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ON THE HEURISTIC DECONSTRUCTION OF THE VOLUME OF A SPHERE

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ABSTRACT

A mathematical deconstruction of the volume formula for the sphere ($\frac{4\pi}{3}r^3$) revealing its pseudo spherical hidden nature is presented in this article. This surprising result rooted in the soil of the Calculus highlights weaknesses of Archimedean and modern analysis techniques used to obtaining the formula. Being in the heart of STEAM, our heuristic surgery bears an exact and self evident formula of the volume for the true sphere. This formula crystallizes an extraordinary mathematical creative power which positively transforms the world scientific, pedagogical, social and Cultural landscape beyond expectations.

1. INTRODUCTION

The formula $V = \frac{4\pi}{3}r^3$ is surely among the best known and the most widely used tools in mathematical sciences. All advanced modern instruments and techniques used for the calculation of the volume of the sphere converge to this famous expression whose the worldwide known oldest mathematical demonstrations are rooted in two classical Archimedes' works, *The Method of Archimedes Treating of Mechanical Problems-To Eratosthenes* and *On the Sphere and the Cylinder, Book 1* [1,2]. This unchanged formula whose demonstration meets all university modern techniques of integration encapsulates its great mathematical and historical power [3].

Possibly for this reason, there has been no critical examination of his range of validity until nowadays.

Being faced to epistemological obstacles in the construction of a simplest and shortest version of the demonstration of the volume of the sphere in the psychology of Archimedes' genius refined by the geometrical intelligence of the Fundamental Theorem of Analysis, we have overcome these barriers though a Deconstruction of the volume of the sphere.

Does the Deconstruction of the volume of the Sphere mean a surgery and an in-depth revision of the classical formula of the sphere or just a mere philosophical sweet seducing approach bearing a scientifically sterile result?

A response to this question is the object of this paper.

2. ARE THERE DARK SIDES IN THE ARCHIMEDES APPROACHES TO FIND THE VOLUME OF THE SPHERE?

In the following section, we will evaluate the mathematical pertinence of tools used in the Archimedean approaches to find the volume of the sphere contained in his works. Let's consider the claim in *On the Sphere and the Cylinder, Book I.*

The proposition 34 entails: "*Any sphere is equal to four times a cone which has its base equal to the greatest circle in the sphere and height equal to the radius of the sphere"*

The logical strength of this theorem depends on the exactness of the formula volume of the cone, $V = \frac{1}{3}\pi R^3$, **which must mathematically be equal to one thirds of the volume of the cylinder of the same base and height**.

Now, applying our mathematical rigor to the accuracy of this claim, the following experimental fact shows the non exactness of the Archimedes' claim:

0) Fill in this empty cone with sand, salt or any liquid
1) Put the content of the cone in the Cylinder (with the same basis and height)
2) Repeat the process three times

What do you find out?

OBSERVATION: There is an exceeding quantity of sand, salt or any liquid. **Three times the Volume of the Cone exceed the volume of the Cylinder**

Expressed mathematically, we have:

3) $\mathbf{3}\, V_{cone} - \Delta V_{cone} = V_{cylinder}$

Hence, the relation implies $V_{cone} > \frac{1}{3}\pi R^3$

In this vein, our unpublished works related to the demonstration of the volume of the cone reveals the exact formula of the cone [5]:

4) $V_{cone} = \frac{\sqrt{2}}{4}\pi R^3$

This discovery is based on a hitherto unknown self evident fact: THE VOLUME OF A CYLINDER CIRCUMSCRIBING A SPHERE IS EQUAL TO THE VOLUME OF A DOUBLE CONE GENERATED BY ROTATING ITS CROSS-SECTIONAL AREA ABOUT ITS DIAGONAL AXIS.

In the other hand, we know that the content of proposition 2 of *The Method* shows the key role of the volume of a in finding an approach to the demonstration of the volume of the sphere whose empirical formula is known before this process:

(1) *Any sphere is (in respect of solid content) equal four times a cone with base equal to the great circle of the sphere and height equal to its radius ; and*
(2) *The cylinder with base equal to a great circle of the sphere and eight equal to the diameter is 3/2 times the sphere.*

But, *extracting* the Analytical intelligence from the Archimedes 'empirical' and geometrical demonstration crystallized in the figure 1 and the figure 2, we have:

1. $r_1^2 + r_2^2 = r_3^2$
2. $\pi r_1^2 + \pi r_2^2 = \pi r_3^2$
3. $\pi r_1^2 dx + \pi r_2^2 dx = \pi r_3^2 dx$
4. $\pi \int_0^r r_1^2\, dx + \pi \int_0^r r_2^2\, dx = \pi \int_0^r r_3^2 dx$

Now

5. $r_3^2 = r^2,\ r_1^2 = x^2\ and\ r_2^2 = r^2 - x^2$

Thence, injecting 5) in 4), it results

6. $\pi \int_0^r r^2 dx = \ \pi \int_0^r x^2 dx + \pi \int_0^r (r^2 - x^2)\, dx$
7. $\pi r^2 \int_0^r dx = Volume\ of\ the\ half\ Cylinder$
8. $\pi \int_0^{h/2} r_1^2\, dx = \pi \int_0^r x^2 dx = Volume\ of\ the\ upper\ cone$
9. $\pi \int_0^{h/2} r_2^2\, dx = \pi \int_0^r (r^2 - x^2)\, dx = Volume\ of\ the\ half\ sphere$

Is anything wrong in these 'firmly established' relations? Interpretation of these coherent results is of higher importance to objectively respond to this metamathematical question.

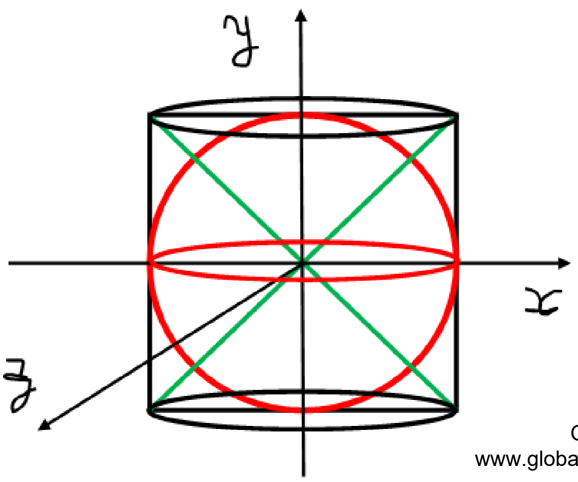
Multiplying relations 5), 6) and 7) by $(\pi r)^{-1}$, we have respectively the following expressions which, interpreted in light of the Fundamental Theorem of Analysis, must be a quarter of the cross sectional areas of the cylinder, the double cone and the sphere.

10. $\int_0^r r dx = Area\ of\ a\ square$
11. $\frac{1}{r} \int_0^r x^2 dx = Area\ of\ the\ down\ parabolic\ region\ in\ the\ square$
12. $\frac{1}{r} \int_0^r (r^2 - x^2)\, dx = Area\ of\ the\ upper\ parabolic\ \ domain\ of\ the\ square$

Now, the first quadrant of the cross-sectional areas of the double cone and the sphere are respectively expressed as follows in a:

13. $\int_0^r x\, dx = Area\ of\ the\ down\ trianggle\ region\ in\ the\ square$
14. $\int_0^r \sqrt{r^2 - x^2}\, dx = area\ of\ the\ first\ quadrant\ of\ circle$

Since the cross-sectional area of the volume of a solid of revolution is invariant by rotation, taking into account the fundamental theorem of analysis, it is impossible to obtain the cross-sectional of a sphere and a cone different to a circle or triangle respectively. It means that classical interpretation of relation $\pi \int_0^r r_1^2\, dx + \pi \int_0^r r_2^2\, dx = \pi r_3^2 \int_0^r dx$ (the volume is of the cylinder is equal to the sum of the volume of an inscribed sphere and the cone of the same basis and height) does not hold.



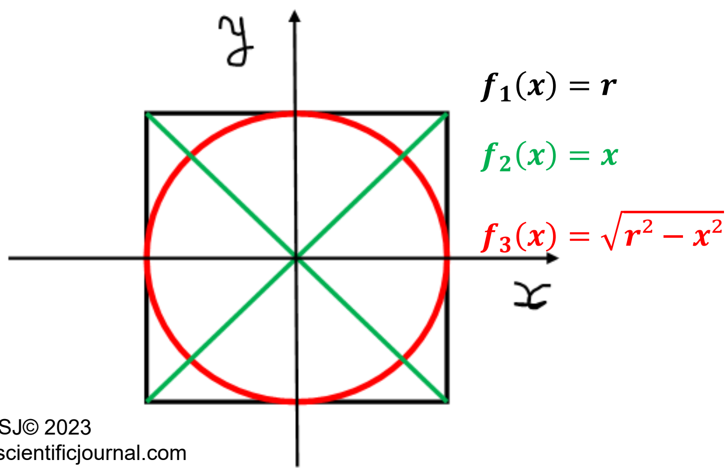


Figure 1 Figure 2

3. THE ALGEBRAIC AND INFINITESIMAL SURGERY OF THE ARCHIMEDEAN VOLUME OF THE SPHERE

This surgery is motivated by our pedagogical project of building advanced mathematical sciences easiest courses compatible with human brain optimal development where all known classical and modern mathematical formulae and techniques are deconstructed. Thus, our metamathematical analysis is on the ground of natural and self evident steps of the logical deconstruction process of the Archimedean formula $\frac{4\pi}{3}r^3$. This heuristic procedure allow us find its easy-to-visualize components in three-dimensional space and their corresponding cross sectional areas.

1. $\frac{4\pi}{3}r^3 = 2\pi r^3 - \frac{2\pi}{3}r^3$
2. $\frac{4\pi}{3}r^3 = \frac{\pi}{2}r(4r^2) - \pi\int_{-r}^{r}x^2dx)$
3. $\frac{4\pi}{3}r^3 = \frac{\pi}{2}r\left(\int_{-r}^{r}dx\int_{-r}^{r}dy\right) - \pi\int_{-r}^{r}x^2dx)$
4. $\frac{4\pi}{3}r^3 = \frac{\pi}{2}r\left(\int_{-r}^{r}dx\int_{-r}^{r}dy\right) - \frac{\pi}{2}(\int_{-r}^{r}x^2dx - \int_{-r}^{r}-x^2dx)$

This last analytical and algebraic decomposition contains all information related to the true geometrical representation the Archimedes' sphere.

4. THE GEOMETRICAL SIDE OF THE FORMULA V= $\frac{4\pi}{3}r^3$

Since the mentioned analytical expression of the volume of the sphere contains geometrical of its components, the visualization of their corresponding shape will help us find the true three dimensional shape of our famous sphere. the following chapter.

We will first of all give geometrical representations of the components of the first term of the second part of the analytical expression of relation 4) after having mentioned them.

4.1. The Main Components of the Expression $\frac{\pi}{2}r\left(\int_{-r}^{r}dx\int_{-r}^{r}dy\right)$

This part consists of two main components $\frac{\pi}{2}r$ and $\int_{-r}^{r}dx\int_{-r}^{r}dy$. Let's have step by step geometrical images of $\int_{-r}^{r}dx$, $\int_{-r}^{r}dy$ and $\int_{-r}^{r}dx\int_{-r}^{r}dy$

4.1.1. Geometrical representation of $\int_{-r}^{r}dx$

In the light of the geometrical version the fundamental theorem of analysis $\int_{-r}^{r}dx$ represents a segment bounded by $-r$ and r in the x axis.

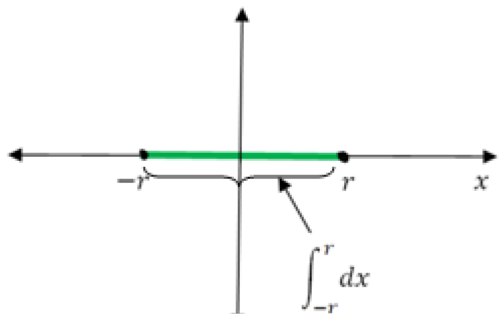


Figure 3

4.1.2. Geometrical representation of $\int_{-r}^{r} dy$

In the light of the geometrical version the fundamental theorem of analysis $\int_{-r}^{r} dy$ represents a segment bounded by $-r$ and r in the y axis.

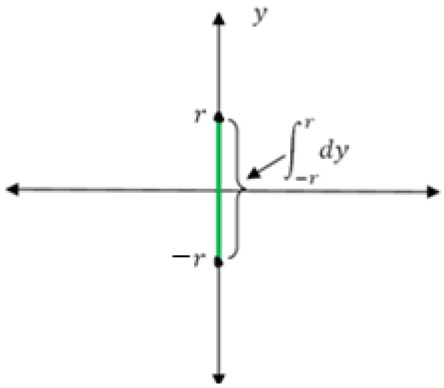


Figure 3

4.1.3. Geometrical representation of the couple $(\int_{-r}^{r} dx , \int_{-r}^{r} dy)$

The geometrical representation of these expressions is two orthogonal segments

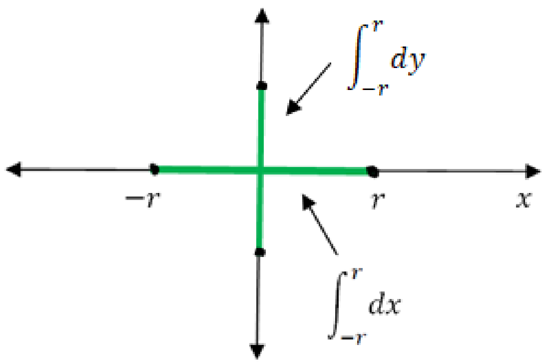


Figure 5

This cross has the following analytic expression

1. $X = \begin{cases} \int_{-r}^{r} dx \\ \int_{-r}^{r} dy \end{cases}$

The former is equivalent the surface area of a square of radius r , the latter is a revolution operator acting on the squared surface to form cylinder.

4.1.4. Geometrical representation of the component $\int_{-r}^{r} dx \int_{-r}^{r} dy$

This expression is an infinitesimal expression of the surface area of a square centered at the origin of a rectangular coordinate system.

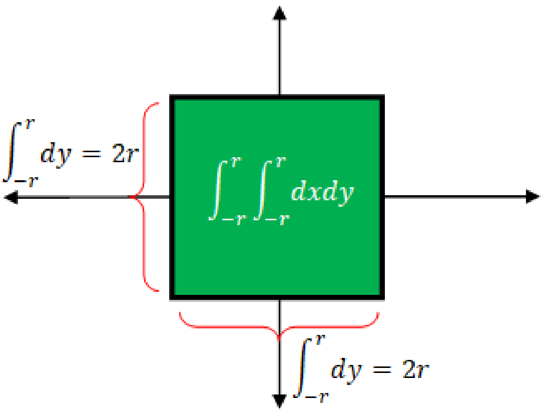


Figure 6: square of side 2r and area 4r²

4.1.5. The Geometrical version of expression $\frac{\pi}{2} r \int_{-r}^{r} dx \int_{-r}^{r} dy$

Since the expression $\frac{\pi}{2} r \int_{-r}^{r} dx \int_{-r}^{r} dy$ equals the volume $V = 2\pi r^3$ of the cylinder whose the cross-sectional area is a square expressed by the formula $\int_{-r}^{r} dx \int_{-r}^{r} dy$, the operator $\frac{\pi}{2} r$ dictates the square to rotate about the y-axis sweeping a half circle area.

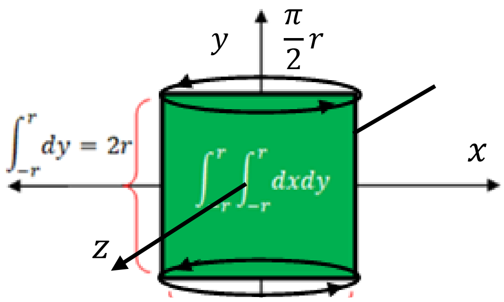


Figure 7: The Square cross-sectional area Cylinder

This cylinder is a geometrical version of the analytical expression $\frac{\pi}{2} r \int_{-r}^{r} dx \int_{-r}^{r} dy$ whose base and height are $B = \pi r^2$ and $h = 2r$ respectively.

4.2. Geometrical representation of the component $\pi \int_{-r}^{r} x^2 dx$

Finding analytical expressions of the components of this expression will help us give their divide geometrical representation. The following reasoning gives this result:

1. $\pi \int_{-r}^{r} x^2 dx = \frac{\pi}{2}(2 \int_{-r}^{r} x^2 dx)$
2. $\frac{\pi}{2}(\int_{-r}^{r} x^2 dx - \int_{-r}^{r} -x^2 dx)$
3. $\pi \int_{-r}^{r} x^2 dx = \frac{\pi}{2}(\int_{-r}^{r} x^2 dx + \int_{r}^{-r} -x^2 dx)$

4. $\pi\int_{-r}^{r}x^2dx=\frac{\pi}{2}r\left[\frac{1}{r}\left(\int_{-r}^{r}x^2dx+\int_{r}^{-r}-x^2dx\right)\right]$

Let's notice that this term ($\pi\int_{-r}^{r}x^2dx$) consists of two principal parts: the hybrid rotational operator $\frac{\pi}{2}r$ and the double surface $\frac{1}{r}\left(\int_{-r}^{r}x^2dx+\int_{r}^{-r}-x^2dx\right)$. The latter has in turn two parts $\frac{1}{r}\int_{-r}^{r}x^2dx$ and $\frac{1}{r}\int_{-r}^{r}x^2dx$ which are represented respectively in following diagrams:

4.2.1. Geometrical representation of the component $\frac{1}{r}\int_{-r}^{r}x^2dx$

This expression represents the blue colored symmetric area limited by the lines $x=-r$ and $x'=r$ and the parabola $f(x)=x^2$.

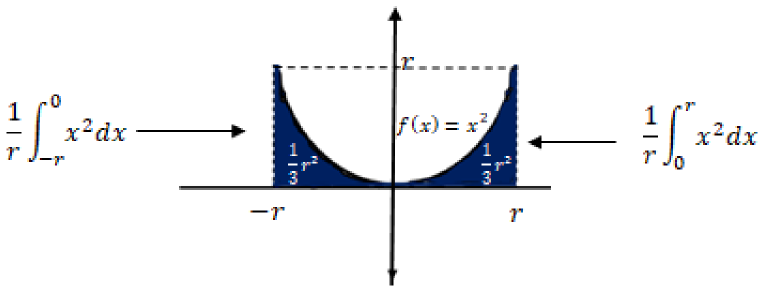


Figure 8: Positive Parabolic blue Area in a $2r\times r$ –Rectangle

4.2.2. Geometrical representation of the component $\frac{1}{r}\int_{r} -x^2 dx = -\frac{\ }{3}r$

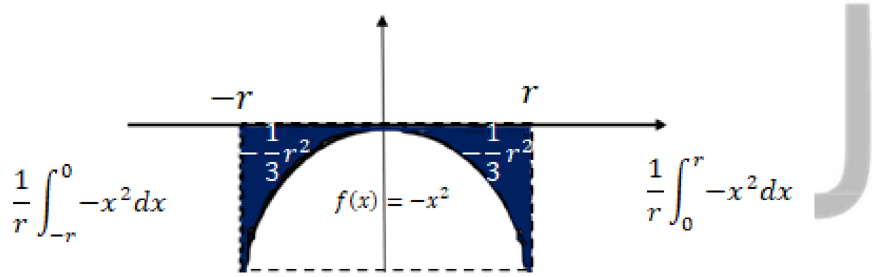


Figure 9: Negative Parabolic blue Area in a $2r\times r$ –Rectangle

4.2.3. Geometrical representation of the component $\frac{1}{r}\left(\int_{-r}^{r}x^2dx+\int_{r}^{-r}-x^2dx\right)$

The geometrical version of this the blue colored surface under the parabolas $f(x)=x^2$ and $f(x)=-x^2$ limited by the lines $x=\pm r$

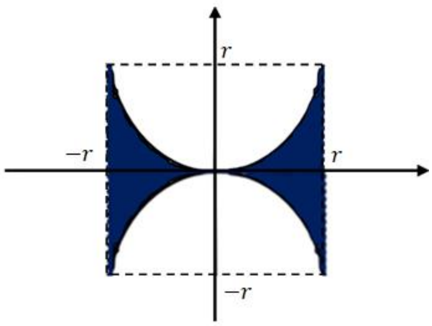


Figure10: Positive and Negative Parabolic blue Areas in the $2r \times 2r$- Square

The $\pm 90°$ rotation of the entire xy-plane gives the following isometric form

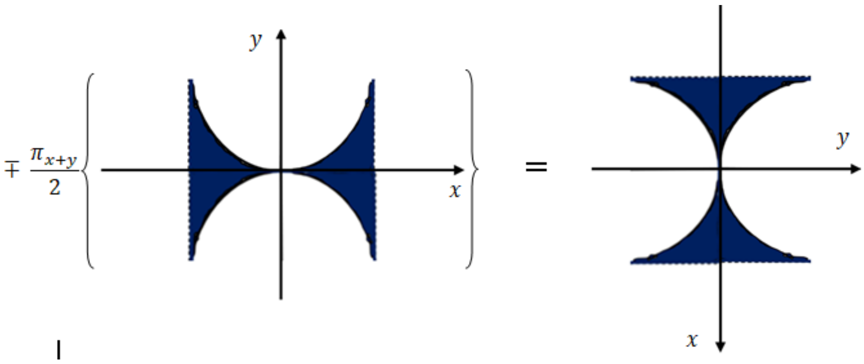


Figure 11

4.2.4. Geometrical representation of the volume $\pi \int_{-r}^{r} x^2 dx$

Since the cross sectional area of the volume $\pi \int_{-r}^{r} x^2 dx$ is the x-y symmetric parabolic bleu colored domain expressed by $\frac{1}{r}(\int_{-r}^{r} x^2 dx + \int_{r}^{-r} -x^2 dx)$, its revolution about y-axis and x-axis generates the following objects with the same volume represented in the above figures:

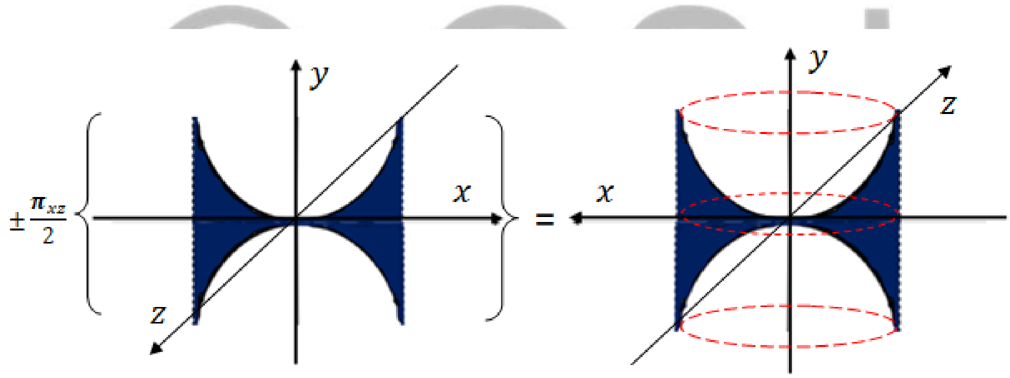


Figure 12: $\frac{\pi}{2}$ rotation of the Parabola-Bounded Area about the y-axis

Double Egyptian Paraboloidal Calabash in a cylinder

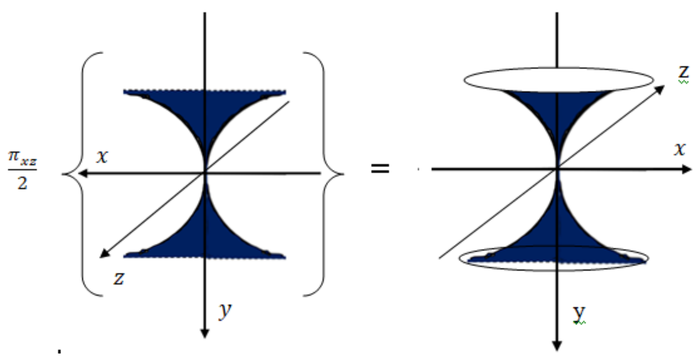


Figure 13: Solid of Revolution generated the by the blue colored area

REMARK: the both volumes of revolution of Figure 12 and Figure 13 are equal to $V = \frac{2}{3}\pi r^3$. This formula would have been different to the volume of double cone whose cross-sectional area is bigger than the former figure.

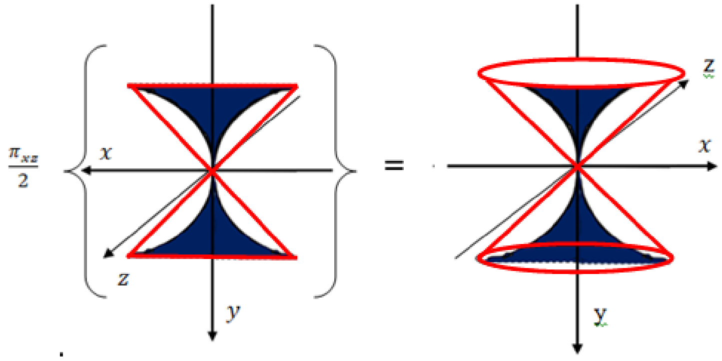


Figure 14: the cone volume versus the pseudo cone volume.

4.2.5. The cross sectional area of the Archimedes' sphere

The Archimedes' sphere is tree-dimensional object whose volume V equals $\frac{4}{3}\pi r^3$. We know that the cross sectional area of any sphere of radius r is a circle of radius r. The following reasoning shows the shape of the cross sectional area of the Archimedes' sphere expressed by the following formula:

1. $C = \left(\int_{-r}^{r} dx \int_{-r}^{r} dy\right) - \frac{1}{r}\left(\int_{-r}^{r} x^2 dx - \int_{-r}^{r} -x^2 dx\right),$

where A and B are the square and the white surface area inside the square respectively.

$A = \left(\int_{-r}^{r} dx \int_{-r}^{r} dy\right)$ =4r²=surface area $of\ the\ square$

$$B = \frac{1}{r}\left(\int_{-r}^{r} x^2 dx - \int_{-r}^{r} -x^2 dx\right) = \frac{4}{3}r^2 = the\ white\ surface\ \ contained\ in\ the\ square$$

$$C = A - B = A \cap \overline{A \cap B} = the\ green\ colored\ surface$$

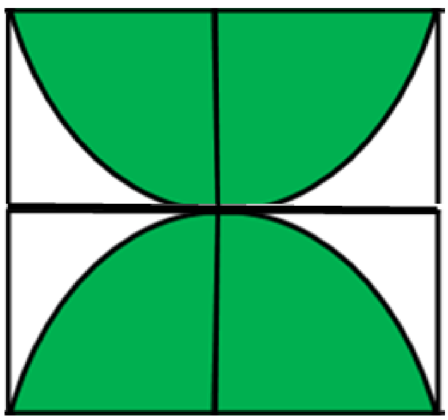
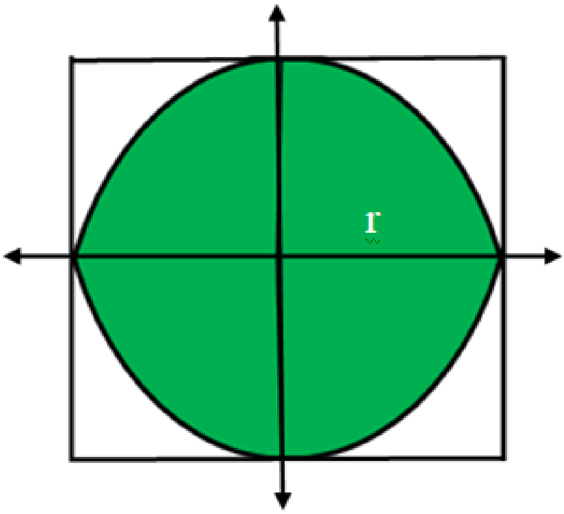
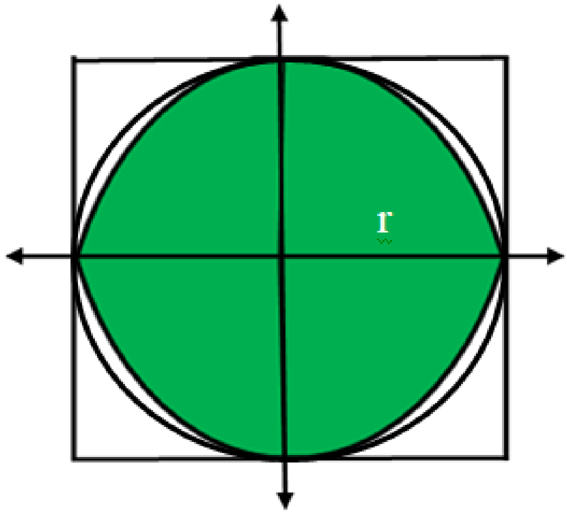


Figure 15: Double 'Archimedean' Parabolic Segment

Rearranging these symmetric surfaces by –r and r translations along the y-axis, we have the corresponding cross sectional area of the Archimedes' sphere (figure 16) which is remarkably contained in a circle of radius r (Figure 17).





4.2.6. The corresponding shape of the volume $\frac{4\pi}{3}r^3$

We know that the $\frac{\pi}{2}$ rotation of the square about the y-axis generates a cylinder whose volume is V= $\frac{\pi}{2}r \times surface\ of\ the\ square$. Likewise, the $\frac{\pi}{2}$ rotation of the green colored pseudo circle about the same axis forms a the green colored pseudo sphere whose volume is the V= $\frac{\pi}{2}r \times surface\ of\ the\ pseudo\ sphere = \frac{4\pi}{3}r^3$ (figure 15)

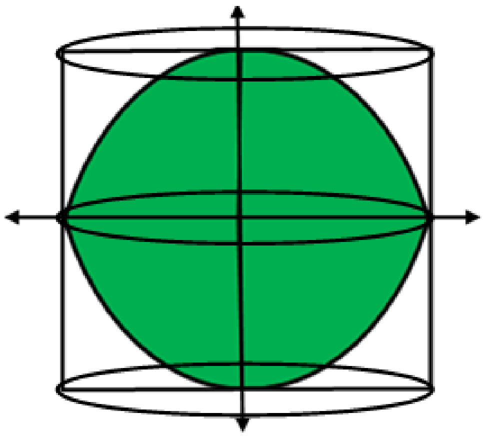


Figure 18: The Pseudo Sphere

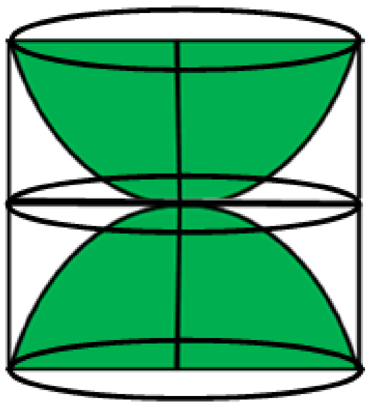


Figure 19: The double Egyptian Calabash

This surface area of shape is the one found in the tenth problem of the Moscow's Papyrus 'rediscovered' centuries later by Archimedes [4].

5. WHAT IS THE VOLUME OF THE SPHERE?

On the light of this New Technique of Calculating the Volume of the Solid of Revolution by which we have determined the respective volumes of the Cylinder and the Pseudo-Sphere crystallized in the formula $\boldsymbol{V_{\Omega}} = \frac{\pi}{2}\boldsymbol{r} \times \boldsymbol{Cross - sectional\ Area\ of\ \Omega}$, the new formula of the sphere can be deduced. Thence, since the cross-sectional area of the sphere of radius is the area of a circle of radius r (πr^2), we have the above formula of the sphere:

1. $V_{SPHERE} = \frac{\pi}{2} r \times Surface\ Area\ of\ the\ the\ Circle$
2. $V_{SPHERE} = \frac{\pi}{2} r \times \pi r^2$
3. $\boldsymbol{V_{SPHERE} = \frac{\pi^2}{2} r^3}$

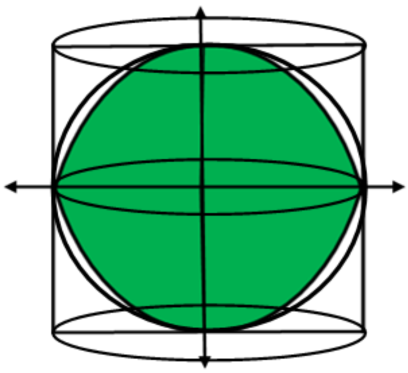


Figure 19: A pseudo sphere in the sphere

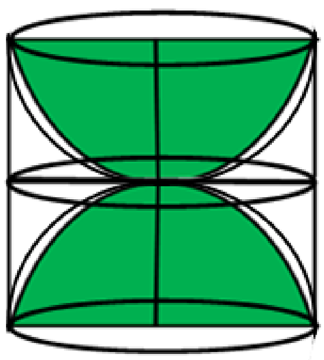


Figure 20: sphere in the sphere

The surface areas of the pseudo sphere and the sphere can be deduced by derivation of their volume with respect to the radius r. Thus, we have the following formulas of the area of these shapes :

1. $\boldsymbol{A_{PSEUDO\ SPHERE} = 4\pi r^2}$
2. $\boldsymbol{A_{SPHERE} = \frac{3}{2}\pi^2 r^2}$

This mathematical deconstruction has brought clearest and deepest methods compatible to natural higher brain cognitive functions rooted in the no man's lands of the world mathematical research mainstream, were $V > \frac{4}{3}\pi r^3$.

6. CONCLUSION

The Deconstruction of the volume of the sphere has been a surgery of one of the most important of Archimedes' works deeply rooted in the formula of the volume of the cone, in the lever technology of weighting geometric figures interacting with the higher intelligence of Classical Greek Geometry.

Containing the genetic material of the Infinitesimal Calculus, our deep desire to construct a simplest deep and a self evident demonstration of the volume of a sphere in the psychology of Archimedean vision (using Analytical Geometry, the Fundamental Theorem of Analysis and Hybrid Rotational Operators), we have been confronted to epistemological obstacles which led to surprising formula: $\boldsymbol{V_{SPHERE} = \frac{\pi^2}{2} r^3}$.

This brainstorming breakthrough has in-depth impacts not only on revision of world scientific and technical university programs of Pure and Applied Mathematics, but also acceleration of advancement of Science, Technology, Engineering, Art and Mathematics with high societal impacts beyond expectations.

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