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On the construction of a fractional normal distribution and a related fractional Brownian motion

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Abstract
A fractional normal distribution is constructed. With the help of this new distribution a fractional Brownian motion is defined. A generalization of the Ito stochastic integration is given and some stochastic properties are studied. Formula Ito and the product rule are generalized.

Keywords: Fractional normal distribution- Riemann stochastic integrals- Fractional Brownian motion- Fractional product rule- Fractional formula Ito.

Mathematics Subject Classifications: 35A05, 47D60, 47D62, 77D09, 60H05, 60H10, 60G18.

1. Introduction
Let $(\Omega, \mathcal{F}, P)$ be a probability space , R be the set of all real numbers. If a random variable $X:\Omega \to R$ has a density :

$f(x) = \int_0^\infty \frac{1}{\sqrt{2\pi t^\alpha \theta}} \zeta_\alpha(\theta) \exp(-\frac{-(x-m)^2}{2t^\alpha \theta}) d\theta$, we say that X has a fractional Gaussian (or fractional normal) distribution with mean m and variance $\frac{t^\alpha}{\Gamma(\alpha+1)}$, where $\zeta_\alpha(\theta)$ is the stable probability density function, $o < \alpha \leq 1, \Gamma(.)$ is the gamma function , see [1-5].

In this case let us write X is $N_\alpha(m, \frac{t^\alpha}{\Gamma(\alpha)})$ random variable.

A real-valued stochastic process is called α −fractional Brownian if

(i) $W_\alpha(0) = 0$,

(ii) $W_\alpha(t) - w_\alpha(s)$ *is* $N_\alpha(0, \frac{t^\alpha - s^\alpha}{\Gamma(\alpha+1)})$,

(iii) for all times $0 < t_1 < \cdots t_n$ the random variables $W_\alpha(t_1), W_\alpha(t_2) - W_\alpha(t_1), \dots, W_\alpha(t_n) - W_\alpha(t_{n-1})$ are independent.

Notice that $E(W_\alpha(t) = 0 \;\; E(W_\alpha^2(t)) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, where $E(X)$ is the expectation of X.

It is easy to prove that:

$$E\big(W_\alpha(t)W_\alpha(s)\big) = \frac{s^\alpha}{\Gamma(\alpha+1)}, \;\; s \le t.$$

In section 2, we shall study some stochastic Riemann sums. In section 3 , we shall study stochastic integrals related to the considered fractional Brownian motion. In section 4 , we generalize the product rule and the formula of Ito.

2- Stochastic Riemann sums

Let $[a, b]$ be an interval in $[0, \infty)$ and suppose that $\wp = \{a = t_o < t_1 < \cdots < t_m$ is an arbitrary partition of $[a, b]$ with mesh $|\wp| = Max_{0 \le k \le m-1}|t_{k+1}^\alpha - t_k^\alpha|$.

Consider the following sum:

$\sum_{k=0}^{m-1}[W_\alpha(t_{k+1}) - W_\alpha(t_k)]^2$ for any partition $\wp$.

Let us take the limit $|\wp| \to 0$ as $m \to \infty$ in $L^2(\Omega)$, ($L^2(\Omega)$ is the set of all random variables X such that $E(X^2)$ exis).

According to the properties of $W_\alpha(t)$, we get:

$$E((Q_m - \frac{b^\alpha - a^\alpha}{\Gamma(\alpha+1)}))^2 = \sum_{k=0}^{m-1} E((Y_k^2 - 1)^2((t_{k+1}^\alpha - t_k^\alpha)/\Gamma(\alpha+1))^2,$$

Where $Q_m = \sum_{k=0}^{m-1}[W_\alpha(t_{k+1}) - W_\alpha(t_k)]^2$, $Y_k = \frac{W_\alpha(t_{k+1}) - W_\alpha(t_k)}{\sqrt{(t_{k+1}^\alpha - t_k^\alpha)/\Gamma(\alpha)}}$

Notice that Y_k is an $N_\alpha(0,1)$

Thus for some positive constant C we have

$$E((Q_m - (\frac{a^\alpha - b^\alpha}{\Gamma(\alpha)})^2 \le C\,|\wp|\,(b-a) \to 0 \; as \; m \to \infty.$$

Consider now the following stochastic Riemann sum:

$R_m = \sum_{k=0}^{m-1} W_\alpha(t_k)((W_\alpha(t_{k+1}) - W_\alpha(t_k)$, $a = 0$, $b = T$.

It is clear that

$R_m = \frac{W_\alpha^2(T)}{2} - \frac{1}{2}\sum_{k=0}^{m-1}[W_\alpha(t_{k+1}) - W_\alpha(t_k)]^2 \to \frac{W_\alpha^2(T)}{2} - \frac{T^\alpha}{2\Gamma(\alpha+1)}$ as the mesh tends to zero and $m \to \infty$.

We can write:

$\int_0^T W_\alpha \, dW_\alpha = \frac{W_\alpha^2(T)}{2} - \frac{T^\alpha}{2\Gamma(\alpha+1)}$.

We can write

$dW_\alpha{}^2(t) = 2W_\alpha dW_\alpha$ *(comp.[6-11]).*

$+\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt.$

It can be proved also that

$d(tW_\alpha) = tdW_\alpha + W_\alpha dt$.

3- Stochastic α –fractional integrals

Let $\mathcal{L}^2(0,T)$ be the space of all real-valued, progressively measurable stochastic processes G(.) such that $E(\int_0^T G^2 dt) < \infty$.

Denote by S the set of all bounded step processes $\mathcal{L}^2(0,T)$.

Let$G \in S$. Then following Ito, we define $\int_0^T G dW_\alpha$, by

$\int_0^T G dW_\alpha = \sum_{k=0}^{m-1} G_k(W_\alpha(t_{k+1}) - W_\alpha(t_k))$,

Where $G(t) = G_k$ for $t_k \le t_{k+1}$, $k = 0, \dots, m-1$.

The random variables $G_1, \dots, G_m$ are independent of $W_\alpha(t_{k+1}) - W_\alpha(t_k)$, $k = 0, \dots, m-1$.

It is easy to see that for all real constants a, b and for all $G, H \in S$,

$\int_0^T (aG + bH)dW_\alpha = a\int_0^T G dW_\alpha + b\int_0^T H dW_\alpha$,

$E(\int_0^T G dW_\alpha) = 0$.

Let us try to find $E[\{\int_0^T G dW_\alpha\}^2]$.

According to the properties of W_α , G we can prove that

$E[\{\int_0^T G dW_\alpha\}^2] = \sum_{k=o}^{m-1} E(G^2)E(W_\alpha(t_{k+1}) - W_\alpha(t_k))^2) = \frac{1}{\alpha\Gamma(\alpha)}\int_0^T E(G^2)dt^\alpha$.

Also, we can see that

$E(\int_0^T GdW_\alpha \int_0^T HdW_\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^T t^{\alpha-1}GHddt$.

Using the fact that $S\ is\ dense\ in\ \mathcal{L}^2(0,T)$, we can see that all the previous relations are valid for all $G\ , H \epsilon\ \mathcal{L}^2(0,T)$.

4-The fractional product rule

Theorem. (fractional product rule). Suppose

$dX_1 = F_1dt + G_1dW_\alpha\ , dX_2 = dF_2dt + G_2dW_\alpha\ , 0 \le t \le T,$

Where $G_1, G_2 \in \mathcal{L}^2(0,T)$, $F_1\ , F_2 \in \mathcal{L}^1(0,T)$, ($\mathcal{L}^1(0,T)$ is the space of all real-valued, progressively measurable stochastic processes F such that $E[\int_0^T |F|dt] < \infty$).

Then

$d(X_1X_2) = X_2dX_1 + X_1dX_2 + \frac{t^{\alpha-1}}{\Gamma(\alpha)}G_1G_2dt.$

Proof. We can write $X_i(t) = X_i(0) + \int_0^t F_i(r)dr + \int_0^t G_i(r)dW_\alpha(r),\ i = 1,2,\ \ 0 \le t \le T.$

I- Let us first consider the case $X_1(0) = X_2(0) = 0, F_i(t) = F_i, G_i(t) = G_i$, where F_i, G_i are time independent. Then

$X_i(t) = F_it + G_iW_\alpha(t), 0 \le t \le T, i = 1,2.$

Thus

$$\int_0^t \left[X_2(r)dX_1(r) + X_1(r)dX_2(r) + \frac{r^{\alpha-1}}{\Gamma(\alpha)}G_1G_2\right]dr$$
$$= F_1F_2t^2 + (G_1F_2 + G_2F_1)\int_0^t d\big(rW_\alpha(r)\big) + 2G_1G_2\int_0^t W_\alpha(r)dW_\alpha(r) + G_1G_2\frac{t^\alpha}{\Gamma(\alpha+1)}.$$

Using the results in section 2, we get

$$\int_0^t \left[X_2(r)dX_1(r) + X_1(r)dX_2(r) + \frac{r^{\alpha-1}}{\Gamma(\alpha)}G_1G_2\right]dr = X_1(t)X_2(t)$$

This is the fractional product formula for the special case that $X_1(0) = X_2(0) = 0$ and F_i, G_i time independent random variables. The case that $X_1(0) \ne 0, X_2(0) \ne 0$ and F_i, G_i are time independent random variables has a similar proof.

II- If F_i, G_i are step processes, we apply step I on each subinterval (t_k, t_{k+1}) on which F_i and G_i are constant random variables, and add the resulting integral expressions.

III- In the general situation, we select step processes $F_i^n \epsilon \mathcal{L}^1(0,T),\ G_i^n \epsilon \mathcal{L}^2(0,T)$ such that $E[\int_0^T |F_i^n - F_i|dt \to 0,\ \ as\ n \to \infty, E[\int_0^T \{G_i^n - G_i\}^2 dt] \to 0, as\ n \to \infty.$

Define

$$X_i^n(t) = X_i(0) + \int_0^t F_i^n dr + \int_0^t G_i^n dW_\alpha\ , i = 1,2.$$

We apply step II to $X_i^n(.)$ on [0,T] and pass to limits to obtain the required formula.
This completes the proof of the theorem.

Theorem (Fractional Ito formula). Suppose that $X(.)$ has a fractional stochastic differential

$dX = Fdt + GdW_\alpha$, $F \in \mathcal{L}^1(0,T)$, $G \epsilon\ \mathcal{L}^2(0,T)$.

Assume u: $(-\infty,\infty)x[0.T] \to (-\infty,\infty)$ is a continuous such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}$ exist and are continuous.
Set

$$Y(t) = u(X(t),t)$$

Then Y has the fractional stochastic differential

$$dY = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{\partial^2 u}{\partial x^2}G^2dt$$

Proof. I- We start with the special case (x^m) , $m = 0,1,2,\ldots$ and first of all claim that

$$dX^m = mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt$$

This is true for $m = 0,1$ and the case $m = 2$ follows from the fractional product rule.

Now assume the stated formula for $m-1$.

$$dX^{m-1} = (m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt$$

And we prove it for m. Using the fractional product rule, we get

$$dX^m = d(XX^{m-1}) = XdX^{m-1} + X^{m-1}dX + (m-1)X^{m-2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt$$

$$= X\left[(m-1)X^{m-2}dX + \frac{1}{2}(m-1)(m-2)X^{m-3}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt\right] + X^{m-1}dX + (m-1)X^{m-2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt$$

$=mX^{m-1}dX + \frac{1}{2}m(m-1)X^{m-2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt.$

Since fractional Ito formula thus holds for the function $u(x) = x^m$, $m = 1,2,\ldots$ it follows that fractional Ito formula is valid for all polynomials u in the variable x.
II- Suppose now $u(x,t) = f(x)g(t)$, where f *and* g are polynomials. Then

$$d\big(u(x,t)\big) = d\big(f(X)g(t)\big)$$
$$= f(X)dg(t) + g(t)df(X)$$
$$= f(X)g'(t)dt + g(t)f'(X)dX + \frac{1}{2}f''(X)\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt$$
$$= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}G^2dt.$$

Thus fractional Ito formula is valid for all polynomial function u of the variables x,t.

III- Given u as in fractional Ito formula, there exists a sequence of polynomials $\{u_n\}$ such that $u_n \to u$, $\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t}$, $\frac{\partial u_n}{\partial x} \to \frac{\partial u}{\partial x}$, $\frac{\partial^2 u_n}{\partial x^2} \to \frac{\partial^2 u}{\partial x^2}, as\ n \to \infty,$
Uniformly on compact subsets of $(-\infty, \infty)x[0, T]$. With the help of step II, we ncan write

$$u_n(X(t), t) - u_n(X(0), 0) = \int_0^t [\frac{\partial u_n}{\partial r} + \frac{\partial u_n}{\partial x} F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{t^{\alpha-1}}{\Gamma(\alpha)} G^2] dr$$

$+ \int_0^t \frac{\partial u_n}{\partial x} G dW_\alpha$ almost surely.
We may pass to limits as $n \to \infty$ in this expression, thereby proving fractional Ito formula in general.

Conclusion

A fractional normal distribution is defined. With the help of this distribution a fractional Brownian motion is studied. The theory of stochastic integrals is generalized. A new fractional product rule and fractional Ito formula are given.

Conflict of interest

There are no conflicts to declare.

Reference

[1] Mahmoud M. El-Borai , Some probability densities and fundamental solution of fractional evolution equations , Chaos, Soliton and Fractals 14 (2002),433-440.

[2] Mahmoud M. El-Borai, Osama Labib and Hamdy M., Asymptotic stability of some stochastic evolution equations , Applied Math. And comp. 144, 2003, 273-286.

[3] Mahmoud M. El-Borai and Khairia El-Said El-Nadi , Osama Labib , Hamdy M., Volterra equations with fractional stochastic integrals, Mathematical problems in Engineering , 5 , (2004), 453-468.

[4] Mahmoud M. El-Borai and Khairia El-Said El-Nadi , On some fractional parabolic equations driven by fractional Gaussian noise, Special Issue SCIENCE and Mathematics with Applications International Journal of Research and reviews in Applied Sciences 6(3). February 2011, 236- 241.

[5] Mahmoud M. El-Borai, and Khairia El-Said El-Nadi, On some stochastic nonlinear equations and the fractional Brownian motion, Caspian Journal of Computational & Mathematical Engineering, 2017, No.1, 20-33.

[6] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, H. M. Ahmed, H. M. El-Owaidy, A. S. Ghanem & R. Sakthivel, Existence and stability for fractional parabolic integro-partial differential equations with fractional Brownian motion and nonlocal condition, Cogent Mathematics & Statistics, Vol.5, 2018, Issue1.

[7] Hamdy M Ahmed, Mahmoud M El-Borai, Hassan M El-Owaidy, Ahmed S Ghanem, Existence Solutions and Controllability of Sobolev Type Delay Nonlinear Fractional Integro-Differential Systems, Mathematics, MDPI, 2019, 7 , 79;doi:10.3390/math7010079 January 2019, 1-14.

[8] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, The parabolic transform and some singular integral evolution equations, Mathematics and Statistics 8(4), 2020, 410-415.
[9] Mahmoud M. El-Borai and Khairia El-Said El-Nadi, Stochastic fractional models of the diffusion of covid-19, Advances in Mathematics: Scientific Journal 9 (2020), no.12, 10267-10280.

[10] Z. Arab and Mahmoud M. El-Borai, Wellposedness and stability of fractional stochastic nonlinear heat equation in Hilbert space, Fractional Calculus and Applied Analysis, 2022, 25(5), 2020-2039.

[11] Mahmoud M. El-Borai, Khairia El-Said El-Nadi, On the stable probability distributions and some abstract nonlinear fractional integral equations with respect to functions, Turkish Journal of Physiotherapy and Rehabilitations; 32(3), 2022, 32858-32863.

[12] Hamdy M. Ahmed, Mahmoud M. El-Borai, Wagdy El-Sayed and Alaa Elbadrawi, Null Controllability of Hilfer Fractional Stochastic Differential Inclusions, Fractal and Fractional. 2022, 6, 721.

[13] A. G. Malliaris, Ito Calculus in fractional decision making, SIAM Review 25 (1983), 481-496.

[14] R. Paley, N. Wiener and A. Zigmond, Notes on random functions, Math. Z. 37 (1959), 647-668.

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