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ON THE P0SITIVE PELL EQUATION $y^2 = 34x^2 + 18$

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ABSTRACT:

The binary quadratic Diophantine equation represented by the positive pellian $y^2 = 34x^2 + 18$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, the solutions of other choices of hyperbolas, parabolas and Pythagorean triangle are obtained.

KEYWORDS:

Binary quadratic, Hyperbola, Parabola, Integral solution, Pell equation.

Introduction

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square negative integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values[1-4].For an extensive review of various problems, one may refer[5-10]. In this communication, yet another interesting equation given by $y^2 = 34x^2 + 18$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

Method of Analysis

The positive Pell equation representing hyperbola under consideration is,

$$y^2 = 34x^2 + 18 \tag{1}$$

The smallest positive integer solutions of (1) are,

$$x_0 = 3, \, y_0 = 18$$

Consider the pellian equation

$$y^2 = 34x^2 + 1 \tag{2}$$

The initial solution of (2) is

$$\tilde{x}_0 = 6, \, \tilde{y}_0 = 35,$$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by,

$$\tilde{x}_n = \frac{1}{2\sqrt{34}} g_n, \, \tilde{y}_n = \frac{1}{2} f_n$$

where,

$$f_n = (35 + 6\sqrt{34})^{n+1} + (35 - 6\sqrt{34})^{n+1}$$

$$g_n = (35 + 6\sqrt{34})^{n+1} - (35 - 6\sqrt{34})^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$2\sqrt{34} x_{n+1} = 3\sqrt{34} f_n + 18 g_n \, ,$$

$$\sqrt{34} y_{n+1} = 9\sqrt{34} f_n + 51 g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 70x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 70y_{n+2} + y_{n+1} = 0$$

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Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	3	18
1	213	1242
2	14907	86922
3	1043277	6083298

From the above table, we observe some interesting relations among the solutions which are presented below.

- x_n values are odd .
- y_n values are even.

- **Each of the following expression is a nasty number:**

  - $\frac{1}{3}\left[6x_{2n+3} - 414x_{2n+2} + 36\right]$
  - $\frac{1}{70}\left[2x_{2n+4} - 9658x_{2n+2} + 840\right]$
  - $\frac{1}{3}\left[36y_{2n+2} - 204x_{2n+2} + 36\right]$
  - $\frac{1}{35}\left[12y_{2n+3} - 4828x_{2n+2} + 420\right]$
  - $\frac{1}{2449}\left[12y_{2n+4} - 337892x_{2n+2} + 29388\right]$
  - $\frac{1}{2}\left[276x_{2n+4} - 19316x_{2n+3} + 24\right]$
  - $\frac{1}{35}\left[828y_{2n+2} - 68x_{2n+3} + 420\right]$
  - $\frac{1}{3}\left[2484y_{2n+3} - 14484x_{2n+3} + 36\right]$
  - $\frac{1}{35}\left[828y_{2n+4} - 337892x_{2n+3} + 420\right]$
  - $\frac{1}{2449}\left[57948y_{2n+2} - 68x_{2n+4} + 29388\right]$
  - $\frac{1}{35}\left[57948y_{2n+3} - 4828x_{2n+4} + 420\right]$
  - $\frac{1}{3}\left[173844y_{2n+4} - 1013676x_{2n+4} + 36\right]$
  - $\frac{1}{6}\left[142y_{2n+2} - 2y_{2n+3} + 72\right]$

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- $\frac{1}{420}\left[9938y_{2n+2}-2y_{2n+4}+5040\right]$
- $\frac{1}{102}\left[108946y_{2n+3}-2414y_{2n+4}+1224\right]$

- **Each of the following expressions is a cubical integer:**

- $\frac{1}{3}\left[3x_{n+2}-207x_{n+1}-69x_{3n+3}+x_{3n+4}\right]$
- $\frac{1}{210}\left[3x_{n+3}-14487x_{n+1}-4829x_{3n+3}+x_{3n+5}\right]$
- $\frac{1}{3}\left[18y_{n+1}-102x_{n+1}-34x_{3n+3}+6y_{3n+3}\right]$
- $\frac{1}{105}\left[18y_{n+2}-7242x_{n+1}-2414x_{3n+3}+6y_{3n+4}\right]$
- $\frac{1}{7347}\left[18y_{n+3}-506838x_{n+1}-168946x_{3n+3}+6y_{3n+5}\right]$
- $\frac{1}{3}\left[207x_{n+3}-14487x_{n+2}-4829x_{3n+4}+69x_{3n+5}\right]$
- $\frac{1}{105}\left[1242y_{n+1}-102x_{n+2}-34x_{3n+4}+414y_{3n+3}\right]$
- $\frac{1}{3}\left[1242y_{n+2}-7242x_{n+2}-2414x_{3n+4}+414y_{3n+4}\right]$
- $\frac{1}{105}\left[1242y_{n+3}-506838x_{n+2}-168946x_{3n+4}+414y_{3n+5}\right]$
- $\frac{1}{7347}\left[86922y_{n+1}-102x_{n+3}-34x_{3n+5}+28974y_{3n+3}\right]$
- $\frac{1}{105}\left[86922y_{n+2}-7242x_{n+3}-2414x_{3n+5}+28974y_{3n+4}\right]$
- $\frac{1}{3}\left[86922y_{n+3}-506838x_{n+3}-168946x_{3n+5}+28974y_{3n+5}\right]$
- $\frac{1}{18}\left[213y_{n+1}-3y_{n+2}+71y_{3n+3}-y_{3n+4}\right]$
- $\frac{1}{1260}\left[14907y_{n+1}-3y_{n+3}+4969y_{3n+3}-y_{3n+5}\right]$
- $\frac{1}{306}\left[253419y_{n+2}-3621y_{n+3}+84473y_{3n+4}-1207y_{3n+5}\right]$

- **Each of the following expressions is a biquadratic integer**:

- $\frac{1}{3}\left[x_{4n+5}-69x_{4n+4}-276x_{2n+2}+4x_{2n+3}+18\right]$

- $\frac{1}{210}\left[x_{4n+6}-4829x_{4n+4}-19316x_{2n+2}+4x_{2n+4}+1260\right]$
- $\frac{1}{3}\left[6y_{4n+4}-34x_{4n+4}-136x_{2n+2}+24y_{2n+2}+18\right]$
- $\frac{1}{105}\left[6x_{4n+5}-2414x_{4n+4}-9656x_{2n+2}+24y_{2n+3}+630\right]$
- $\frac{1}{7347}\left[6y_{4n+6}-168946x_{4n+4}-675784x_{2n+2}+24y_{2n+4}+44082\right]$
- $\frac{1}{6}\left[138x_{4n+6}-9658x_{4n+5}-38632x_{2n+3}+552x_{2n+4}+36\right]$
- $\frac{1}{105}\left[414y_{4n+4}-34x_{4n+5}-136x_{2n+3}+1656y_{2n+2}+630\right]$
- $\frac{1}{3}\left[414y_{4n+5}-2414x_{4n+5}-9656x_{2n+3}+1656y_{2n+3}+18\right]$
- $\frac{1}{105}\left[414y_{4n+6}-168946x_{4n+5}-675784x_{2n+3}+1656y_{2n+4}+630\right]$
- $\frac{1}{7347}\left[28974y_{4n+4}-34x_{4n+6}-136x_{2n+4}+115896y_{2n+2}+44082\right]$
- $\frac{1}{105}\left[28974y_{4n+5}-2414x_{4n+6}-9656x_{2n+4}+115896y_{2n+3}+630\right]$
- $\frac{1}{3}\left[28974y_{4n+6}-168946x_{4n+6}-675784x_{2n+4}+115896y_{2n+4}+18\right]$
- $\frac{1}{18}\left[71y_{4n+4}-y_{4n+5}+284y_{2n+2}-4y_{2n+3}+108\right]$
- $\frac{1}{1260}\left[4969y_{4n+4}-y_{4n+6}+19876y_{2n+2}-4y_{2n+4}+7560\right]$
- $\frac{1}{306}\left[84473y_{4n+5}-1207y_{4n+6}+337892y_{2n+3}-4828y_{2n+4}+1836\right]$

- **Each of the following expression is a quintic integer:**

- $\frac{1}{3}\left[x_{5n+6}-69x_{5n+5}-690x_{n+1}+10x_{n+2}-345x_{3n+3}+5x_{3n+4}\right]$
- $\frac{1}{210}\left[x_{5n+7}-4829x_{5n+5}-48290x_{n+1}+10x_{n+3}-24145x_{3n+3}+5x_{3n+5}\right]$
- $\frac{1}{3}\left[6y_{5n+5}-34y_{5n+5}-340x_{n+1}+60y_{n+1}-170x_{3n+3}+30y_{3n+3}\right]$
- $\frac{1}{105}\left[6y_{5n+6}-2414x_{5n+5}-24140x_{n+1}+60y_{n+2}-12070x_{3n+3}+30y_{3n+4}\right]$
- $\frac{1}{7347}\left[6y_{5n+7}-168946x_{5n+5}-1689460x_{n+1}+60y_{n+3}+844730x_{3n+3}+30y_{3n+5}\right]$
- $\frac{1}{6}\left[138x_{5n+7}-9658x_{5n+6}-96580x_{n+2}+1380x_{n+3}-48290x_{3n+4}+690x_{3n+5}\right]$

- ❖ $\frac{1}{105}\left[414y_{5n+5}-34x_{5n+6}-340x_{n+2}+4140y_{n+1}-170x_{3n+4}+2070y_{3n+3}\right]$
- ❖ $\frac{1}{3}\left[414y_{5n+6}-2414x_{5n+6}-24140x_{n+2}+4140y_{n+2}-12070x_{3n+4}+2070y_{3n+4}\right]$
- ❖ $\frac{1}{105}\left[414y_{5n+7}-168946x_{5n+6}-1689460x_{n+2}+4140y_{n+3}-844730x_{3n+4}+2070y_{3n+5}\right]$
- ❖ $\frac{1}{7347}\left[28974y_{5n+5}-34x_{5n+7}-340x_{n+3}+289740y_{n+1}-170x_{3n+5}+144870y_{3n+3}\right]$
- ❖ $\frac{1}{105}\left[28974y_{5n+5}-2414x_{5n+7}-24140x_{n+3}+289740y_{n+2}-12070x_{3n+5}+144870y_{3n+4}\right]$
- ❖ $\frac{1}{3}\left[28974y_{5n+7}-168946x_{5n+7}-1689460x_{n+3}+289740y_{n+3}-844730x_{3n+5}+144870y_{3n+5}\right]$
- ❖ $\frac{1}{18}\left[71y_{5n+5}-y_{5n+6}+710y_{n+1}-10y_{n+2}+355y_{3n+3}-5y_{3n+4}\right]$
- ❖ $\frac{1}{1260}\left[4969y_{5n+5}-y_{5n+7}+49690y_{n+1}-10y_{n+3}+24845y_{3n+3}-5y_{3n+5}\right]$
- ❖ $\frac{1}{306}\left[84473y_{5n+6}-1207y_{5n+7}+844730y_{n+2}-12070y_{n+3}+422365y_{3n+4}-6035y_{3n+5}\right]$

➢ **Relations among the solutions are given below:**

- ❖ $x_{n+3}+x_{n+1}-70x_{n+2}=0$
- ❖ $6y_{n+1}+35x_{n+1}-x_{n+2}=0$
- ❖ $6y_{n+2}+x_{n+1}-35x_{n+2}=0$
- ❖ $6y_{n+3}+35x_{n+1}-2449x_{n+2}=0$
- ❖ $420y_{n+1}+2449x_{n+1}-x_{n+3}=0$
- ❖ $12y_{n+2}+x_{n+1}-x_{n+3}=0$
- ❖ $420y_{n+3}+x_{n+1}-2449x_{n+3}=0$
- ❖ $y_{n+2}-204x_{n+1}-35y_{n+1}=0$
- ❖ $y_{n+3}-14280x_{n+1}-2449y_{n+1}=0$
- ❖ $35y_{n+3}-204x_{n+1}-2449y_{n+2}=0$
- ❖ $6y_{n+1}+2449x_{n+2}-35x_{n+3}=0$
- ❖ $6y_{n+2}+35x_{n+2}-x_{n+3}=0$
- ❖ $6y_{n+3}+x_{n+2}-35x_{n+3}=0$
- ❖ $35x_{n+3}-2449x_{n+3}-6y_{n+1}=0$
- ❖ $35y_{n+2}-204x_{n+2}-y_{n+1}=0$
- ❖ $y_{n+3}-408x_{n+2}-y_{n+1}=0$
- ❖ $y_{n+1}+204x_{n+2}-35y_{n+2}=0$
- ❖ $y_{n+3}-204x_{n+2}-35y_{n+2}=0$
- ❖ $2449y_{n+2}-204x_{n+3}-35y_{n+1}=0$
- ❖ $24449y_{n+3}-14280x_{n+3}-y_{n+1}=0$
- ❖ $35y_{n+3}-204x_{n+3}-y_{n+2}=0$

- $204x_{n+3} - 11995201y_{n+1} + 171395y_{n+2} = 0$
- $y_{n+3} + y_{n+1} - 70y_{n+2} = 0$
- $408x_{n+1} - 4897y_{n+1} + y_{n+3} = 0$
- $99960x_{n+2} - 83966407y_{n+1} + 17143y_{n+3} = 0$

Remarkable Observations

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$18x^2 - 17y^2 = 648$	$(69x_{n+1} - x_{n+2}, 71x_{n+1} - x_{n+2})$
2	$18x^2 - 17y^2 = 3175200$	$(4829x_{n+1} - x_{n+3}, 4969x_{n+1} - x_{n+3})$
3	$2x^2 - 153y^2 = 162$	$(51x_{n+1} - 9y_{n+1}, 6x_{n+1} - y_{n+1})$
4	$2x^2 - 153y^2 = 198450$	$(3621x_{n+1} - 9y_{n+2}, 414x_{n+1} - y_{n+2})$
5	$2x^2 - 153y^2 = 971611362$	$(253419x_{n+1} - 9y_{n+3}, 28974x_{n+1} - y_{n+3})$
6	$x^2 - 34y^2 = 11664$	$(86922x_{n+2} - 1242x_{n+3}, 14907x_{n+2} - 213x_{n+3})$
7	$2x^2 - 17y^2 = 198450$	$(51x_{n+2} - 621y_{n+1}, 18x_{n+2} - 213y_{n+1})$
8	$2x^2 - 17y^2 = 162$	$(3621x_{n+2} - 621y_{n+2}, 1242x_{n+2} - 213y_{n+2})$
9	$2x^2 - 17y^2 = 198450$	$(253419x_{n+2} - 621y_{n+3}, 86922x_{n+2} - 213y_{n+3})$
10	$2x^2 - 17y^2 = 971611362$	$(51x_{n+3} - 43461y_{n+1}, 18x_{n+3} - 14907y_{n+1})$
11	$2x^2 - 17y^2 = 198450$	$(3621x_{n+3} - 43461y_{n+2}, 1242x_{n+3} - 14907y_{n+2})$
12	$2x^2 - 17y^2 = 162$	$(253419x_{n+3} - 43461y_{n+3}, 86922x_{n+3} - 14907y_{n+3})$
13	$289x^2 - 306y^2 = 374544$	$(71y_{n+1} - y_{n+2}, 69y_{n+1} - y_{n+2})$
14	$14161x^2 - 14994y^2 = 89928014400$	$(4969y_{n+1} - y_{n+3}, 4829y_{n+1} - y_{n+3})$
15	$x^2 - 34y^2 = 3370896$	$(253419y_{n+2} - 3621y_{n+3}, 43461y_{n+2} - 621y_{n+3})$

II. Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$54x-17y^2=324$	$(x_{2n+3}-69x_{2n+2},71x_{n+1}-x_{n+2})$
2	$3780x-17y^2=1587600$	$(x_{2n+4}-4829x_{2n+2},4969x_{n+1}-x_{n+3})$
3	$x-17y^2=18$	$(9y_{2n+2}-51x_{2n+2},6x_{n+1}-y_{n+1})$
4	$35x-17y^2=11025$	$(9y_{2n+3}-3621x_{2n+2},414x_{n+1}-y_{n+2})$
5	$2449x-17y^2=53978409$	$(9y_{2n+4}-253419x_{2n+2},28974x_{n+1}-y_{n+3})$
6	$27x-17y^2=2916$	$(1242x_{2n+4}-86922x_{2n+3},14907x_{n+2}-213x_{n+3})$
7	$315x-17y^2=99225$	$(621y_{2n+2}-51x_{2n+3},18x_{n+2}-213y_{n+1})$
8	$9x-17y^2=81$	$(621y_{2n+3}-3621x_{2n+3},1242x_{n+2}-213y_{n+2})$
9	$315x-17y^2=99225$	$(621y_{2n+4}-253419x_{2n+3},86922x_{n+2}-213y_{n+3})$
10	$22041x-17y^2=485805681$	$(43461y_{2n+2}-51x_{2n+4},18x_{n+3}-14907y_{n+1})$
11	$315x-17y^2=99225$	$(43461y_{2n+3}-3621x_{2n+4},1242x_{n+3}-14907y_{n+2})$
12	$9x-17y^2=81$	$(43461y_{2n+4}-253419x_{2n+4},86922x_{n+3}-14907y_{n+3})$
13	$289x-17y^2=10404$	$(71y_{2n+2}-y_{2n+3},69y_{n+1}-y_{n+2})$
14	$20230x-17y^2=50979600$	$(4969y_{2n+2}-y_{2n+4},4829y_{n+1}-y_{n+3})$
15	$459x-17y^2=842724$	$(253419y_{2n+3}-3621y_{2n+4},43461y_{n+2}-621y_{n+3})$

Consider $p=x_{n+1}+y_{n+1}$, $q=x_{n+1}$. Note that $p>q>0.$

Treat p,q as the generators of the Pythagorean triangle $T(X,Y,Z)$

where $X=2pq,Y=p^2-q^2,Z=p^2+q^2$.

Then the following results are obtained:

a) $X - 17Y + 16Z + 18 = 0.$

b) $\frac{2A}{P} = x_{n+1} y_{n+1}.$

c) $3(Z - Y)$ is a nasty number.

d) $3(X - \frac{4A}{P})$ is a nasty number.

e) $X - \frac{4A}{P} + Y$ is written as the sum of two squares.

Conclusion

In this paper, we have presented infinitely many integer solutions for the positive Pell Equation $y^2 = 34x^2 + 18$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

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