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ORBITAL ANGULAR MOMENUM IN PROLATE SPHEROIDAL COORDINATE

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ABSTRACT

The quantum angular momentum operator is one of several operators analogous to classical angular momentum. The angular momentum operator plays a key role in the theory of atomic physics and other quantum problems that involve rotational symmetry. These operators are well known in Cartesian and spherical coordinate as widely used in quantum mechanics. More so, the quantum commutation relation which tell if we can measure two quantum observables at the same time are equally expressed in Cartesian and spherical coordinate. However, a more controversial point is whether we can perfectly describe elementary particles properties using the proposed spherical coordinate. Consequently in this paper, we derived the various components of the orbital angular momentum operators in quantum mechanics as well as the commutation relations in the oblate spheroidal coordinate to expand the scope of mechanics.

Keywords: Orbital angular momentum, commutation relation, prolate spheroidal coordinate, spherical polar coordinate, Cartesian coordinate.

1.0 INTRODUCTION

In Cartesian coordinate (x, y, z), the orbital angular momentum operators are well known and are given by [1,2,3]

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{1.1}$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{1.2}$$

$$\hat{L}_z = -\text{iħ}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad (1.3)$$

Similarly, the commutation relation between the components of the orbital angular momentum operators are [1,4,5]

$$\left[\hat{L}_x, \hat{L}_z\right] = -\text{iħ}\hat{L}_y \quad (1.4)$$

$$\left[\hat{L}_y, \hat{L}_x\right] = -i\hbar\hat{L}_z \quad (1.5)$$

$$\left[\hat{L}_z, \hat{L}_y\right] = -\text{iħ}\hat{L}_x \quad (1.6)$$

Also,

$$\left[\hat{L}^2, \hat{L}_z\right] = \left[\hat{L}, \hat{L}_y\right] = [\hat{L}^2, \hat{L}_x] = 0 \quad (1.7)$$

where,

$$\hat{L}^2 = \hat{L}^2{}_x + \hat{L}^2{}_y + \hat{L}^2{}_z \quad (1.8)$$

In spherical polar coordinate (r, θ, ϕ), the orbital angular momentum operators are given by[2,3,4]

$$\hat{L}_x = \text{iħ}\left(sin\phi\frac{\partial}{\partial\theta} + cot\theta cos\phi\frac{\partial}{\partial}\right) \quad (1.9)$$

$$\hat{L}_y = \text{iħ}\left(-cos\phi\frac{\partial}{\partial\theta} + cot\theta sin\phi\frac{\partial}{\partial}\right) \quad (2.0)$$

$$\hat{L}_z = -\text{iħ}\frac{\partial}{\partial\phi} \quad (2.1)$$

Since

$$\hat{L}^2 = \hat{L}^2{}_x + \hat{L}^2{}_y + \hat{L}^2{}_z$$

Then in spherical polar coordinate, [3,5]

$$\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{2.10}$$

Equation (1.1) – (2.10) are the orbital angular momentum operators in Cartesian and spherical coordinate as known in quantum mechanics. These equations constitute the basis of the study of structure of elementary particles such as the atom, molecules, nucleus, as well as other quantum problems that involve rotational symmetry. These equations also constitute the basis of the study of particle's eigenvalue which corresponds to the experimental measured value of the observable, and the eigenfunction (spherical harmonics) for a particle which configuration space is a sphere (rigid rotor)[1,5]. However, it is now well known that elementary particles such as the atom, molecule, as well as the nucleus, experiences shape deformation as result of repulsive forces between proton and quantum shell effect[7]. The most commonly agreed shape is the flattened oblate or the elongated prolate spheroidal depending on the detailed movement of the nucleons. Therefore, treating the elementary particles as perfect sphere is at best an approximation [7]. for mathematical conveniences. There is need to express the quantum angular momentum operators in the spheroidal coordinate system so as to record geometrically approved information about elementary particles. Consequently in this paper, we develop the various components of the quantum orbital angular momentum operators as well as the quantum commutation relations in the prolate spheroidal coordinate system (η, ξ, ϕ).

2:0 MATHEMATICAL FORMULATIONS

The Cartesian coordinate (x, y, z) is related to the prolate spheroidal coordinate (η, ξ, ϕ) as [9,10]

$$X = a(1 - \eta^2)^{1/2}(\xi^2 - 1)^{1/2}\cos\phi \tag{2.11}$$

$$Y = a(1 - \eta^2)^{1/2}(\xi^2 - 1)^{1/2}\sin\phi \tag{2.12}$$

$$Z = a\eta\xi \tag{2.13}$$

Where 'a' is the scale factor and

$$-1 \leq \eta \leq 1 ; \ 0 \leq \phi \leq 2\pi ; \ 0 \leq \xi < \infty \qquad (2.14)$$

Similarly, the spherical polar coordinate (r, θ, ϕ) is related to the prolate spheroidal coordinate (η, ξ, ϕ) as

$$r = a(\eta^2 + \xi^2 - 1)^{½} \qquad (2.15)$$

$$\theta = \cos^{-1}\left(\frac{\eta\xi}{(\eta^2+\xi^2-1)^{½}}\right) \qquad (2.16)$$

$$\phi = \phi \qquad (2.17)$$

Therefore,

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad (2.18)$$

$$\frac{\partial \eta}{\partial r} = \frac{(\eta^2+\xi^2-1)^{½}}{a\eta} \qquad (2.19)$$

$$\frac{\partial \xi}{\partial \theta} = \frac{-(1-\eta^2)^{½}(\eta^2+\xi^2-1)}{\eta(\eta^2-1)^{½}} \qquad (2.20)$$

$$\frac{\partial \theta}{\partial x} = \frac{x\eta\xi}{a^2(1-\eta^2)^{½}(\xi^2-1)^{½}(\eta^2+\xi^2-1)} \qquad (2.21)$$

$$\frac{\partial \phi}{\partial x} = \frac{-y}{a^2(1-\eta^2)(\xi^2-1)} \qquad (2.22)$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad (2.23)$$

$$\frac{\partial \theta}{\partial y} = \frac{y\eta\xi}{a^2(1-\eta^2)^{½}(\xi^2-1)^{½}(\eta^2+\xi^2-1)} \qquad (2.24)$$

$$\frac{\partial \phi}{\partial y} = \frac{x}{a^2(1-\eta^2)(\xi^2-1)} \qquad (2.25)$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \tag{2.26}$$

$$\frac{\partial \phi}{\partial z} = 0 \tag{2.27}$$

$$\frac{\partial \theta}{\partial z} = \frac{-(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}{a(\eta^2+\xi^2-1)} \tag{2.28}$$

It therefore follows from (2.18 – 2.28) that;

$$\frac{\partial}{\partial x} = \frac{x}{a^2\eta}\frac{\partial}{\partial \eta} - \frac{x\xi(\xi^2-1)^{1/2}}{a^2(\eta^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}\frac{\partial}{\partial \xi} - \frac{y}{a^2(1-\eta^2)(\xi^2-1)}\frac{\partial}{\partial \phi} \tag{2.29}$$

$$\frac{\partial}{\partial y} = \frac{y}{a^2\eta}\frac{\partial}{\partial \eta} - \frac{y\xi(\xi^2-1)^{1/2}}{a^2(\eta^2-1)^{1/2}(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}\frac{\partial}{\partial \xi} + \frac{x}{a^2(1-\eta^2)(\xi^2-1)}\frac{\partial}{\partial \phi} \tag{2.30}$$

$$\frac{\partial}{\partial z} = \frac{z}{a^2\eta}\frac{\partial}{\partial \eta} + \frac{(1-\eta^2)^{1/2}(1-\xi^2)^{1/2}(\xi^2-1)^{1/2}}{a\eta(\eta^2-1)^{1/2}}\frac{\partial}{\partial \xi} \tag{2.31}$$

Putting equation (2.29 – 2.31) into (1.1 – 1.30), we obtained;

$$\hat{L}_{x\,(\eta,\xi,\phi)} = -i\hbar\left\{\frac{(1-\xi^2)^{1/2}(\eta^2+\xi^2-1)}{\eta(\eta^2-1)^{1/2}}\sin\phi\frac{\partial}{\partial \xi} - \frac{\eta\xi\cos\phi}{(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}\frac{\partial}{\partial \phi}\right\} \tag{2.32}$$

$$\hat{L}_{y\,(\eta,\xi,\phi)} = -i\hbar\left\{\frac{(1-\xi^2)^{1/2}(\eta^2+\xi^2-1)}{\eta(\eta^2-1)^{1/2}}\cos\phi\frac{\partial}{\partial \xi} - \frac{\eta\xi\sin\phi}{(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}}\frac{\partial}{\partial \phi}\right\} \tag{2.33}$$

$$\hat{L}_{z\,(\eta,\xi,\phi)} = -i\hbar\frac{\partial}{\partial \phi} \tag{2.34}$$

It therefore follows that the total angular momentum operator given by;

$$\hat{L}^2 = \hat{L}^2{}_x + \hat{L}^2{}_y + \hat{L}^2{}_z$$

Is obtained in the prolate spheroidal coordinate as;

$$\hat{L}^2 = -\hbar^2\left[\frac{(1-\eta^2)(\eta^2+\xi^2-1)^2}{\eta^2(\eta^2-1)}\frac{\partial^2}{\partial\xi^2} + \frac{(\eta^2+\xi^2-1)}{\eta^2(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}\right] \qquad (2.35)$$

Equation (2.32 – 2.34) represent the components of the orbital angular momentum operator in prolate spheroidal coordinate.

To compute the commutation relations, we use equation (1.4 – 1.6) and the following properties were formulated

$$[\hat{L}_{x\,(\eta,\xi,\phi)}, \hat{L}_{y\,(\eta,\xi,\phi)}] = -i\hbar\left(\frac{\eta^2\xi^2}{(1-\eta^2)(\xi^2-1)}\right)\hat{L}_{z\,(\eta,\xi,\phi)} \qquad (2.36)$$

$$[\hat{L}_{y\,(\eta,\xi,\phi)}, \hat{L}_{z\,(\eta,\xi,\phi)}] = i\hbar\hat{L}_{x\,(\eta,\xi,\phi)} \qquad (2.37)$$

$$[\hat{L}_{z\,(\eta,\xi,\phi)}, \hat{L}_{x\,(\eta,\xi,\phi)}] = i\hbar\hat{L}_{y\,(\eta,\xi,\phi)} \qquad (2.37)$$

$$\left[\hat{L}^2, \hat{L}_{x\,(\eta,\xi,\phi)}\right] = \left[\hat{L}^2, \hat{L}_{y\,(\eta,\xi,\phi)}\right] = \left[\hat{L}^2, \hat{L}_{z\,(\eta,\xi,\phi)}\right] = 0 \qquad (2.38)$$

Equation 2..36 – 2.38 are some of the quantum commutation relations in the prolate spheroidal coordinate system which relates the uncertainty relation between the various components of the orbital angular momentum operators as well as the total angular momentum.

3:0 SUMMARY AND CONCLUSION

In this paper, we derived the various components of the orbital angular momentum operators as well as the total angular momentum operator, $\hat{L}^2$ in prolate spheroidal coordinate. Their commutation relations were equally obtained.

The orbital angular momentum operators derived in this paper are purely in prolate spheroidal coordinate. These results are available for applications in mathematics and physics. For example, the atomic nuclei of actinide elements are prolate spheroidal in geometry. Hence, the door is opened for the solution of the total angular momentum operator, $\hat{L}^2$ to obtain the corresponding eigenvalue as well as the eigenfunction (spheroidal harmonics) for orbiting particles in quantum mechanics which configuration space is a prolate spheroidal.

REFEREENCES

[1] David J. Grifiths, (2004), *Introduction to Quantum mechanics (2nd Edition),* Prentice Hall, England, pp 123 – 124, 135 – 147.

[2] Blockhintsev D.I., Reidel D., (1964), *Quantum Mechanics,* Dorddrecht, Germany, pp. 237 – 239

[3] Emonds A.R., (1974), *Angular Momentum in Quantum Mechanics, 2nd* ed., Princeton University Press, Princeton, NJ. U.S.A., pp. 2 – 37.

[4] Dirac P.A.M., (1947), *The Principle of Quantum Mechanics,* 3rd ed., Clarendon Press, Oxford, U.K. pp. 18, 36, 48.

[5] Baym G., *Lectures on Quantum Mechanics,* Benjamin, New York, NY, U.S.A., 1969, Chapter 1.

[6] Croxton, (1974), *Introduction to Eigenphysics,* John Wiley, London, pp 194 – 199

[7] Atkins P.W., Friedman R.S. (2005), *Molecular Quantum Mechanics,* 4th ed., Oxford University Press, Oxford, U.K. pp. 89, 126 – 128.

[8] Vinti J.P., (1960), *New Approach in Theory of Satellite Orbits,* Physical Review Letter, Volume 3, 1,8 – 15.

[9] Hilderbrand F.B., (1962), *Advanced Calculus for Application*, Prentice Hall, England Cliff,

pp. 234 – 298

[10] Arken G., (1968), *Mathematical Methods for Physicists,* Academic Press, New York, 1968, pp. 211 - 249