represented by the negative pellian $y^2 = 102x^2 - 18$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

KEYWORDS: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

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Introduction

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square negative integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 102x^2 - 18$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of Analysis

The negative Pell equation representing hyperbola under consideration is,

$$y^2 = 102x^2 - 18 \qquad (1)$$

The smallest positive integer solutions of (1) are,

$$x_0 = 3, y_0 = 30$$

Consider the pellian equation

$$y^2 = 102x^2 + 1 \qquad (2)$$

The initial solution of (2)

$$\tilde{x}_0 = 10, \tilde{y}_0 = 101$$

The general solution (x_n, y_n) of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{102}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

where,

$$f_n = (101 + 10\sqrt{102})^{n+1} + (101 - 10\sqrt{102})^{n+1}$$

$$g_n = (101 + 10\sqrt{102})^{n+1} - (101 - 10\sqrt{102})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{3}{2} f_n + \frac{30}{2\sqrt{102}} g_n$$

$$y_{n+1} = \frac{30}{2} f_n + \frac{306}{2\sqrt{102}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 202x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 202y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

N	x_n	y_n
0	3	30
1	603	6090
2	121803	1230150
3	12423906	248484210

From the above table, we observe some interesting relations among the solutions which are presented below.

- y_n values are even.
- **Each of the following expression is a nasty number:**
  - $\frac{6}{9}\left[18+306x_{2n+2}-30y_{2n+2}\right]$
  - $\frac{6}{90}\left[180+6090x_{2n+2}-30x_{2n+3}\right]$
  - $\frac{6}{36360}\left[72720+2460300x_{n+1}-60x_{n+3}\right]$
  - $\frac{6}{1818}\left[3636+123012x_{2n+2}-60y_{2n+3}\right]$
  - $\frac{6}{183609}\left[367218+12423906x_{2n+2}-30y_{2n+4}\right]$
  - $\frac{6}{909}\left[1818+306x_{2n+3}-6090y_{2n+2}\right]$
  - $\frac{6}{183609}\left[367218+306x_{n+3}-1230150y_{n+1}\right]$
  - $\frac{6}{9180}\left[18360+306y_{2n+3}-61506y_{2n+2}\right]$
  - $\frac{6}{1854360}\left[3708720+306y_{2n+4}-12423906y_{2n+2}\right]$
  - $\frac{6}{90}\left[180+1230150x_{2n+3}-6090x_{2n+4}\right]$
  - $\frac{6}{9}\left[18+61506x_{2n+3}-6090y_{2n+3}\right]$
  - $\frac{6}{909}\left[1818+12423906x_{2n+3}-6090\,y_{2n+4}\right]$
  - $\frac{6}{909}\left[1818+61506x_{2n+4}-1230150y_{2n+3}\right]$
  - $\frac{6}{9180}\left[18360+61506y_{2n+4}-12423906x_{2n+3}\right]$

GSJ: Volume 7, Issue 4, April 2019
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- **Each of the following expression is a cubical integer.**
  - $\frac{1}{9}\left[306x_{3n+3}-30y_{3n+3}+918x_{n+1}-90y_{n+1}\right]$
  - $\frac{1}{90}\left[6090x_{3n+3}-30x_{3n+4}+18270x_{n+1}-90x_{n+2}\right]$
  - $\frac{1}{36360}\left[2460300x_{3n+3}-60x_{3n+5}+7380900x_{n+1}-180x_{n+3}\right]$
  - $\frac{1}{1818}\left[123012x_{3n+3}-60y_{3n+4}+369036x_{n+1}-180y_{n+2}\right]$
  - $\frac{1}{183609}\left[12423906x_{3n+3}-30y_{3n+5}+37271718x_{n+1}-90y_{n+2}\right]$
  - $\frac{1}{909}\left[306x_{3n+4}-6090y_{3n+3}+918x_{n+2}-18270y_{n+1}\right]$
  - $\frac{1}{183609}\left[306x_{3n+5}-1230150y_{3n+3}+918x_{n+3}-3690450y_{n+1}\right]$
  - $\frac{1}{9180}\left[306y_{3n+4}-61506y_{3n+3}+918y_{n+2}-184518y_{n+1}\right]$
  - $\frac{1}{1854360}\left[306y_{3n+5}-12423906y_{3n+3}+918y_{n+3}-37271718y_{n+1}\right]$
  - $\frac{1}{90}\left[1230150x_{3n+4}-6090x_{3n+5}+3690450x_{n+2}-18270x_{n+3}\right]$
  - $\frac{1}{9}\left[61506x_{3n+4}-6090y_{3n+4}+184518x_{n+2}-18270y_{n+2}\right]$
  - $\frac{1}{909}\left[12423906x_{3n+4}-6090y_{3n+5}+372711718x_{n+2}-18270y_{n+3}\right]$
  - $\frac{1}{909}\left[61506x_{3n+5\backslash}-1230150y_{3n+4}+184518x_{n+3}-3690450y_{n+2}\right]$
  - $\frac{1}{9180}\left[61506y_{3n+5\backslash}-12423906y_{3n+4}+184518y_{n+3}-37271718y_{n+2}\right]$

- **Each of the following expression is a biquadratic integer.**
  - $\frac{1}{9}\left[306x_{4n+4}-30y_{4n+4}+1224x_{2n+2}-120y_{2n+2}+54\right]$
  - $\frac{1}{90}\left[6090x_{4n+4}-30x_{4n+5}+24360x_{2n+2}-120x_{2n+3}+540\right]$
  - $\frac{1}{36360}\left[2460300x_{4n+4}-60x_{4n+6}+9841200x_{2n+2}-240x_{2n+4}+218160\right]$
  - $\frac{1}{1818}\left[123012x_{4n+4}-60y_{4n+5}+492048x_{2n+2}-240y_{2n+3}+10908\right]$
  - $\frac{1}{183609}\left[12423906x_{4n+4}-30y_{4n+6}+49695624x_{n+1}-120y_{n+3}+1101654\right]$
  - $\frac{1}{909}\left[306x_{4n+5}-6090y_{4n+4}+1224x_{2n+3}-24360y_{2n+2}+5454\right]$
  - $\frac{1}{183609}\left[306x_{4n+6}-1230150y_{4n+4}+1224x_{2n+4}-4920600y_{2n+2}+1101654\right]$

GSJ: Volume 7, Issue 4, April 2019
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- $\frac{1}{9180}\left[306y_{4n+5}-61506y_{4n+4}+1224y_{2n+3}-246024y_{2n+2}+55080\right]$
- $\frac{1}{1854360}\left[306y_{4n+6}-12423906y_{4n+4}+1224y_{2n+3}-49695624y_{n+1}+11126160\right]$
- $\frac{1}{90}\left[1230150x_{4n+5}-6090y_{4n+6}+4920600x_{2n+3}-24360x_{2n+4}+540\right]$
- $\frac{1}{9}\left[61506x_{4n+5}-6090x_{4n+6}+246024x_{2n+3}-24360y_{2n+3}+54\right]$
- $\frac{1}{909}\left[12423906x_{4n+5}-6090y_{4n+6}+49695624x_{n+2}-24360y_{n+3}+5454\right]$
- $\frac{1}{909}\left[61506x_{4n+6}-1230150y_{4n+5}+246024x_{2n+4}-4920600y_{2n+3}+5454\right]$
- $\frac{1}{9180}\left[61506y_{4n+6}-12423906y_{4n+5}+246024y_{2n+4}-49695624y_{2n+3}+55080\right]$

- **Each of the following expression is a quintic integer:**

- $\frac{1}{9}\left[306x_{5n+5}-30y_{5n+5}+1530x_{3n+3}-150y_{3n+3}+3060x_{n+1}-300y_{n+1}\right]$
- $\frac{1}{9}\left[6090x_{5n+5}-30x_{5n+6}+30450x_{3n+3}-150x_{3n+4}+60900x_{n+1}-300x_{n+2}\right]$
- $\frac{1}{36360}\left[2460300x_{5n+5}-60x_{5n+7}+12303500x_{3n+3}-300x_{3n+5}+24603000x_{n+1}-600x_{n+3}\right]$
- $\frac{1}{1818}\left[123012x_{5n+5}-60y_{5n+6}+615060x_{3n+3}-300y_{3n+4}+1230120x_{n+1}-600y_{n+2}\right]$
- $\frac{1}{183609}\left[12423906x_{5n+5}-30y_{5n+7}+62119530x_{3n+3}-150y_{3n+5}-124239060x_{n+1}-300y_{n+2}\right]$
- $\frac{1}{909}\left[306x_{5n+6}-6090y_{5n+5}+1530x_{3n+4}-30450y_{3n+3}+3060x_{n+2}-60900y_{n+1}\right]$
- $\frac{1}{183609}\left[306x_{5n+7}-1230150y_{5n+5}+1530x_{3n+5}-7380900y_{3n+3}+3060x_{n+3}-11071350y_{n+1}\right]$
- $\frac{1}{9180}\left[306y_{5n+6}-61506y_{5n+5}+1530y_{3n+4}-307530y_{3n+3}+3060y_{n+2}-615060y_{n+1}\right]$
- $\frac{1}{1854360}\left[306y_{5n+7}-12423906y_{5n+5}+1530y_{3n+5}-62119530y_{3n+3}+3060y_{n+3}-124239060y_{n+1}\right]$
- $\frac{1}{90}\left[1230150x_{5n+6}-6090x_{5n+7}+6150750x_{3n+4}-30450x_{3n+5}+12301500x_{n+2}-60900x_{n+3}\right]$
- $\frac{1}{9}\left[61506x_{5n+6}-6090y_{5n+6}+307530x_{3n+4}-30450y_{3n+4}+615060x_{n+2}-60900y_{n+2}\right]$
- $\frac{1}{909}\left[1243906x_{5n+6}-6090y_{5n+7}+62119530x_{3n+4}-30450y_{3n+5}+124239060x_{n+2}-60900y_{n+3}\right]$
- $\frac{1}{909}\left[61506x_{5n+7}-1230150y_{5n+6}+307530x_{3n+5}-6150750y_{3n+4}+615060x_{n+3}-12301500y_{n+2}\right]$

$\frac{1}{9180}\left[61506y_{5n+7}-12423906y_{5n+6}+307530y_{3n+5}-62119530y_{3n+4}+615060y_{n+3}-124239060y_{n+3}\right]$

- **Relations among the solutions are given below:**
  - $x_{n+2} = 101x_{n+1} + 10y_{n+1}$
  - $x_{n+3} = 20401x_{n+1} + 2020y_{n+1}$
  - $y_{n+2} = 1020x_{n+1} + 101y_{n+1}$
  - $y_{n+3} = 206040x_{n+1} + 20401_{n+1}$
  - $x_{n+3} = 202x_{n+2} - x_{n+1}$
  - $10y_{n+2} = 101x_{n+2} - x_{n+1}$
  - $10y_{n+3} = 20401x_{n+2} - 101x_{n+1}$
  - $20y_{n+2} = x_{n+3} - x_{n+1}$
  - $2020y_{n+3} = 20401x_{n+3} - x_{n+1}$
  - $101y_{n+3} = 1020x_{n+1} + 20401y_{n+2}$
  - $101x_{n+3} = 20401x_{n+2} + 10y_{n+1}$
  - $101y_{n+2} = 1020x_{n+2} - y_{n+1}$
  - $y_{n+3} = 2040x_{n+2} + y_{n+1}$
  - $20401y_{n+2} = 1020x_{n+3} + 101y_{n+1}$
  - $20401y_{n+3} = 206040x_{n+3} + y_{n+1}$
  - $y_{n+3} = 202y_{n+2} - y_{n+1}$
  - $10y_{n+2} = x_{n+3} - 101x_{n+2}$
  - $10y_{n+3} = 101x_{n+3} - x_{n+2}$
  - $y_{n+3} = 1020x_{n+2} + 101y_{n+2}$
  - $101y_{n+3} = 1020x_{n+3} + y_{n+2}$

Remarkable Observations

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$4Y^2 - 4131X^2 = 1296$	$\left(y_{n+1} - 10x_{n+1}, 306x_{n+1} - 30y_{n+1}\right)$
2	$Y^2 - 51X^2 = 32400$	$\left(3x_{n+2} - 603x_{n+1}, 6090x_{n+1} - 30x_{n+2}\right)$
3	$4Y^2 - 408X^2 = 5288198400$	$\left(3x_{n+3} - 121803x_{n+1}, 1230150x_{n+1} - 30x_{n+3}\right)$
4	$4Y^2 - 408X^2 = 13220496$	$\left(3y_{n+2} - 6090x_{n+1}, 61506_{n+1} - 30y_{n+2}\right)$
5	$Y^2 - 102X^2 = 1.348490595 \times 10^{11}$	$\left(3y_{n+3} - 1230150x_{n+1}, 12423906x_{n+1} - 30y_{n+3}\right)$

6	$Y^2-102X^2=3305124$	$(603y_{n+1}-30x_{n+2},306x_{n+2}-6090y_{n+1})$
7	$Y^2-102X^2=1.348490595\times10^{11}$	$(121803y_{n+1}-30x_{n+3},306x_{n+3}-1230150y_{n+1})$
8	$Y^2-102X^2=337089600$	$(6090y_{n+1}-30y_{n+2},306y_{n+2}-61506y_{n+1})$
9	$Y^2-102X^2=1.375460404\times10^{13}$	$(1230150y_{n+1}-30y_{n+3},306y_{n+3}-12423906y_{n+1})$
10	$Y^2-102X^2=32400$	$(603x_{n+3}-121803x_{n+2},1230150x_{n+2}-6090x_{n+3})$
11	$Y^2-102X^2=324$	$(603y_{n+2}-6090x_{n+2},61506x_{n+2}-6090y_{n+2})$
12	$Y^2-102X^2=3305124$	$(603y_{n+3}-1230150x_{n+2},12423906x_{n+2}-6090y_{n+3})$
13	$Y^2-102X^2=3305124$	$(121803y_{n+2}-6090x_{n+3},61506x_{n+3}-1230150y_{n+2})$
14	$Y^2-102X^2=337089600$	$(1230150y_{n+2}-6090y_{n+3},61506y_{n+3}-12423906y_{n+2})$
15	$Y^2-23X^2=8940100$	$(518y_{n+3}-24852y_{n+2},119186y_{n+2}-2484y_{n+3})$

II. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$4Y-459X^2=144$	$(y_{n+1}-10x_{n+1},306x_{2n+2}-30y_{2n+2}+18)$
2	$90Y-51X^2=32400$	$(3x_{n+2}-603x_{n+1},6090x_{2n+2}-30x_{2n+3}+180)$
3	$36360Y-408X^2=5288198400$	$(3x_{n+3}-121803x_{n+1},2460300x_{2n+2}-60x_{2n+4}+72720)$
4	$1818Y-408X^2=13220496$	$(3y_{n+2}-6090x_{n+1},123012x_{2n+2}-60y_{2n+3}+3636)$
5	$Y-102X^2=734436$	$(3y_{n+3}-1230150x_{n+1},12423906x_{2n+2}-30y_{2n+4}+367218)$
6	$909Y-102X^2=3305124$	$(603y_{n+1}-30x_{n+2},306x_{2n+3}-6090y_{2n+2}+1818)$
7	$Y-102X^2=734436$	$(306x_{n+3}-1230150y_{n+1},306x_{n+3}-1230150y_{n+1}+367218)$
8	$9180Y-102X^2=337089600$	$(6090y_{n+1}-30y_{n+2},306y_{2n+3}-61506y_{2n+2}+18360)$
9	$1854360Y-102X^2=1.375460404\times10^{13}$	$(1230150y_{n+1}-30y_{n+3},306y_{2n+4}-12423906y_{2n+2}+3708720)$

10	$90Y-102X^2=32400$	$(603x_{n+3}-121803x_{n+2},1230150x_{2n+3}-6090x_{2n+4}+180)$
11	$9Y-102X^2=324$	$(603y_{n+2}-6090x_{n+2},61506x_{2n+3}-6090y_{2n+3}+18)$
12	$909Y-102X^2=3305124$	$(603y_{n+3}-1230150x_{n+2},12423906x_{2n+3}-6090y_{2n+4}+1818)$
13	$909Y-102X^2=3305124$	$(121803y_{n+2}-6090x_{n+3},61506x_{2n+4}-1230150y_{2n+3}+1818)$
14	$9180Y-102X^2=337089600$	$(1230150y_{n+2}-6090y_{n+3},61506y_{2n+4}-12423906y_{2n+3}+18360)$
15	$1495Y-23X^2=8940100$	$(518y_{n+3}-24852y_{n+2},119186y_{2n+3}-2484y_{2n+4}+2990)$

SPECIAL CASES:

- $P_y^{10}(t_{3,x+1})^2=918P_x^6(t_{3,y})^2+18(t_{3,y})^2(t_{3,x+1})^2$
- $9P_y^6(t_{3,x})^2=102P_x^{10}(t_{3,y+1})^2+18(t_{3,x})^2(t_{3,y+1})^2$
- $P_y^{10}(t_{3,2x-2})^2=102(6P_{x-1}^4)^2(t_{3,y})^2+18(t_{3,y})^2(t_{3,2x-2})^2$
- $36P_{y-1}^8(t_{3,x})^2=102P_x^{10}(t_{3,2y-2})^2+18(t_{3,x})^2(t_{3,2y-2})^2$
- $9P_y^6(t_{3,2x-2})^2=102(36P_{x-1}^8)(t_{3,y+1})^2+18(t_{3,2x-2})^2(t_{3,y+1})^2$
- $(6P_{y-1}^4)^2(t_{3,x+1})^2=102(3P_x^3)^2(t_{3,2y-2})^2+18(t_{3,x+1})^2(t_{3,2y-2})^2$

Conclusion

In this paper, we have presented infinitely many integer solutions for the negative Pell Equation $y^2=102x^2-18$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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