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ON THE SOLUTIONS OF A LINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract: In this paper, a Modified Adomian Decomposition method (MADM) was introduced and applied to a reduced Fractional differential equation. The fractional derivative is considered in the Caputo sense [6]. The proposed method reduces the equation into an integral equation. Some test problems are considered to demonstrate the accuracy and the convergence of the presented method. Numerical results show's that this approach is easy and accurate when applied to fractional differential equations.

Keywords: Modified Adomian Decomposition method; fractional differential equation.

I. Introduction

In this paper we introduced a modified Adomian decomposition method for the numerical solution of initial value problems of the form

$$D^{\alpha}y(t) = f\big(t, y(t)\big),\ \ y(0) = y_0\,, \qquad 0 < \propto \geq 1 \qquad \dots (1)$$

Fractional calculus has a long history from 30 September 1695, when the derivative of order $\alpha = 1/2$ has been described by Leibniz [7]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Grùnwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [7]. Our main focus is the Caputo definition which turns out to be of great usefulness in real-life applications.

In the past decade, mathematicians have devoted effort to the study of explicit and numerical solutions to linear and nonlinear fraction differential equations [3-4]. An extensive amount of research has been done on fractional calculus, such as [5]. Over the past few years, a number of fractional calculus applications are being used and in the field of science, engineering and economics [6]. Research on linear and non-linear differential equations and linearization techniques has gained quite a momentum due the rapidly proliferating use and recent developments of fractional calculus in these fields [5].

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II. Preliminaries

Definition 1

The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of the function $f \in C\mu, \mu \geq -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-\mu)^{\alpha-1} f(\mu) d\mu, \quad \dots (2)$$

$\alpha > 0, x > 0$

$$J^{\alpha} f(x) = f(x) \quad \dots (3)$$

When we formulate the model of real-world problems with fractional calculus, The Riemann–Liouville have certain disadvantages. Caputo proposed in his work on the theory of viscoelasticity [9] a modified fractional differential operator D_{*}^{α}.

Definition 2

The fraction derivative of $f(x)$ in Caputo sense is defined as

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-\eta)^{m-\alpha-1} f^{(m)}(\eta)\, d\eta$$

For

$$m-1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^{m} \quad \dots (4)$$

Definition 3

The left sided Riemann–Liouville fractional integral operator of the order $\mu^{3}0$ of a function $f \in C_{\mu}$,
$\mu^{3} - 1$, is defined as

$$j^{\alpha} f(x) = \frac{1}{\Gamma(x)} \int_{0}^{1} \frac{f(x)}{(x-t)^{1-\alpha}} dt, \qquad \alpha > 0, x > 0, \quad \dots (5)$$

$$j^{\alpha} f(x) = f(x) \qquad \dots (6)$$

Definition 4

Let $\in C_{-1}^{m}$ $1, m \in N \cup \{0\}$. Then the Caputo fractional derivative of $f(x)$ is defined as

$$D^{\alpha} f(x) = \begin{cases} j^{m-\alpha} f^{m,(x),\ m-1 < \alpha \leq m,\ meN,} \\ \frac{D^m f(x)}{Dx^m}, \alpha = m \end{cases} \qquad \dots (7)$$

Hence, we have the following properties:

(1). $j^{\alpha} j^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu,} \mu > 0$

$$j^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$$

$$j^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{k}(0^{+}) \frac{x^k}{k},\ \ x > 0,\ \ m-1 < \alpha \leq m$$

$$D j^{\alpha} f(x) = f(x), x > 0, m-1 < \alpha \leq m$$

$$DC = 0, C \text{ is the constant}$$

$$\begin{cases} 0,\ \ \beta \in N_0, \quad \beta < [\alpha], \\ D^{\alpha} x^{\beta} = \frac{\tilde{A}(\beta+1)}{\tilde{A}(\beta-\alpha+1) X^{\beta-\alpha}},\ \ \beta \in N_0, \quad \beta \geq [\alpha] \end{cases} \qquad \dots (8)$$

Where [α] denoted the smallest integer greater than or equal to α and $N_0 = \{0,1,2, \dots\}$

Definition 5

An integral equation is an equation in which the unknown function $y(t)$ appears under the integral-signs.

A standard integral equation $y(t)$ is of the form:

$$y(t) = f(t) + \lambda \int_{g(x)}^{h(x)} K(x,t) y(t) dt \qquad \dots (9)$$

Where $g(t)$ and $y(t)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variable x and t called the kernel or the nucleolus of the integral equation. The function $y(t)$ that will be determined appears under the integral sign and also appears inside and outside the integral sign as well. It is to be noted that the limits of integration $g(t)$ and $h(t)$ may be both variables, constants or mixed [9].

III. Fundamentals of The Modified Adomian Decomposition Method

The modified Adomian decomposition method (MADM) provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by splitting the forcing terms into two. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence relation to determine the other approximations [9].

In this paper, we consider the caputo fractional integro-differential equation of the form:

$$D^{\propto} y(x) = f_1(t) + f_2(t) + \lambda \int_0^x K(x,t) u(t) dt, \qquad \dots (10)$$

Where $y(t)$ is the unknown function to be determined, $K(x,t)$is the kernel, and λ is a parameter. The MADM introduces the recurrence relation
Operating with $J^{\propto}$ on both sides of (10), gives

$$y(x) = J^{\propto}(f_1(t) + f_2(t) + \lambda \int_0^x K(x,t)\, y_{n-1}(t) dt), n \geq 1, \qquad \dots (11)$$

Where

$$y_0(x) = J^{\propto}(f_1(t)),$$
$$y_n(x) = J^{\propto}(f_2(t) + \lambda \int_0^x K(x,t)\, y_{n-1}(t) dt), n \geq 1, \qquad \dots (12)$$

IV. Numerical Examples

Problem 1:

Consider the following fractional differential equation:

$$D^{\propto}(y(t)) = \frac{2}{\Gamma(3-\propto)} t^{2-\propto} - \frac{1}{\Gamma(2-\propto)} t^{1-\propto} + y(t) + t^2 - t,$$

$$y(0) = 0, t > 0, where\ 0 < \propto \leq 1 \qquad \ldots(13)$$

With exact solution

$y(t) = t^2 - t$

Applying $J^{1/2}$ to both sides of (13) yields

$$J^{\frac{1}{2}}\left(D^{\propto}(y(t)\right) = J^{\frac{1}{2}}\left(\frac{2}{\Gamma(3-\propto)} t^{2-\propto} - \frac{1}{\Gamma(2-\propto)} t^{1-\propto} + y(t) + t^2 - t\right) \qquad \ldots(14)$$

$$y(t) = (t^2 - t) - J^{\frac{1}{2}} y(t) + J^{\frac{1}{2}}(t^2 - t) \qquad \ldots(15)$$

Now we apply the MADM

$y_0(t) = 0.9999999998\, t^2 \ - 1.000000000\, t$

$y_1(t) = -J^{\frac{1}{2}} y_0(t) + J^{\frac{1}{2}}(t^2 - t) = -1.203604445\, 10^{-10}\, t^{2.5}$

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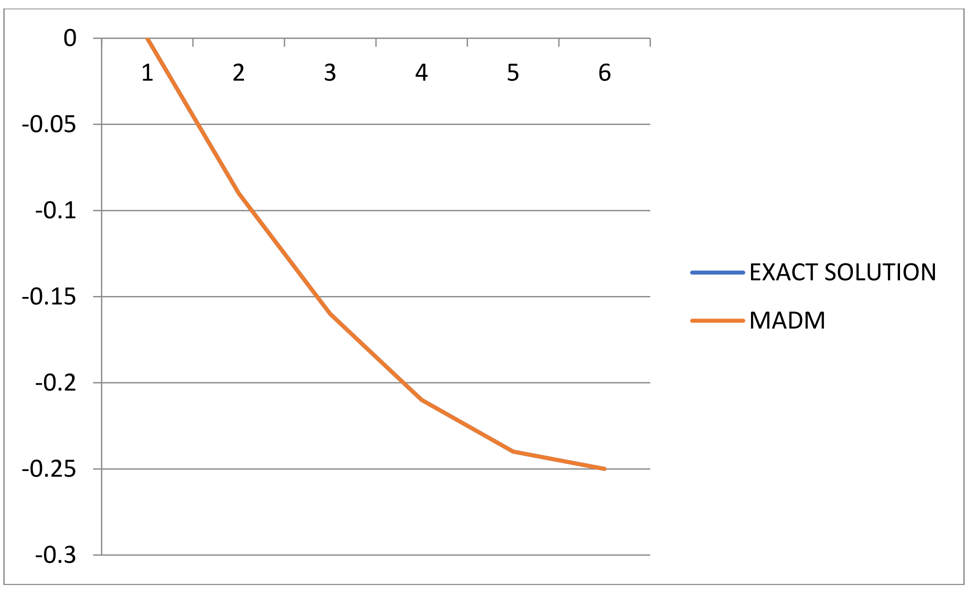
The approximate solution is

$y(t) = t^2 - t - 1.203604445\, 10^{-10}\, t^{2.5}$

Table of Numerical results for Problem 1

t	EXACT SOLUTION	MADM	Abs. Error
0	0	0	0
0.1	-0.09	-0.09	3.81E-13
0.2	-0.16	-0.16	2.15E-12
0.3	-0.21	-0.21	5.93E-12
0.4	-0.24	-0.24	1.22E-11
0.5	-0.25	-0.25	2.13E-11
0.6	-0.24	-0.24	3.36E-11
0.7	-0.21	-0.21	4.93E-11
0.8	-0.16	-0.16	6.89E-11
0.9	-0.09	-0.09	9.25E-11

Graphical Representation of Problem 1:



Problem 2:

Consider the following fractional differential equation

$$D^{\propto}y(t) = -y(t) \ , \qquad y(0) = 1 \ , \quad t > 0 \ , where\ 0 \ < \propto \leq 1. \qquad \dots (16)$$

The exact solution is $y(t) = E_{\propto}(-t^{\propto})$

Applying $J^{1/2}$(Half fractional integral) to both sides of (16) and applying (6) yields

$$y(t) = -J^{1/2}[y(t)] + 1 \qquad \dots (17)$$

Applying (12), yields

$y_0(t) = \ 1$

$y_1(x) = -1.128379167\sqrt{t}$

$y_2(x) = t$

$y_3(x) = -0.7522527782\, t^{3/2}$

$y_4(x) = 0.5000000000\, t^2$

$y_5(x) = -0.2769459142\, t^{5/2}$

Therefore, the series solution after four iterations is

$$y(t) = 1 - 1.128379167\sqrt{t} + t - 0.7522527782\, t^{3/2} \ + 0.5000000000\,(t)^2 - 0.2769459142\,(t)^{5/2}$$

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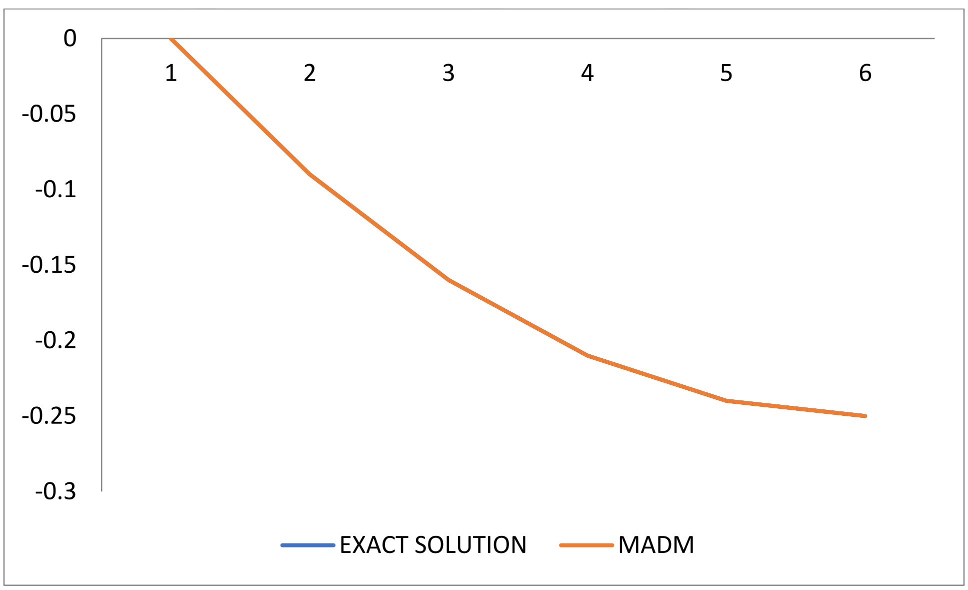
Table of Numerical results for Problem 2

t	EXACT SOLUTION	MADM	Abs. Error
0	1	1	0
0.1	0.723955	0.723511075	0.000444
0.2	0.644094	0.643135803	0.000958
0.3	0.592277	0.589701475	0.002576
0.4	0.553831	0.548018825	0.005812
0.5	0.523355	0.512196335	0.011159
0.6	0.498203	0.47911806	0.019085
0.7	0.476864	0.446826683	0.030037
0.8	0.458394	0.413945628	0.044448
0.9	0.442157	0.379426338	0.062731

Graphical Representation of Problem 2:



V. Conclusion

In this paper, the modified Adomian Decomposition Method (MADM) was applied to a transformed fractional differential equations, the results when compared with other methods yields better results and also in a very good agreement with the exact solution.

VI. Reference

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