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OPTIMIZATION METHODS IN THE RECONSTRUCTION OF SPARSE SIGNAL.

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Abstract

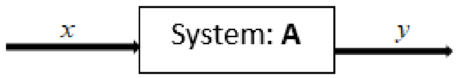
In this paper, sparse signal problem is defined and formulated. Constrained optimization and unconstrained optimization problems that solve sparse signal problem are also modeled. The main objective of the paper is to reconstruct sparse signal that was corrupted by a white noise. A white Gaussian noise was added to the original signal for an observed sparse signal and the split Bregman iterations is an optimization method for regularization of inverse problem used to solve and recover the sparse signal which is closer to the original signal. This is obtained by varying the regularization parameters and choose the best one to reconstruct the signal as required comparatively to the original signal.

KeyWords
Inverse problem, sparse signal, convex optimization, split Bregman, reconstruction of signal.

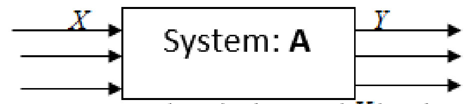
Introduction
Sparse signals were studied recently as shown by different authors. The problem solved by the sparse representation is to be search for the most compact representation of a signal in terms of linear combination of atoms in an over complete dictionary [1].The sparse signal representation is well applied where the original signal needs to be reconstructed as accurately as possible, such as denoising, image inpainting and coding [1]. Finding sparse solutions to undetermined linear systems may become better-behaved and be a much more practical [2]. The insight has been developing in signal and image processing, where it has been found that many media types (still imagery, video, acoustic,…) can be sparsely represented using transform-domain methods and many important tasks dealing with such medial can fruitfully be viewed as finding sparse solutions to undetermined systems of linear equations [2]. Different methods were applied to reconstruct a sparse signal such as: principal component analysis ([1] [3]), independent component analysis ([1] [4]), transform-domain [2] and split Bregman iterations ([5] [6] [7]). This last method is used in this paper as one of optimization methods to solve sparse signal problem.

Signal system representation

As defined in [8], a system is a mathematical model of a physical process that relates the input signal to the output or response signal. If x and y are the input and output signals respectively of a system, the system is a transformation of x into y represented mathematically as $y = Ax$, where A is the operator representing some well-defined rule by which x is transformed into y as indicated by the figures below. See [8].



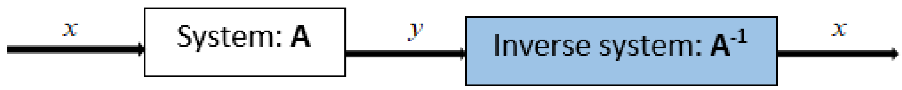
A system with single input (x) and output (y) signals.



A system with multiple input (X) and output (Y) signals.

Inverse system and inverse problem

A special class of systems of great importance consists of systems having feedback. In a feedback system, the output signal is fed back and added to the input to the system. That is an inverse problem, See [8].



From [7]and [9], an inverse problem in science is the process of calculating from a set of observations the causal factors that produced them. Inverse problems are the opposites of direct problems. In a direct problem, one finds an effect from a cause and in an inverse problem one is given the effect and want to recover the cause.

That is: $x \approx A^{-1} y$.

An inverse problem is said to be well-posed if the following conditions are satisfied: (a) Existence of solution, (b) Uniqueness of solution, and (c) Stability where the solution's behavior changed continuously with the initial conditions. See the details in [9].

Sparse signals processing and Problem formulation.

The resolution of the signal problem needs the notions of different norms such as ℓ_0, ℓ_1, ℓ_2 and ℓ_p in general for $p > 0$ [7]. The ℓ_0 norm as defined in [2] is as follows:

$$\|x\|_0 = \lim_{p\to 0} \|x\|_p^p = \lim_{p\to 0} \sum_{k=1}^{m} |x_k|^p , \tag{1}$$

where $\forall i, x_i \neq 0$. This is to count the number of nonzero entries or components of the vector x in the measurement of sparsity of x.

The problem of sparse representation is an inverse problem aimed to find an unknown sparse $\bar{x} \in \Re^{m\times 1}$ such that $y = Ax$ and $\|x\|_0$ is minimized with $y \in \Re^n$ representing a set of n linear projections of $\bar{x}$. That is

$$\begin{aligned} &\bar{x} = \arg\min_x \|x\|_0 \\ &s.t \;\; y = Ax \end{aligned} \tag{2}$$

A practical alternative is to solve the ℓ_1 norm optimization problem of the form

$$\min_x \|x\|_1 \quad s.t \;\; y = Ax \tag{3}$$

or equivalently,

$$\min_x \|x\|_1 + \frac{\lambda}{2}\|y - Ax\|_2^2 \tag{4}$$

where, y is the vector signal values and A is the matrix whose columns are the elements of the different bases to be used in the representation [2].

The problem (2) posed offers literally the sparsest representation of the signal content [2]. The use of sparsity in signal processing leads to solve the minimization problem:

$$\arg\min_x \frac{1}{2}\|y - Ax\|_2^2 + \lambda\|x\|_1 \tag{5}$$

where, $\lambda > 0$ is a regularization parameter that balances the trade-off between reconstruction error and sparsity. The signal $y \in \Re^n$ is assumed to be generated by the model

$$y = Ax + \eta$$

where, $x \in \Re^{m\times 1}$, $A \in \Re^{n\times m}$ and η is the noise model, and $\|x\|_2^2 = \sum_{k=1}^{n}|x_k|^2$ and $\|x\|_1 = \sum_{k=1}^{n}|x_k|$, denote the energy of x and the ℓ_1 norm of x respectively [2].

In sparse signal processing, the general problem is to find the value of x that satisfies

$$\arg\min_x \frac{1}{2}\|y - Ax\|_2^2 + \sum_{i=1}^{n}\phi(x_i) \tag{6}$$

where, the function $\phi(\cdot)$ is a penalty function or regularization function. If the function $\phi(x) = \lambda|x|$ which is a convex function, then the problem (6) becomes automatically the problem (5) for which we need to solve in sparse signal processing, see [10]. The convex functions are preferable for regularization of inverse problem since they are more reliable to be minimized than the non-convex functions.

Split Bregman iterations

The split Bregman algorithms that are detailed in [6] are experimented numerically in different applications but only sparse signal reconstruction is considered in this paper. Consider the generalized constrained optimization problem:

$$\min_u E(u) \quad s.t \;\; Au = b \tag{7}$$

where, E is a convex functional and $A : \Re^n \to \Re^m$ a linear function. The corresponding unconstrained optimization problem is

$$\min_u E(u) + \frac{\lambda}{2}\|Au - b\|_2^2, \tag{8}$$

where, λ is a penalty function weight.

The split Bregman iterations in [6] to solve the constrained optimization problem (7) when A is linear with the corresponding unconstrained optimization problem (8) need the Bregman iteration:

$$u^{k+1} = \min_u E(u) + \frac{\lambda}{2}\left\|Au - b^k\right\|_2^2$$
$$b^{k+1} = b^k + b - Au^k, \tag{9}$$

and we have the convergence in the 2-norm sense: $\lim_{k\to\infty} Au^k = b.$

Consider the general L_1-regularized optimization problem:

$$\min_u |\Phi(u)| + H(u), \tag{10}$$

where $|\cdot|$ denotes the L_1-norm and both $|\Phi(\cdot)|$ and $H(\cdot)$ are convex functions and assume that $|\Phi(\cdot)|$ is differentiable. Consider also the constrained optimization problem

$$\begin{aligned} &\min_{u,d} |d| + H(u) \\ &s.t \;\; d = \Phi(u), \end{aligned} \tag{11}$$

with its corresponding unconstrained problem

$$\min_{u,d} |d| + H(u) + \frac{\lambda}{2}\|d - \Phi(u)\|_2^2 .$$

Let $E(u,d) = |d| + H(u)$ and define $A(u,d) = d - \Phi(u)$, then with the above Bregman formulation in (9) we get the simplified split Bregman iteration in two-phase algorithm:

$$\begin{aligned} &\left(u^{k+1}, d^{k+1}\right) = \min_{u,d} |d| + H(u) + \frac{\lambda}{2}\left\|d - \Phi(u) - b^k\right\|_2^2, \\ &b^{k+1} = b^k + \Phi(u^{k+1}) - d^{k+1}. \end{aligned} \tag{12}$$

This is a reduced sequence of unconstrained optimization problems and Bregman updates of the L_1-Regularized problem (10), see [6]. Solving the first equation of above algorithm by iteratively minimizing with respect to u and d separately decoupling L_1 and L_2 components:

Step 1: $u^{k+1} = \min_u H(u) + \frac{\lambda}{2}\left\|d - \Phi(u) - b^k\right\|_2^2,$

Step 2: $d^{k+1} = \min_d |d| + \frac{\lambda}{2}\left\|d - \Phi(u^{k+1}) - b^k\right\|_2^2 .$

To solve the Step 1, it is possible to use any convenient optimization technique such as Fourier transform method. In Step 2, it is possible to find d explicitly using Shrinkage operator

$$d_j^{k+1} = Shrink\left(\Phi(u_j) + b_j^k, \frac{1}{\lambda}\right),$$

where, $Shrink(x,\gamma) = \frac{x}{|x|} * \max\left(|x| - \gamma, 0\right)$.

To implement the algorithm above in (12), we have the following algorithm as shown in [6].

Algorithm 1 : Generalized Split Bregman Algorithm.

Input : f, d^0 *and* b^0

Output : u

while $\left\|u^k - u^{k-1}\right\|_2 > tol$ do

$$\begin{aligned} &u^{k+1} = \min_u H(u) + \frac{\lambda}{2}\left\|d^k - \Phi(u) - b^k\right\|_2^2; \\ &d^{k+1} = \min_d |d| + \frac{\lambda}{2}\left\|d - \Phi(u^{k+1}) - b^k\right\|_2^2; \\ &b^{k+1} = b^k + \Phi(u^{k+1}) - d^{k+1}. \end{aligned}$$

end

Split Bregman algorithm for Sparse Signal Reconstruction.

As shown in [7] and [5], the sparse signal x is recovered from the observed signal $y = Ax + \eta$ with a known linear operator A and this can be achieved by solving the minimization problem in (4) to get

$$\hat{x} = \arg\min_x \|x\|_1 + \frac{\lambda}{2}\|y - Ax\|_2^2.$$

By the split Bregman iteration and solving for $x^{k+1}, d^{k+1}, b^{k+1}$ respectively, we have

$$x^{k+1} = \arg\min \frac{\lambda}{2}\|Ax - y\|_2^2 + \frac{\alpha}{2}\|d^k - x - b^k\|_2^2.$$

Then solving we get

$$x^{k+1} = \left(\lambda A^T A + \alpha I\right)^{-1}\left(\lambda A^T y + \alpha\left(d^k - b^k\right)\right), \text{ and}$$

$$d^{k+1} = \arg\min \frac{\lambda}{2}\|d\|_1 + \frac{\alpha}{2}\|d - x^{k+1} - b^k\|_2^2,$$

using the Shrinkage operator formula

$$d^{k+1} = Shrink\left(x^{k+1} + b^k, \frac{1}{\alpha}\right), \text{ and } b^{k+1} = b^k + x^{k+1} - d^{k+1}.$$

Therefore, the split Bregman algorithm to implement the results as shown in [7] and [5] is as follows:

Algorithm 2 : Split Bregman Sparse Signal recovering algorithm.

Input : $d^0 = 0$ *and* $b^0 = 0$

Output : x

while $\|x^k - x^{k-1}\|_2 > tol$ do

$x^{k+1} = \left(\lambda A^T A + \alpha I\right)^{-1}\left(\lambda A^T y + \alpha(d^k - b^k)\right);$

$d^{k+1} = Shrink\left(x^{k+1} + b^k, \frac{1}{\alpha}\right);$

$b^{k+1} = b^k + x^{k+1} - d^{k+1}.$

end

Numerical Experiment

As shown in [7], a numerical experiment for sparse signal reconstruction is made by presenting an example of an original sparse signal to be reconstructed by creating the spike signal as the original signal with length $N = 150$ and it is presented in figure 1 below. The 5-point impulse response or point spread function (PSF) f is chosen to be a uniform noise model of the form:

$$f(t) = \begin{cases} \frac{1}{L} & if \quad \frac{-L}{2} \le t \le \frac{L}{2} \\ 0, & otherwise \end{cases}$$

with the length of this PSF to be used $L = 5$. The additive zero-mean $\mu = 0$ Gaussian noise is created with standard deviation $\sigma = 0.03$ and it is added to the convolution of original signal x and impulse response to get the observed signal y and it is presented in figure 2 below.

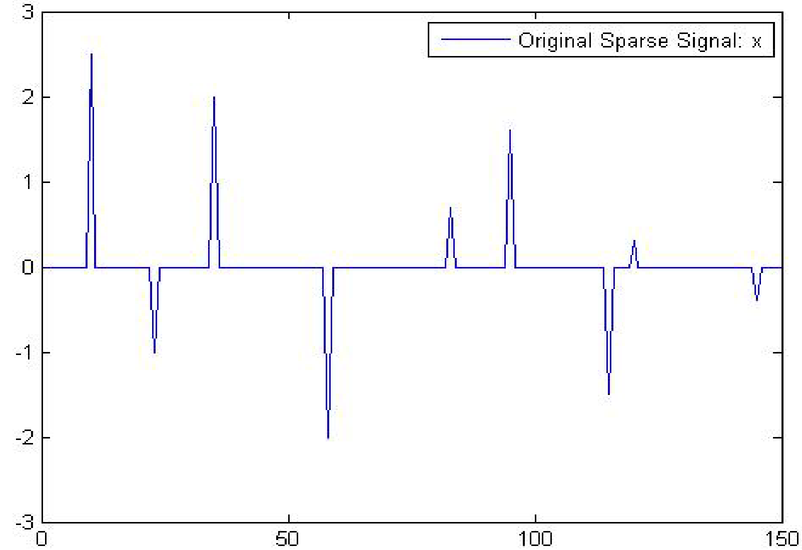


Figure 1: Spike signal as the original sparse signal.

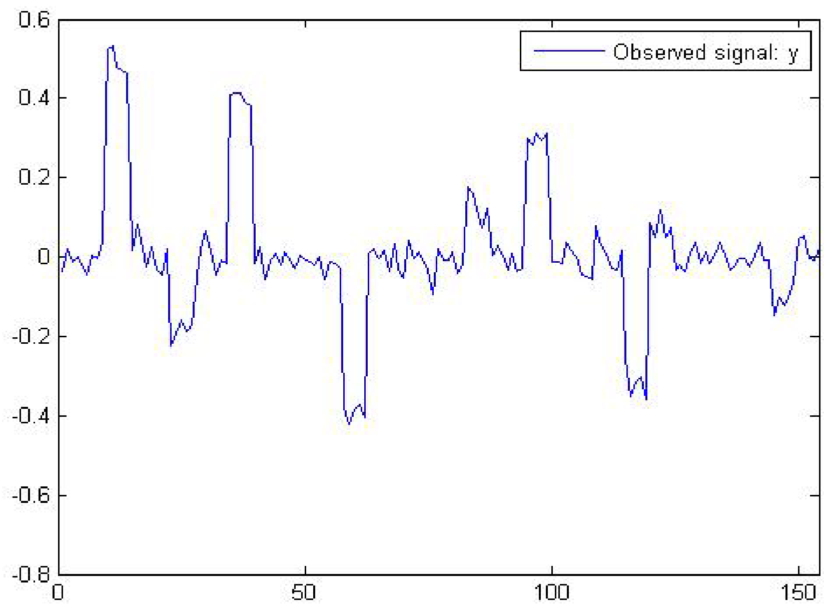


Figure 2: Observed sparse signal.

The regularization parameter λ varied for different values $\lambda = 5, \lambda = 15, \lambda = 20, \lambda = 30$ as shown in figure 3. It is observed that for $\lambda = 30$ with splitting regularization parameter $\alpha = 1$, the reconstructed sparse signal is quite similar to the original signal as required and 20 iterations are enough for a good reconstruction of the signal with tolerance error 0.0021.

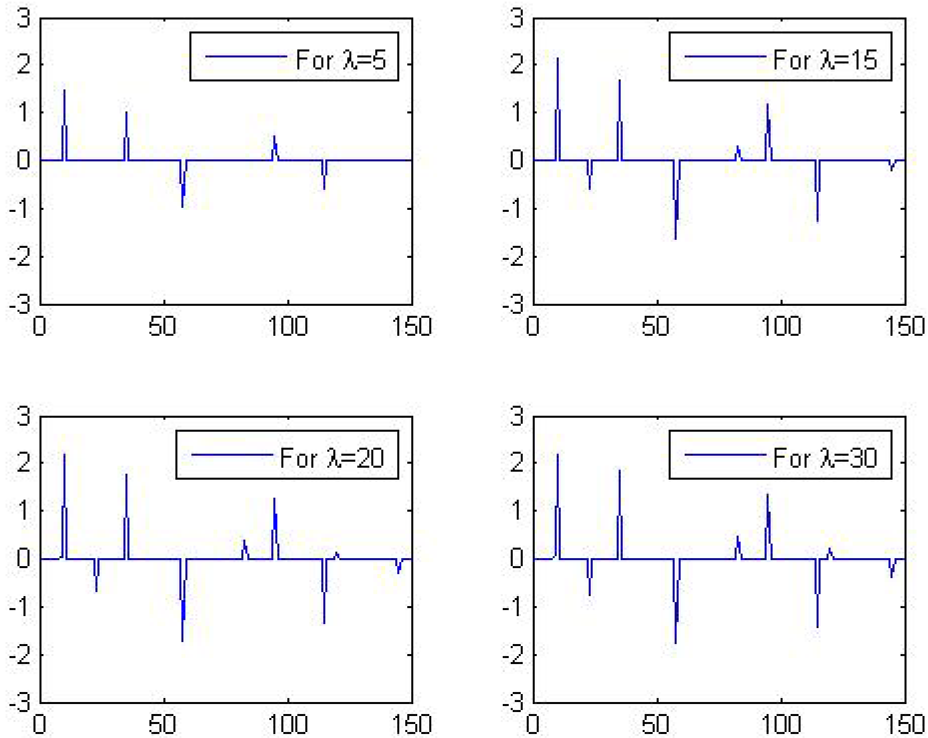


Figure 3: The reconstructed sparse signals with different values of regularization parameter.

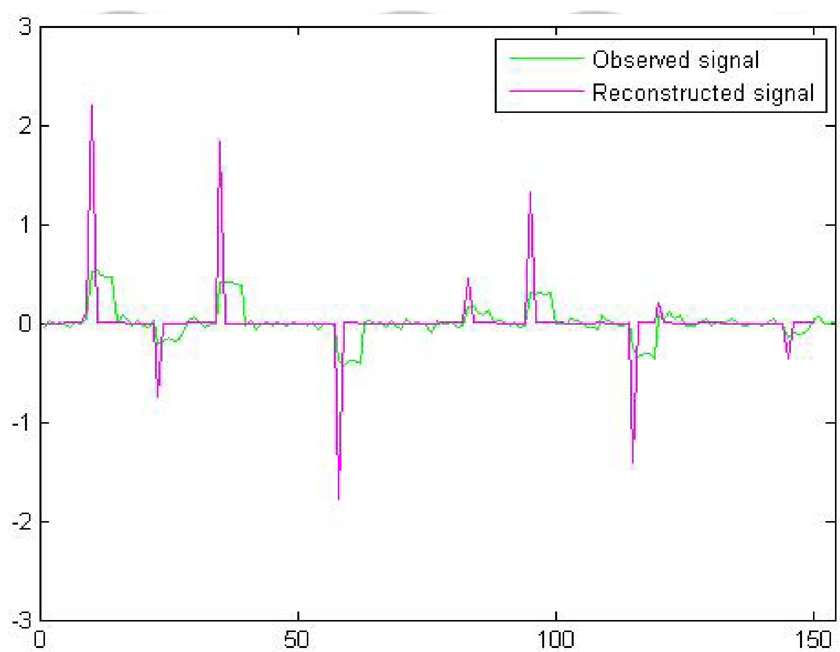


Figure 4: The comparison of the observed sparse signal and the reconstructed sparse signal.

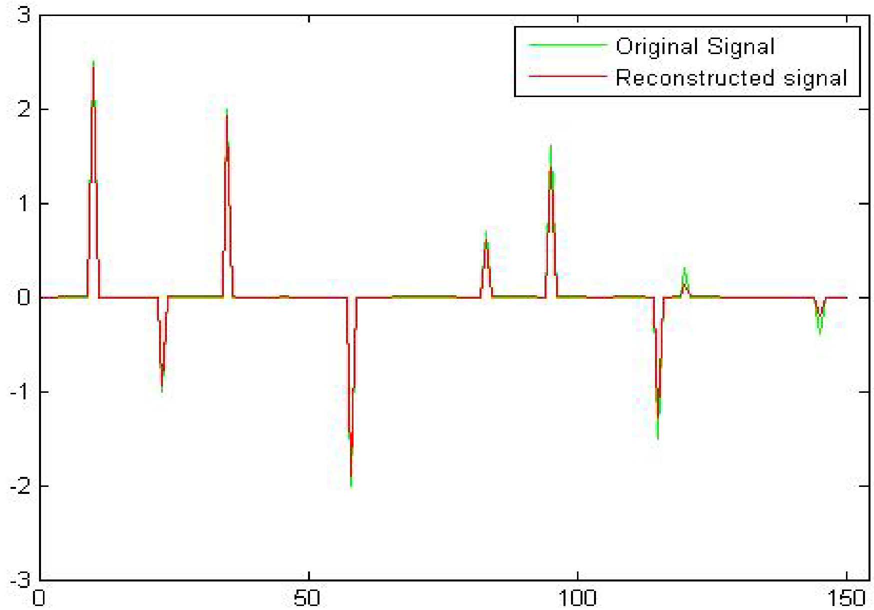


Figure 5: The comparison of the original sparse signal and the reconstructed sparse signal.

This figure (figure 5) shows the fitness of the original sparse signal and the reconstructed sparse signal, where the good similarity of the original signal and reconstructed signal for $\lambda = 30$ is observed.

Conclusion

The sparse signal problem defined and formulated in this paper is the main problem targeted to be solved. The split Bregman iterations was used as an optimization technique to solve sparse signal problem for the sparse signal recovering. From the previous studies, split Bregman was designed for L_1-regularized optimization problems applied in image denoising, and this method was applied in this paper for reconstruction of sparse signal as it is also good for it. The observed signal was reconstructed by varying the regularization parameter and the targeted best estimated reconstructed signal was obtained comparatively to the original signal targeted. The split Bregman is observed as a good method that solves L_1-regularized optimization problems because it is quietly quick to converge with a very minimum tolerance error and it is easy to code when solving problems.

References

[1] K. Huang and S. Aviyente, "Sparse representation for Signal Classification," *Department of Electrical and Computer Engineering. Michigan State University, East Lansing, MI 48824,* 2006.

[2] M. Elad, Sparse and Redundant Representations, From Theory to Applications in Signal and Image Processing, New York, USA: Springer Science+Business Media, LLC, 2010.

[3] M. Mudrová and A. Procházka, "Principal Component Analysis in Image Processing," 2005.

[4] A. Tharwat, "Independent component analysis: An introduction," *Appied Computing and Informatics,* pp. 222-249, 2021.

[5] J. Gilles, "The Bregman CookBook," 2011.

[6] T. Goldstein and S. Osher, "The Split Bregman Method for L1-Regularized Problems," *SIAM J.*

IMAGING SCIENCES, pp. 323-343, 2009.

[7] I. Viateur, Analysis of Some Optimization Techniques for Regularization of Inverse Problems, University of Rwanda, 2016.

[8] P. H. P. Hsu, SCHAUM'S OUTLINES OF Theory and Problems of Signals and Systems, USA: McGraw-Hill Companies, Inc., 1995.

[9] S. Siltanem, "MAT-52506 Inverse Problems, Version 12," 2009.

[10] I. Selesnick, "Penalty and Shrinkage Functions for Sparse Signal Processing," 2013.