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On the contributions of the Arzela-Ascoli theorem to Analysis

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Abstract

In this paper we single out some of the crucial contributions of the Arzela- Ascoli theorem to Analyis. It is demonstrated on how it resonates in the proof of Peano existence theorem, Peter-Weyl theorem and establishment of Bolzano-Weistrass theorem. It is also justified that in mathematics, the Arzela- Ascoli theorem of functional analysis gives necessary and sufficient condition of equicontinuity to the family of functions.

Keywords: Equicontinuity, Initial value problem, Peano existence theorem, Peter-Weyl theorem, Bolzano-Weistrass theorem.

Introduction:

The notion of equicontinuity was introduced at around the same time by Ascoli (1883-1884) and Arzela (1882-1883). A weak form of the theorem was proved by Ascoli (1883-1884), who established the sufficient condition for compactness and by Arzela (1895), who established the necessary condition and the gave the first clear presentation of the result.

A further generalization of the theorem was proven by Frechet (1906) to sets of real-valued continuous functions with domain a compact metric space (Dunford and Schwartz,1958). Modern formulations of the theorem allow for the domain to be compact Hausdorff and for the range to be an arbitrary metric space.

More general formulations of the theorem exist that give necessary and sufficient conditions for a family of functions from a compactly generated Hausdorff space into a uniform space to be compact in the compact-open topology, Kelley (1991, page234).

Research Methodology

Definition 1.0 (Equicontinuity)

A family of functions F is said to be equicontinuos on [a,b] if for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\| x(t) - x(s) \| < \varepsilon$, whenever $|t - s| < \delta$ for every function x∈F and t,s∈[a,b].

Definition 1.1(Initial value problem)

Let X(x,t) be a continuous function. Then a function x(t) is a solution of the initial value problem

$$\begin{cases} \frac{dx}{dt} = X(x,t) \\ x(a) = C \end{cases} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots * $$

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If and only if it is a solution of the integral equation

$$x(t) = C + \int_a^t X(x(s), s)ds \ldots\ldots\ldots\ldots\ldots\ldots\ldots ** $$

Theorem1.2(The Arzela- Ascoli theorem)

If a sequence $\{f_n\}_1^{\infty}$ in C(x) is bounded and equicontinuous then it has a uniformly convergent subsequence.

Theorem 1.3 (The Arzela-Ascoli theorem)

Assume that the sequence $\{x^n(t)\}$ is bounded and equicontinuous on [a,b], then there exists a subsequence $\{x^{ni}(t)\}$ that is uniformly convergent on [a,b]

Theorem 1.4(Peano Existence theorem)

Assume that X(x,t) is continuous in the closed domain $\| x - c \| \leq T$. Then the initial value problem ... $*$ has at least one solution in the interval |t-a|≤ min{T,$\frac{k}{m}$} where

$$m = \underbrace{Sup}_{\substack{\|x-c\|\leq k \\ |t-a|=T}} \| X(x,t) \|$$

Theorem 1.5 (Peter-Weyl theorem)

The theorem has three parts:

i. The first part states that the matrix coefficients of irreducible representations of compact topological group G are dense in the space C(G) of continuous complex-valued functions on G and thus also in the L_2(G) of square-integrable functions.
ii. The second part asserts the complete reducibility of unitary representations of G.Let P be a unitary representation of a compact group G on a complex Hilbert space H.Then H splits into an orthogonal direct sum of irreducible finite-dimensional unitary representation of G.
iii. The third part asserts that the regular representation of G on L_2(G) decomposes as the direct sum of all irreducible unitary representations, Moreover, the matrix coefficients of the irreducible unitary representations form an orthogonal basis of L_2(G)

Theorem 1.6(Bolzano-Weierstrass theorem)

Every bounded infinite set has at least one limit point

2.0 MAIN RESULT (CONTRIBUTIONS TO ANALYSIS)

2.0.1: The Arzela-Ascoli theorem is a fundamental result in mathematics. In particular, it forms the basis for the proof of **the Peano existence theorem, theorem 1.4** in the theory of ordinary differential equations as below:

Proof of the Peano existence theorem:

Denote $T_1 = \min\{T, \frac{k}{m}\}$.Let $M = \underbrace{Sup}_{|t-a|\le T} |X(c,t| < +\infty$. Without loss of generality, we can assume that a=0 and t$\geq a$. We need to prove the theorem on $0 \le t \le T_1$ with a=0. We first construct a sequence of bounded equicontinuous functions $\{x^n(t)\}$ on $[0, T_1]$. For each n, define

$$x^n(t) = \begin{cases} C & \forall\ 0 \le t \le \frac{T_1}{n.} \\ C + \int_0^{t-\frac{T_1}{n.}} X(x^n(s), S)ds & \forall\ \frac{T_1}{n.} \le t \le T_1 \end{cases}$$

The above formula defines the value of $x^n(t)$ recursively in terms of the previous values of $x^n(t)$. We can use mathematical induction to show that

$$\| x^n(t) - c \| \le K \text{ on } [0, T_1].$$

Indeed on $[0, \frac{T_1}{n.}]$, it is trivial since $x^n(t) = C$. If we assume that the inequality holds on $[0,\ k\frac{T_1}{n.}]$

$0 \le k < n$, then on $[k\frac{T_1}{n.}, (k+1)\frac{T_1}{n.}]$, $\| x^n(t) - c \| = \int_0^{t-\frac{T_1}{n.}} X(x^n(s), S)ds$

$$\| x^n(t) - c \| \le M|t - \frac{T_1}{n.}| \le MT_1 \le K$$

Hence the sequence$\{x^n(t)\}$ is uniformly bounded on $[0, T_1]$: $\| x^n(t) \| \le \| C \| + K$

The equicontinuity of the sequence $\{x^n(t)\}$ on $[0, T_1]$ can be proven by the following estimates: $\forall\ t_1, t_2 \in 0, [0, T_1]$

$$\| x^n(t_1) - x^n(t_n) \| = \begin{cases} 0 & if\ \ t_1, t_2 \in \left[0, \frac{T_1}{n.}\right], \\ \| \int_0^{t_2-\frac{T_1}{n.}} X(x^n(s), S)ds \|, & if\ t_1 \in [0, \frac{T_1}{n.}] and\ t_2 \in (\frac{T_1}{n.}, T_1), \\ \| \int_0^{t_1-\frac{T_1}{n.}} X(x^n(s), S)ds \|, & if\ \ t_2 \in [0, \frac{T_1}{n.}]\ and\ \ t_1 \in [\frac{T_1}{n.}, T_1], \\ \| \int_{t_1}^{t_2-\frac{T_1}{n.}} X(x^n(s), S)ds \|, & if\ \ \ \ t_1, t_2 \in [\frac{T_1}{n.}, T_1] \end{cases}$$

I $\le$M|t-s|

By Arzela-Ascoli theorem, theorem 1.3, we know that there exists a uniformly convergent sequence $\{x^{ni}(t)\}$ that converges to a continuous function $\{x^{\infty}(t)\}$ on $[0,\ T_1]$ as $ni \to \infty$.We can show that the function $x^{\infty}(t)$ is actually a solution of the initial value problem….∗. Indeed for any fixed $t \in [0,\ T_1]$, we can take ni sufficiently large such that $\frac{T_1}{ni} < t$. Thus by the definition of $\{x^n(t)\}$, we have

$$x^{ni}(t) = C + \int_0^t X(x^{ni}(s), S)ds - \int_{t-\frac{T_1}{ni}}^t X(x^{ni}(s), S)ds$$

As $ni \to \infty$, since X(x,t) is uniformly continuous, we have

$$\int_0^t X(x^{ni}(s), S)ds \to \int_0^t X(x^{\infty}(s), S)ds;$$

The second integral of the last equation tends to zero, since

$$\left|\int_{t-\frac{T_1}{ni}}^t X(x^{ni}(s), S)\right| ds| \le \int_{t-\frac{T_1}{ni}}^t M\, ds = M\, \frac{T_1}{ni} \to 0$$

Hence we know that the function $x^{\infty}(t)$ satisfies the integral equation

$$x^{\infty}(t) = C + \int_0^t X(x^{\infty}(s), S)ds \quad \square$$

2.0.2: In mathematics, the Arzela-Ascoli theorem of functional analysis gives necessary and sufficient conditions to decide whether every sequence of a given family of real-valued continuous functions defined on a and bounded interval has a uniformly convergent sub-sequence. The main condition is the equicontinuity of the family of functions.

2.0.3 It also plays a decisive role in the proof of the Peter-Weyl theorem, theorem 1.5

2.0.4 The Arzela-Ascoli theorem is the key to the following result: Asub-set F of C(X) is compact iff it is closed, bounded and equicontinuous.

2.0.5 The establishing of the Bolzano-Weierstrass theorem, theorem 1.6 is a generalization of the Arzela-Ascoli theorem.

Conclusion

In this paper, we have summarized and singled out some of the contributions of the Arzela-Ascoli theorem and how it resonates the field of mathematical analysis.

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