

ON THE USE OF THE SCHWARZ-CHRISTOFFEL TRANSFORMATION IN THE SOLUTION AND ANALYSIS OF HARMONIC DIRICHLET PROBLEMS OF IDEAL FLUID FLOWS

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Abstract

A concise, elegant, and purely complex variable method for solving harmonic Dirichlet problems of ideal fluid flows in domains whose boundaries have inconvenient geometries consisting of polygonal paths is presented. The method which is conformal based and uses the appropriately determined Schwarz-Christoffel map and then its inverse as mapping functions, was applied to a number of selected flow problems and their complex potentials determined and each flow characterized on the basis of it. More specifically, the stream function which is the imaginary part of the complex potential and solution of the flow problem was isolated and the streamlines of each flow generated to show the flow field and the flow pattern analyzed in terms of fluid speed by the spacing of the streamlines. The fluid velocity was also determined from the complex potential and the fluid speed was also shown to be in agreement with the streamline pattern. This method could therefore be a useful alternative choice to the powerful Fourier method in solving Laplace's equation for two dimensional flows whose boundaries consist of straight line segments.

Keywords and Phrases: Conformal Map, Schwarz-Christoffel Map, Analytic Function, Branch of a Multiple Valued Function, Inviscid, Incompressible.

Introduction

The problems of fluid dynamics when modelled mathematically under the assumptions that the flows are inviscid and incompressible lead to Laplace's equation

$$\nabla^2 \psi = 0 \tag{1}$$

subject to some specified boundary conditions which depend on the problem in question. Two such boundary value problems of great importance in applied mathematics are the Dirichlet and Neumann problems also known as boundary value problems of the first and second kinds, respectively. In this paper we shall however focus on harmonic Dirichlet problems only. This problem seeks the determination of a function ψ which satisfies equation (1) in a domain Ω (that is, ψ is harmonic in Ω) and takes prescribed values on the boundary $\partial \Omega$ of Ω . It is well known in the theory of analytic functions of a complex variables that if a function f(z) = u(x, y) + iv(x, y)is analytic in a domain Ω then its component functions u(x, y) and v(x, y) are harmonic there. The solution to problem (1) therefore reduces to finding a function which is analytic in Ω and whose real and imaginary parts satisfy the boundary conditions. The complex variable method of conformal mapping is a useful intermediate step in the solution and analysis of ideal flows in general and in particular those of the Dirichlet type as well as other none flow Dirichlet problems in electrostatics, electromagnetism, and thermal physics as is evident in the works of Churchill and Brown (1984), Spiegel (1974), Tobin and Lloyd (2002), Etsuo (2004), Etsuo (2015), Swem et al. (2017), Ganzolo et al. (2008), Tao et al. (2008), Anders (2008), Weiman et al. (2016), Yariv and Sherwood (2015), Andreas and Yorgos (2004), Deglaire et al. (2008), Kapania et al. (2008), Chattot and Hafez (2006), Xu et al. (2015), and the National Aeronautics and Space Administration (NASA) website (2015). The technique involves the transformation of the problem

from a domain with an inconvenient geometry in one complex plane into a domain with a simpler geometry in another complex plane by means of an appropriate mapping function which preserves the magnitude of the angles between curves as well as their orientation. Using this technique, the fluid flow around the oblong shape of an airfoil or flat plate placed perpendicular to the incoming flow can be analyzed as the flow exterior to a circle or in the upper half Im z > 0 of the *z* plane for which an analytic form of the solution for potential flow is well known. Amongst a variety of conformal transformations, the ones commonly used in the analysis of ideal fluid flows are the Joukowski map, the Karman-Trefftz map (a generalization of the Joukowski map), and the Schwarz-Christoffel map. In this paper, we shall focus on the Schwarz-Christoffel map only. This transformation which is given by Churchill and Brown (1984) as

$$w = f(z) = A \int_{z_0}^{z} \prod_{j=1}^{n-1} (s - x_j)^{-k_j} ds + B$$
(2)

or

$$\frac{dw}{dz} = f'(z) = A \prod_{j=1}^{n-1} (z - x_i)^{-k_j}$$
(3)

is one that conformally maps the upper half Im z > 0 of the z plane and the entire x axis except for a finite number of points $x_1, x_2, ..., x_{n-1}, \infty$ in a one-to-one correspondence onto the interior of simple closed polygon and its boundary, respectively, such that given а $w_i = f(x_i)(i = 1, 2, ..., n-1)$ and $w_n = f(\infty)$ are the vertices of the polygon. The points $z = x_j$ (j = 1, 2, ..., n - 1) are arranged such that the order relation $x_1 < x_2 < \cdots < x_{n-1}$ is satisfied. The complex constants A and B in formula (1) determine the size, orientation and position of the polygon, the k_i 's are real constants between -1 and 1 determined from the relation $-\pi < k_j \pi < \pi$, where $k_j \pi (j = 1, 2, ..., n - 1)$ are the exterior angles at the vertices w_j (j = 1, 2, ..., n - 1) of the polygon, while the limits of integration z_0 and z are respectively

fixed and variable points in the region Im $z \ge 0$ of analyticity of the Schwarz-Christoffel function. In order to make the function in (3) analytic everywhere in the region Im $z \ge 0$ except at the n - 1points $z = x_j$ (j = 1, 2, ..., n - 1), we introduce branch lines or cuts extending below those points and normal to the real axis and letting

$$(z - x_j)^{-k_j} = |z - x_j|^{-k_j} e^{-ik_j\theta_j} \left(|z - x_j| > 0, -\frac{\pi}{2} < \theta_j < \frac{3\pi}{2} \right)$$
(6)

where $\theta_j = \arg(z - x_j)$ and j = 1, 2, ..., n - 1. It then follows that the function

$$G(z) = \int_{z_0}^{z} f'(z) dz$$
 (7)

is analytic in the region Im $z \ge 0$ and that G'(z) = f'(z). Furthermore, the function G(z) is defined at the points $z = x_j$ (j = 1, 2, ..., n - 1) such that it continuous there (Churchill and Brown 2013) so that the Schwarz-Christoffel transformation (2) is continuous throughout the region Im $z \ge 0$ and conformal there except for the points $z = x_j$ (j = 1, 2, ..., n - 1).

In applications the domains usually encountered are simply connected (Churchill and Brown, 1984) and for such domains the existence of conformal maps is guaranteed by the Riemann mapping theorem which asserts that there exists a unique one to one conformal map from any simply connected domain *D* which is not the whole of the *z* plane onto the unit disc |w| < 1 in the *w* plane. It is also well known that if z_0 is any point in the upper half Im z > 0, then the bilinear transformation

$$w = e^{i\theta_0} \left(\frac{z - z_0}{z - \overline{z_0}} \right)$$

where θ_0 is a constant, conformally maps the upper half of the *z* plane in a one-to-manner onto the unit disc |w| < 1 and conversely. Thus the Riemann mapping theorem also asserts that there exists a unique one-to-one conformal map from the upper half Im z > 0 of the *z* plane onto any simply connected domain which is not the whole of the *z* plane. Although the Riemann mapping theorem demonstrates the existence of a mapping function, it does not produce it. However, for maps of the upper half Im z > 0 of the *z* plane onto the interior of a polygon the Schwarz-Christoffel formula provides explicit formulae that work. In this research paper, we shall apply the transformation in the solution and analysis of Dirichlet harmonic problems of ideal fluid flows in domains consisting of straight line segments for the cases of an infinite strip, infinite sector of Angle πm (0 < m < 1), flow over a flat plate placed perpendicular to an incoming flow, and flow over a step in the bed of a deep stream.

METHODOLOGY

Consider an ideal fluid flow in a domain Ω of the *w* plane whose boundary $\partial \Omega$ is a polygonal path. The flow field is the solution of the mathematical problem (1) subject to some conditions on the boundary $\partial \Omega$ for which ψ takes prescribed values. In order to solve this problem the specific Schwarz-Christoffel transformation w = f(z) that maps the upper half Im z > 0 of the *z* plane in a one-to one manner onto Ω which satisfies the boundary conditions

$$w_j = f(x_j) \ (j = 1, 2, ..., n - 1) \text{ and } w_n = f(\infty)$$

where

$$x_1 < x_2 < \cdots < x_{n-1}$$
 , $x_n = \infty$

is first determined from the generalized form of the transformation (2). Solving for z in terms of w, the inverse function z = g(w) which transforms the domain of the given flow problem and hence the flow field onto one in the upper half Im z > 0 of the z plane is then obtained. Now, the complex potential for a uniform flow to the right in the upper half Im z > 0 of the z plane is analytic and well known and given by Kapania *et al*;(2008) and Churchill and Brown (1984) as

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$$H(z) = v_0 z \tag{8}$$

where v_0 is a positive real constant corresponding to the speed of the flow. If z = x + iy then the velocity potential $\Phi(x, y)$ and stream function $\Psi(x, y)$ of the uniform flow are

$$\Phi(x, y) = v_0 x \tag{9}$$

and

$$\Psi(x,y) = v_0 y \tag{10}$$

respectively. If the inverse function z = g(w) is analytic in the problem domain or made so using the complex variable method of branch cuts in case it is multiple valued, then its composition with the complex potential for the uniform flow in the upper half Im z > 0 of the z plane, namely,

$$H[g(w)] = v_0 g(w) = F(w)$$
 (11)

is also analytic in the domain of the flow problem and hence represents the complex potential for the flow there. The imaginary part of the complex potential function in (11) represents the stream function $\psi(u, v)$ or solution of the flow problem. On setting $\psi(u, v) = c$, where c is a constant and assigning different values to c, the streamlines of the flow are obtained. The streamlines represent the actual path taken by the fluid particles in a steady flow. Alternatively, the streamlines of flow can also be generated by finding the images of the streamlines of the uniform flow $\Psi(x, y) = v_0 y = c_1$ (where c_1 is a constant) represented by the horizontal lines $y = c_1/v_0$ under the Schwarz-Christoffel transformation. The fluid velocity v(w) is simply the conjugate of the derivative of the complex potential and is obtained from equation (11) as

$$v(w) = \overline{F'(w)} = v_0 \overline{g'(w)} = v_0 \overline{\left(\frac{dz}{dw}\right)} = \frac{v_0}{\overline{(dw/dz)}}$$
(12)

where $\frac{dw}{dz}$ is the derivative of the Schwarz-Christoffel transformation. The fluid speed is the modulus of the fluid velocity and from equation (12) we have that

$$|v(w)| = |F'(w)| = v_0 \left| \frac{dz}{dw} \right| = \frac{v_0}{\left| \frac{dw}{dz} \right|}$$
 (13)

Formula (13) is important and shows that the fluid speed in the problem domain of the w plane is proportional to that of the corresponding uniform flow in the z plane where the constant of proportionality is the reciprocal of the modulus of the derivative of the Schwarz-Christoffel map.

RESULTS

In this section we present the solution to some harmonic Dirichlet problems of ideal fluid flows based on the purely complex variable method outlined in the methodology. This is accomplished by first transforming the given problem via the inverse Schwarz-Christoffel map onto one in the upper half Im z > 0 of the z plane where the simplified problem is then efficiently solved.

Problem 1: (Flow in an Infinite Strip of Width *a*)

We first consider the harmonic Dirichlet problem in equation 1(a) for the ideal fluid flow in an infinite strip described by the equation

$$0 < v < a, -\infty < u < \infty \tag{14}$$

as shown in Figure 1.



Figure 1(a): One-to-one mapping of the upper half Im z > 0 of the *z* plane onto an infinite strip in the *w* plane.



Figure 1(b): Streamlines of flow in the Interior of an Infinite Strip.

The Schwarz-Christoffel transformation w = f(z) that maps the half plane Im z > 0 and the entire real axis except the origin in a one-to-one manner onto the strip and its boundary, respectively is found to be

$$w = \frac{a}{\pi} \ln z \ (|z| > 0, 0 \le \arg z \le \pi)$$
(15)

by considering the strip as a limiting form of a rhombus with vertices at $w_1 = ai, w_2, w_3 = 0$, and $w_4 = \infty$ respectively or using the table of transforms given by Spiegel (1974). In this problem, we note that the point x_1 is to be determined while the values $x_2 = 0, x_3 = 1$, and $x_4 = \infty$ are given. The inverse transformation is therefore

$$z = e^{\frac{\pi}{a}w} = g(w) \tag{16}$$

and maps the strip in a one-to-one manner onto nonzero points in the half plane Im z > 0. Note that the image of the boundary of the strip under the inverse map in (16) is the entire x axis except the point z = 0 which is not in the range of the function. Since the inverse map is analytic, the function

$$F(w) = v_0 e^{\frac{\pi}{a}w} \tag{17}$$

is the complex potential for the flow in the strip. On letting w = u + iv and $F(w) = \phi(u, v) + i\psi(u, v)$, we obtain the stream function of the flow as

$$\psi(u,v) = v_0 e^{\frac{\pi u}{a}} \sin \frac{\pi v}{a} \tag{18}$$

while

$$\psi(u,v) = v_0 e^{\frac{\pi u}{a}} \sin \frac{\pi v}{a} = c \tag{19}$$

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where c is a real constant, is the expression for the streamlines of the flow. The velocity of the flow is

$$v(w) = \overline{F'(w)} = \frac{\pi}{a} v_0 e^{\frac{\pi \overline{w}}{a}}$$
(20)

while the fluid speed is

$$|v(w)| = |F'(w)| = \frac{\pi}{a} v_0 e^{\frac{\pi u}{a}}$$
(21)

Problem 2: (Flow in an Infinite Sector of Angle πm)

We next consider the harmonic Dirichlet problem in equation (1) for the ideal fluid flow in an angular region $|w| \ge 0, 0 \le \arg w \le \pi m$ (0 < m < 1) of Figure 2(a).



Figure 2(a): One-to-one mapping of the upper half Im $z \ge 0$ of the *z* plane onto an infinite sector of angle $m\pi$ in the *w* plane.



Figure 2(b): Streamlines of flow in the Interior of an Infinite Sector of Angle $\pi/4$ in the *w* plane.

The Schwarz-Christoffel transformation that maps the half plane Im $z \ge 0$ onto the infinite sector and the point z = 1 into the point w = 1 is found to be

$$w = z^m \ (0 < m < 1) \tag{22}$$

by considering the angular region as the limiting case of the triangle shown in Figure 2(a) as the angle α tends to zero or is given by Churchill and Brown (1984). The inverse of transformation (22) is

$$z = w^{\frac{1}{m}} = g(w) \tag{23}$$

and maps the angular region in the *w* plane onto the upper half $\text{Im } z \ge 0$ of the *z* plane. The complex potential for the flow in the angular region $|w| \ge 0, 0 \le \arg w \le \pi m \ (0 < m < 1)$ is thus

$$F(w) = v_0 w^{\frac{1}{m}} \tag{24}$$

If $w = \rho e^{i\sigma}$ and $F(w) = \phi(\rho, \sigma) + i\psi(\rho, \sigma)$, then the stream function of the flow is

$$\psi(\rho,\sigma) = v_0 \rho^{\frac{1}{m}} \sin \frac{\sigma}{m} \ (\rho \ge 0, 0 \le \sigma \le \pi m)$$
(25)

while

$$\psi(\rho,\sigma) = v_0 \rho^{\frac{1}{m}} \sin \frac{\sigma}{m} = c$$
(26)

where c is a real constant, is the equation of the streamlines of flow.

The fluid velocity and speed are, respectively

and

$$v(w) = \overline{F'(w)} = \frac{v_0}{m} \left(w^{\frac{1}{m}-1} \right)$$

$$|v(w)| = \frac{v_0}{m} \rho^{\left(\frac{1-m}{m}\right)}$$
(27)
(28)

Problem 3: (Flow over a flat plate $0 \le v \le a$ in the upper half of the *w* plane)

We first note that the Schwarz-Christoffel transformation that maps the upper half Im $z \ge 0$ of the *z* plane onto the shaded region and its boundary in the *w* plane of figure 3(a) is found to be

$$w = K \int_0^z \frac{\int_{2\alpha}^{2\alpha} \frac{s\pi}{\pi}}{(1-s^2)^{\frac{\alpha}{\pi}}} ds + ai$$
(29)

where

$$K = \frac{(b-a)\sqrt{\pi}}{\Gamma\left(\frac{\alpha}{\pi} + \frac{1}{2}\right)\Gamma\left(1 - \frac{\alpha}{\pi}\right)}$$



Figure 3(a): Mapping of the Upper Half Im $z \ge 0$ of the z plane onto the Shaded Region in the w plane.



Figure 3(b): Flow over a flat plate $0 \le v \le a$ in the upper half of the *w* plane placed perpendicular to an incoming flow.



Figure 3(c): Streamlines of Flow over a flat plate $0 \le v \le a$ in the upper half of the *w* plane placed perpendicular to an incoming flow.

As $b \to 0$ and $\alpha \to \frac{\pi}{2}$, problem (3) reduces to that for flow over a flat plate placed perpendicular to an incoming flow as shown in figure 3(b). In this case the transformation (29) simplifies to

$$w = a\sqrt{z^2 - 1} \tag{30}$$

By introducing a branch cut consisting of the line segment $-1 \le x \le 1$ and letting

$$|z-1| = r_1$$
, $|z+1| = r_2$, $\arg(z-1) = \theta_1$, $\arg(z+1) = \theta_2$

we obtain the branch f_1 of the multiple valued function (30) as

$$f_1(z) = a\sqrt{r_1 r_2} e^{\left(\frac{\theta_1 + \theta_2}{2}\right)}$$
 (31)

where

$$r_k > 0$$
, $0 \le \theta_k < 2\pi$, $r_1 + r_2 > 2$ $(k = 1, 2)$

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The function in equation (31) is analytic in the entire *z* plane except on the line segment $-1 \le x \le 1$ and maps that domain in a one-to-one manner onto the entire *w* plane except the line segment $-a \le v \le a$. The positive and negative *y* axis are mapped by $w = f_1(z)$ onto parts of the *v* axis for which v > a and v < -a, respectively. Each point in the upper half Im z > 0 is mapped into the upper half Im w > 0 except for points on the line segment $0 < v \le a$ while points in the lower half Im z < 0 map into points in the lower half Im w < 0 except for points on the line segment $-a \le v < 0$. The ray $r_1 > 0$, $\theta_1 = 0$ map onto the positive real axis in the *w* plane, while the ray $r_2 > 0$, $\theta_2 = \pi$ is mapped onto the negative real axis. Solving for *z* in equation (30) we obtain the double valued function

whose branch

$$z = \frac{1}{a}\sqrt{(w - ai)(w + ai)} = g(w)$$
(33)
$$g_1(w) = \frac{1}{a}\sqrt{\rho_1\rho_2}e^{\left(\frac{\phi_1 + \phi_2}{2}\right)}$$
(34)

where

 $w - ai = \rho_1 e^{i\phi_1}, w + ai = \rho_2 e^{i\phi_2}, \rho_k > 0, \rho_1 + \rho_2 > 2a, -\frac{\pi}{2} \le \phi_k < \frac{3\pi}{2}$ (k = 1,2) is the inverse of the branch in equation (31) of the double valued function in equation (30). When the values of θ_k (k = 1,2) are restricted in the range $0 < \theta_k < \pi$, the domain of $f_1(z)$ becomes the half plane Im z > 0 while the restriction $\rho_2 > \rho_1$ on the function $g_1(w)$ in equation (34) limits its domain of definition to the upper half Im w > 0 except the line segment $0 < v \le a$. The complex potential for the flow over the flat plate in figure 3(c) is therefore the branch of the double valued function

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$$F(w) = \frac{v_0}{a} \sqrt{(w - ai)(w + ai)}$$
(35)

obtained as

$$F_1(w) = \frac{v_0}{a} \sqrt{\rho_1 \rho_2} e^{\left(\frac{\phi_1 + \phi_2}{2}\right)}$$
(36)

where

$$w - ai = \rho_1 e^{i\phi_1}, w + ai = \rho_2 e^{i\phi_2}, \rho_k > 0, \rho_1 + \rho_2 > 2a, \ \rho_2 > \rho_1, -\frac{\pi}{2} \le \phi_k < \frac{3\pi}{2} \ (k = 1, 2)$$

The stream function of the flow is

$$\psi(u,v) = \frac{v_0}{a} \sqrt{\rho_1 \rho_2} \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \tag{37}$$

By differentiating equation (36) we obtain the conjugate of the velocity field

$$F'(w) = \frac{v_0}{a} \frac{w}{\sqrt{(w-ai)(w+ai)}} = \frac{1}{2} \frac{v_0}{a} \left(\frac{\rho_1 e^{i\phi_1} + \rho_2 e^{i\phi_2}}{\sqrt{\rho_1 \rho_2} e^{i\left(\frac{\phi_1 + \phi_2}{2}\right)}} \right)$$
(38)

and hence the fluid velocity and speed respectively as

$$\overline{F'(w)} = \frac{1}{2} \frac{v_0}{a} \left(\frac{\rho_1 e^{-i\phi_1} + \rho_2 e^{-i\phi_2}}{\sqrt{\rho_1 \rho_2} e^{-i\left(\frac{\phi_1 + \phi_2}{2}\right)}} \right)$$
(39)

and

$$|v(w)| = |F'(w)| = \frac{1}{2} \frac{v_0}{a} \left(\sqrt{\frac{\rho_1}{\rho_2} + \frac{\rho_1}{\rho_2} + 2\cos(\phi_1 - \phi_2)} \right)$$
(40)

Problem 4: (Flow Over a Step in the Bed of a Deep Stream)

Figure 4(a) shows the flow over a step in the bed of a deep stream represented by the shaded region in the *w* plane.



Figure 4(b): Streamlines of Flow Over the Step at the Bed of a Deep Strem.

As the point *z* moves to the right along the negative part of the real axis where $x \le -1$, its image point *w* is to move to the right along the half line $u \le 0$, v = h. As the point *z* moves to the right along the line segment $-1 \le x \le 1$ of the *x* axis, , its image point *w* is to move to the direction of decreasing *v* along the segment $0 \le v \le h$ of the *v* axis. Finally, as *z* moves to the right along the positive part of the real axis where $x \ge 1$, its image point *w* is to move to the right along the positive real axis. By noting the changes in the direction of motion of *w* at the images of the points z = -1 and z = 1, the derivative of the mapping function might be

$$\frac{\mathrm{dw}}{\mathrm{dz}} = A \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}}$$
(41)

where A is some constant. From equation (41), the Schwarz-Christoffel transformation that maps the upper half Im $z \ge 0$ of the z plane onto the shaded region in the w plane is formally found to be

$$w = \frac{h}{\pi} \left[(z^2 - 1)^{\frac{1}{2}} + \cosh^{-1} z \right]$$
(42)

or,

$$w = \frac{h}{\pi} \left[(z-1)^{\frac{1}{2}} (z+1)^{\frac{1}{2}} + \log \left[z + (z-1)^{\frac{1}{2}} (z+1)^{\frac{1}{2}} \right] \right]$$
(43)

where $0 \le \arg(z \mp 1) \le \pi$. The form of equation (43) is particularly useful because it shows the manner in which the boundary of the *x* axis is mapped onto the boundary in the problem domain of the *w* plane.

In this problem, we seek to obtain an expression for the conjugate of the velocity field \bar{v} using the expression in equation (12) where it is supposed that the fluid velocity v(w) approaches a real constant v_{∞} as $|w| \rightarrow \infty$ in that region. Substituting the expression for dw/dz from equation (41)

into equation (12) and noting that $A = h/\pi$ from the Schwarz-Christoffel transformation (42) or (43)

$$\overline{v(w)} = \frac{\pi}{h} v_0 \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}$$
(44)

Now we must select u so that when w is near ∞ the velocity is v_{∞} . When w is near ∞ , z is also near ∞ and we have

$$\lim_{|w|\to\infty}\overline{v(w)} = v_0 \frac{\pi}{h} \lim_{|z|\to\infty} \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}$$

Thus we have that near infinity



In term of the points z = x whose images are points along the bed of the stream, equation (45) becomes

$$\overline{\nu(w)} = \nu_{\infty} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}$$
(46)

Hence the fluid speed is

$$|v(w)| = |v_{\infty}| \sqrt{\left|\frac{x-1}{x+1}\right|}$$
(47)

Finally, the streamlines of flow over the step in the bed of a deep stream were generated by finding the images of the lines y = c where *c* is a constant under the Schwarz-Christoffel transformation (43).

DISCUSSIONS

Flow in an Infinite Strip of Width a

In analyzing this flow, we first note that the stream function (equation (18)) is indeed the solution of the inviscid and incompressible flow (potential flow) in the infinite strip since it satisfies Laplace's equation

$$\nabla^2 \psi = \psi_{uu} + \psi_{vv} = \left(\frac{\pi}{a}\right)^2 v_0 e^{\frac{\pi u}{a}} \sin\left(\frac{\pi v}{a}\right) - \left(\frac{\pi}{a}\right)^2 v_0 e^{\frac{\pi u}{a}} \sin\left(\frac{\pi v}{a}\right) = 0$$

throughout that domain. Since $\psi(u, 0) = \psi(u, a) = 0$ it follows that the stream function vanishes at all points on the boundary of the strip consisting of the lines v = 0 (the *u* axis of the *w* plane) and v = a. Figure 1(b) shows the streamlines of flow generated by setting $a = \pi, v_0 = 1$, and varying the values of *c* from 0.01 to 2 in steps of 0.08 in the expression (19) for the general equation of the streamline. Closer streamlines in the flow indicate regions of higher fluid speed. The fluid speed therefore increases from left to right as it is also evident from equation (20) which shows that the fluid speed increases exponentially as *u* increases.

Flow in an Infinite Sector of Angle πm

Here too the stream function (equation (25)) for this flow is harmonic throughout the indicated region since it satisfies the polar form of Laplace's equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{r} \frac{\partial \psi}{\partial \rho} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \sigma^2} = \frac{v_0}{m} \left(\frac{1}{m} - 1\right) \rho^{\frac{1}{m} - 2} \sin\left(\frac{\sigma}{m}\right) + \frac{v_0}{m} \rho^{\frac{1}{m} - 2} \sin\left(\frac{\sigma}{m}\right) - \frac{v_0}{m^2} \rho^{\frac{1}{m} - 2} = 0$$

and also vanishes on the boundary where $\psi(\rho, 0) = \psi(\rho, m\pi) = 0$ when $\rho \ge 0$. The value of the stream function at a point with polar coordinates (ρ, σ) can be interpreted as the rate of flow across a line segment extending from the origin to that point. Figure 2(b) show the streamlines of flow in the interior of the angular region generated by assigning real values to *c* in steps of 0.05 in the interval [0.01,3] and setting m = 1/4, $v_0 = 1$. Also for these values of *m* and v_0 , the expression for the fluid speed reduces to $|v| = 4\rho^3$ and shows that the speed of a particle in the flow is directly proportional to the cube of its distance from the origin of the *w* plane.

Flow over a flat plate $0 \le v \le a$ in the upper half of the *w* plane

First observe that the determined Schwarz-Christoffel transformation that maps the half plane Im $z \ge 0$ onto the shaded region and its boundary in the *w* plane was expressed in terms of the gamma function and not elementary functions as in the other previous cases in which the polygon was degenerate. This is usually the case when the polygon is simple and closed or part of the boundary is in form of a polygon as it is the case here (Churchill and Brown, 1984). Problem 3(a) can be visualized as a two dimensional flow over a triangular prism placed at the bed of a deep stream. When $b \rightarrow 0$ and $\alpha \rightarrow \frac{\pi}{2}$, problem (3) reduces to that for flow over a flat plate placed perpendicular to an incoming flow as shown in figure 3(b). The MATLAB plot in Figure 3(c) show the streamlines of flow around the flat plate. From the flow pattern, we conclude that the fluid speed is highest at points around the tip of the flat plate where the streamlines are closer. This is also evident from equation (40) which shows that the fluid speed is highest when $\phi_1 = \phi_2$ corresponding to points on the part of the *v* axis for which v > a. Further away from the plate where the flow is least disturbed the streamlines show that the fluid speed reduces to the normal speed of the flow as one would expect.

Flow Over a Step in the Bed of a Deep Stream

Fiagure 4(b) show the streamlines of flow over a step in the bed of a deep stream. Closer streamlines indicate regions of higher flow speed as expected in a normal flow of that nature. Observe from equation (47) that at the points z = x whose images are points along the bed of the stream the fluid speed increases from $|v_{\infty}|$ along A'B' until $|v| = \infty$ at B', then diminishes to zero at C', and increases to $|v_{\infty}|$ from C' to D'. Note too that the fluid speed is $|v_{\infty}|$ at the point



between B' and C'.

Conclusion

In this research paper, a simple but efficient method for solving harmonic Dirichlet problems of ideal fluid flows in domains whose boundaries consists of straight line segments is presented. The method which is conformal based was then applied to ideal flows in an infinite strip, infinite sector of angle $m\pi(0 < m < 1)$, flow over a flat plate, and flow over a step at the bed of a deep stream and the solution for each flow problem obtained. Using these solutions the streamlines of flow were generated to show the various flow patterns and the fluid speed analyzed in terms of the streamlines spacing.

We however note here that although the method in this research paper gives exact analytical solutions and has interesting features, it is not without limitations. A serious limitation of the method has to do with the evaluation of the integral involved in the Schwarz-Christoffel transformation, particularly when the result of the integral cannot be expressed in terms of elementary functions as in the first part of problem (3). Consequently, we recommend the use of numerical techniques in such situations as presented in a paper by Thomas and Everett (2011). Secondly, because the method is conformal based, it is limited to two dimensional flow problems only and in particular to problems whose boundaries consist of straight line segments only. We therefore suggest that further research in this field should focus on extending the work to include flows in domains with complicated boundaries such as airfoils.

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