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Semigroup with Bi-ideals

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Abstract: In this paper we introduce the strongly prime, semi prime, strongly irreducible and irreducible bi-ideals of semi groups. The space of strongly prime bi ideals is topologies,the notation of (m, n) ideal of semi groups ,(m, n) bi ideals and generalized (m, n) bi ideals and that of generalized (m, n) *- bi ideals. We also characterized these semi groups for which each bi-ideal is strongly prime.

Introduction: Semi group is anon empty set S endowed with a binary operation '.'Such that '.'Is associative.A non empty sub set A of a semi group $(S, .)$ is called a sub semi group of S if $ab \in A$ for all a, b in A.A sub semi group B of semi group S is called a bi ideal of S if $BSB \subseteq B$. An element a of of a semigroup S is called a regular element if there exists x in S such that$axa = a$. A semi group Sis called regular if every element of S is regular. An element a of a semi group S is called intra regular if there exist elements x and y in S such that$a = xa^2y$.

Definition.1:A bi- ideal B of a semi group S is called a prime (strongly prime) bi- ideal if $B_1B_2 \subseteq B(B_1B_2 \cap B_2B_1 \subseteq B)$implies$B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi ideal B of a semi group S is called a semi prime bi ideal if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi ideal B_1 of S.

Note: every strongly prime bi ideal of a semi group S is a prime bi ideal and every prime bi ideal is a semi prime bi ideal. A prime bi ideal is not necessary strongly prime and a semiprime bi-ideal is not necessary prime.

Example: Consider the following semi group$S = \{0, a, b\}$.

.	0	a	b

0	0	0	0
a	0	a	a
b	0	b	b

It is evident that S is both regular and intra – regular. the bi ideals of S are $\{0\}, \{0, a\}, \{0, b\}$ and $\{0, a, b\}$ all bi ideals are prime bi ideals and hence semi prime bi- ideals. However, the prime bi ideal $\{0\}$ is not strongly prime bi –ideal because

$$\{0, a\}\{0, b\} \cap \{0, a\}\{0, b\} = \{0, a\}\{0, b\} = \{0\} \subseteq \{0\}$$

But neither $\{0, a\}$ or $\{0, b\}$ is contained in $\{0\}$.

Definition.2:A bi ideal B of a semi group S is called irreducible (strongly irreducible) bi- ideal if $B_1 \cap B_2 = B(B_1 \cap B_2 \subseteq B)$ implies $B_1 = B$ or $B_2 = B(B_1 \subseteq B \; or \; B_2 \subseteq B)$

Every strongly irreducible bi- ideal of a semi group is an irreducible bi- ideal but the converse is not true. The following example illustrates that an irreducible bi-ideal of a semi group may not be a strongly irreducible bi-ideal.

Example Consider the following semi group$S = \{0,1,2,3,4,5\}$.

.	**0**	**1**	**2**	**3**	**4**	**5**
0	**0**	**0**	**0**	**0**	**0**	**0**
1	**0**	**1**	**1**	**1**	**1**	**1**
2	**0**	**1**	**2**	**3**	**1**	**1**
3	**0**	**1**	**1**	**1**	**2**	**3**
4	**0**	**1**	**4**	**5**	**1**	**1**
5	**0**	**1**	**1**	**1**	**4**	**5**

Bi ideal in S are

$\{0\}, \{0,1\}, \{012\}, \{0,1,2, \}, \{0,1,3\}, \{0,1,4\}, \{0,1,5\}, \{0,1,2,4\}, \{0,1,3,5\}, \{0,1,2,3\}, \{0,1,4,5\}$,and$S$.

Irreducible bi ideals are :$\{0\}, \{0,1,2,4\}, \{0,1,3,5\}, \{0,1,2,3\}, \{0,1,4,5\}$, and S.

Strongly irreducible bi ideal is$\{0\}$.

Theorem.3: Every strongly irreducible, semi prime bi ideal of bi ideal of a semi group S is a strongly prime bi ideal.

Proof: Let B be a strongly irreducible semi prime bi ideal of S. letB_1, B_2 be any two bi ideals of S such that$B_1B_2 \cap B_2B_1 \subseteq B$. Since $(B_1 \cap B_2)^2 \subseteq B_1B_2$and

$(B_1 \cap B_2)^2 \subseteq B_2B_1, (B_1 \cap B_2)^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B$.

Since B is semi prime bi ideal,$B_1 \cap B_2 \subseteq B$. Because B is a strongly irreducible bi ideal of S, so either $B_1 \subseteq B$ or$B_2 \subseteq B$. Thus B is a strongly prime bi ideal of S.

Theorem. 4: Let B be a bi ideal of a semi group S and $a \in S$ such that $a \notin B$.then there exists an irreducible bi ideal I of S such that $B \subseteq I$and $a \notin I$.

Proof: Let A be the collection of all bi ideals of$S$ which contain B and do not contain$a$. Then A is non- empty, because$B \in A$. The collection A is a partially ordered set under inclusion. If C is any totally ordered sub set of A then $\cup C$ is bi ideal of S containing B.hence by **Zorn's lemma**there exists a maximal element I in A. We show that I is an irreducible bi ideal. Let C and D be two bi ideals of S such that $I = C \cap D$. If Both C and D properly contain I then $a \in C$and $a \in D$. Hence $a \in C \cap D = I$. This contradicts the fact that $a \in I$. Thus $I = C$or $I = D$.

Theorem.5: For a semi group$S$, the following assertions are equivalent:

i. S is both regular and intra- regular.
ii. $B^2 = B$for every bi- ideal B of S.
iii. $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$for all bi ideals B_1 and B_2 of S.
iv. Each bi –ideal of S is semi prime.
v. Each proper bi – ideal of S is the intersection of irreducible semi prime bi- ideals of Swhich contains it.

Proof $(ii) \rightarrow (iii)$Let B_1 and B_2 be any two bi – ideals of semi group S.then by our hypothesis

$$B_1 \cap B_2 = (B_1 \cap B_2)\hat{}2$$
$$= (B_1 \cap B_2)(B_1 \cap B_2)$$
$$\subseteq B_1B_2$$

Similarly

$$B_1 \cap B_2 \subseteq B_2B_1$$

Thus

$$B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(*)$$

Now B_1B_2 and B_2B_1 are bi- ideals being the products of bi- ideals. Also $B_1B_2 \cap B_2B_1$ is a bi-ideal. Thus

$$B_1B_2 \cap B_2B_1 = (B_1B_2 \cap B_2B_1)(B_1B_2 \cap B_2B_1)$$

$$\subseteq B_1B_2.B_2B_1$$

$$\subseteq B_1SB_1$$

$$\subseteq B_1$$

Similarly

$$B_1B_2 \cap B_2B_1 \subseteq B_2$$

Thus

$$B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(**)$$

Hence by (*) and (**)

$$B_1 \cap B_2 = B_1B_2 \cap B_2B_1.$$

$(iii \rightarrow iv)$letB_1 and B be bi-ideals of Ssuch that $B_1^2 \subseteq B$. By our hypothesis

$$B_1 = B_1 \cap B_1 = B_1B_1 \cap B_1B_1 = B_1^2$$

Thus

$$B_1 \subseteq B$$

Hence every bi – ideal of S is semi prime.

$(iv) \rightarrow (v)$let$B$ be a proper bi- ideal of S. Then B is contained in the intersection of all irreducible bi – ideals of S which containing$B$. By theorem (3) guarantees the existence of such irreducible bi- ideals. If $a \notin B$ then there existsall irreducible bi – ideal of S which contains B but does not contain$a$. Hence B is the intersection of all bi ideals of S which contains it. By our hypothesis every bi ideal is semi prime, and so each bi ideal is the intersection of irreducible semi prime bi ideals of S containing it.

$(v) \rightarrow (ii)$let$B$ be a bi –ideal of S. If $B^2 = S$ then clearly B is idempotent, that is, $B^2 = B$. If $B^2 \neq S$, then B^2 is a proper bi ideal of S containing B^2 and so by our hypothesis.

$$B^2 = \bigcap_{\alpha} \{B_\alpha : B_\alpha \, is \; irreducible \; semi \; prime \; bi \; ideal \; of \; S\}$$

Since each B_α is a semi prime bi-ideal, $B \subseteq B_\alpha$,for all α and so $B \bigcap_\alpha B_\alpha = B^2$.

Hence each bi ideal in S is idempotent.

Theorem.6: Each bi – ideal of a semi group S is strongly prime if and only if S is regular, intra regular, and the set of bi ideals of S is strongly ordered by inclusion.

Proof : Suppose that each bi- ideal of S is strongly prime. Then each bi ideal of S is semi prime. Thus by Theorem (4),S is both regular and intra regular. We show that the set of bi ideals of S is totally ordered. Let B_1 and B_2 be any two bi ideals of S. Then by Theorem (4) $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$. As Such bi ideal is strongly prime. $B_1 \cap B_2$is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$. If $B_2 \subseteq B_1 \cap B_2$then $B_2 \subseteq B_1$.

Conversely assume that S is regular, intra regular and the set of bi ideals of S is totally ordered under inclusion. Then we want to show that each bi ideal of S is strongly prime. Let B be an arbitrarily bi ideal of S and B_1 and B_2 bi ideals of S such that

$$B_1B_2 \cap B_2B_1 \subseteq B$$

Since S is both regular and intra regular, by theorem (4)

$$B_1 \cap B_2 = B_1B_2 \cap B_2B_1$$

Also

$$B_1B_2 \cap B_2B_1 \subseteq B$$

Implies

$$B_1 \cap B_2 \subseteq B \; .$$

Since the set of bi ideals of S is totally ordered, so either $B_1 \cap B_2 = B_1$or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$.hence B is strongly prime.

Theorem.7: If the set of bi ideals of a semi group S is totally ordered, then S is bothregular and intra regular if and only if each bi ideal of S is prime.

Proof: Suppose that S is both regular and intra regular. Let B be any bi ideal of $S$and B_1,B_2 be bi ideals of S such that $B_1B_2 \subseteq B$. Since the set of bi ideals of S is totally ordered, either $B_1 \subseteq B_2$or $B_2 \subseteq B_1$. Suppose that $B_1 \subseteq B_2$then $B_1^2 \subseteq B_1B_2 \subseteq B$ by theorem (4) B is semi prime.So $B_1 \subseteq B$. Hence B is a prime bi ideal of S.

Conversely, assume that every bi ideal of S is prime. Since the set of bi ideals of S is totally ordered so the concepts of prime and strongly prime coincide. Now by theorem (5), we see that S is both regular and intra regular.

Theorem.8: For a semi group S the following assertions are equivalent:

i. The set of bi ideals of S is totally ordered under inclusion.
ii. Each bi ideal of S is strongly irreducible.
iii. Each bi ideal of S is irreducible.

Proof: $(i) \Longrightarrow (\boldsymbol{ii})$let B an arbitrary bi ideal of $S$and B_1, B_2 two bi ideals of S such that $B_1 \cap B_2 \subseteq B_2$. Since the set of bi ideals of S is totally ordered, either $B_1 \subseteq B_2$or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$or $B_2 \subseteq B$. This shows that B is strongly irreducible.

$(ii) \Longrightarrow (iii)letB$ be an arbitrarly bi ideal of S and B_1 ,B_2 two bi ideals of S such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$and $B \subseteq B_2$. By hypothesis rather $B_1 \subseteq B$or $B_2 \subseteq B$. Hence either $B_1 = B$or $B_2 = B$. That is B is irreducible bi ideal.

$(iii) \Longrightarrow (\boldsymbol{i})LetB_1$ and B_2 be any two bi ideals of S.then $B_1 \cap B_2$ is bi ideal of S.also $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$that is either $B_1 \subseteq B_2$or $B_2 \subseteq B_1$. Hence the set of bi ideals of S is totally ordered.

Definition.9: let S be a semi group and A be be anon empty sub set of$S$ thn A is said to be a generalized $(\boldsymbol{m}, \boldsymbol{n})$ bi ideal of S if $A^mSA^n \subseteq A$. Where m, n are arbitrarily non negative integers and the powerA^m is collected when$m = 0$. Here $m = n = 1$ then A is called generalized bi ideal of S. If A is a sub semi group of S then A is called an (m, n) bi ideal of S. If $m = n = 1$ then it is called a bi- ideal. If $A \in \overline{G(S)}$ then A is called semi prime if $\forall K \in \overline{G(S)}$, $K^2 \subseteq A \Longrightarrow K \subseteq A$. $A$is said to be prime if $\forall K, L \in \overline{G(S)}$,$KL \subseteq A \Longrightarrow K \subseteq A$ or $L \subseteq A$. A is said to be strongly prime

$\forall K, L \in \overline{G(S)}, \Theta \neq \mathrm{KL} \cap \mathrm{LK} \subseteq \mathrm{A} \Longrightarrow \mathrm{K} \subseteq \mathrm{A}$ or $L \subseteq A$. The set of semi prime, prime and strongly prime generalized (m,n) bi ideal of S are denoted by $\overline{SePG(S)}, \overline{PG(S)}and\overline{StPG(S)}$.

Theorem.10:let S be a semi group then the following assertions are equivalent

i. $\boldsymbol{A^2} = A$For every choice of generalized bi ideal A from $\overline{G(S)}$of S.
ii. $K \cap L = KL \cap LK$For all bi ideals $K, L$of S.
iii. Each collection of generalized bi ideal $\overline{G(S)}$ of S is semi prime.

Proof.$(i) \Longrightarrow (\boldsymbol{ii})letK$ and L be any two generalized bi ideals of semi group S.then the our first hypothesis we have $K \cap L = (K \cap L)^2 = (K \cap L)(K \cap L) \subseteq KL$

In the similar way one can show that $K \cap L \subseteq LK$. Thus $(K \cap L) \subseteq KL \cap LK$. Hence both KL and LK are generalayized bi ideal as they are product of two generalayzed bi ideals. Moreover $KL \cap LK$ is also generalazied bi ideals. Thereore we have

$$KL \cap LK = (KL \cap LK)(KL \cap LK)$$

$$\subseteq KL.LK$$

$$\subseteq KSK \subseteq K$$

Dually

$$KL \cap LK \subseteq L$$

Therefore$KL \cap LK \subseteq L \cap K$.Hence,$K \cap L = KL \cap LK$. This proves our second assertion.

$(ii) \Longrightarrow (\boldsymbol{iii})$To show this assertion we let that$K$ and A be any two generalized bi ideal of S from the collection $\overline{G(S)}$ such that $K^2 \subseteq A$. Therefore by hypothesis we have $K = K \cap K = KK \cap KK = K^2$ and hence $K \subseteq A$. Thus it shows that every generalized bi ideal of S is semi prime.

$(iii) \Longrightarrow (\boldsymbol{i})letA$ be a generalized bi ideal of S from the collection $\overline{G(S)}$. Now if $A^2 = S$ that is $A^2 = A$, then it is obvious that A is an idempotent. If $A^2 \neq S$ then A^2 is a power generalized bi ideal of S.which containsA^2 and therefore by hypothesis$A^2 = \bigcap_a \{ \overline{G(S)_a} : \overline{G(S)}$ is the collection of the entire generalized bi ideal$\}$. Since each $\overline{G(S)_a}$ is a semi prime generalizd bi ideal therefore

$A \subseteq \overline{G(S)}_a$ for all a and so $A \subseteq \overline{G(S)}_a = A^2$. Hence each generalized bi ideal in S is indepotent that is $A^2 = A$.

Theorem.11: Let $a, b \in \{a, b\}S\{a, b\}$ for every choice of a and b in a semi group S. Then every generalized bi ideal of S is a bi ideal of S.

Proof : Let A be a generalized bi ideal. So A is a non empty sub set such that $ASA \subseteq A$. Now we want to show that A is a semi group. Let $a, b \in A$. Then as $ab \in \{a, b\}S\{a, b\}$ and as $\{a, b\} \subseteq A$so $ab \in \{a, b\}S\{a, b\} \subseteq ASA \subseteq A$. So A is a subsemigroup and hence a bi ideal of S.

Theorem.12: Let Sa semi group. Suppose that every generalized (m, n) bi ideal of S is an(m, n) bi ideal of S.then $a^m b^m \in \{a^m, b^n\}S\{a^m, b^n\}$ for every choice of a and b in a semi group$S$.

Proof: Suppose$a, b \in S$. Now, consider the generalized (m, n)bi ideal generated by the sub set $\{a^m, b^n\}$ of S.by definition it is the set A such that $A = \bigcap_{\alpha \in I} N_\alpha$ where N_α is a generalized (m, n) bi ideal containing$\{a^m, b^n\}$. We claim that

$A = \{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$ .Clearly$A$contains$\{a^m, b^n\}$. Now suppose that N is some other generalized (m, n) bi ideal containing$\{a^m, b^n\}$. Then to prove our, claim it is sufficient to show that $A \subseteq N$. Let $x = a^m$ for some$s \in S$. This is because if $x = a^m$or $x = b^m$ then clearly$x \in N$. Now, as N is generalized (m, n) bi ideal so $N^m S N^n \subseteq N$ and so $a^m s b^n \subseteq N$. Hence $x \in N$and $A \subseteq N$.

Now $\{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$ is also a generalized (m, n) bi ideal containing$\{a^m, b^n\}$. It follows that,$A = \{a^m, b^n\} \cup \{a^m, b^n\}S\{a^m, b^n\}$.

Since every generalized (m, n) bi ideal of S is an (m, n) bi ideal of S.Thus,$a^m b^n$ clearly a generated element of the set (a^m, b^n) and so $a^m, b^n \in A$. As a^m, b^ndoes not belongs to $\{a^m, b^n\}$. So $a^m, b^n \in \{a^m, b^n\}S\{a^m, b^n\}$ and this completes the proof.

Definition.13:A * semi group is a set S equipped with a binary operations $\star$ and a unary opretions $*: S \to S$ satisfying the following three axioms

i. $(x.y)z = x.(y.z)$ ii) $(x^*)^* = x^{**} = x iii)$ $(xy)^* = y^* x^*$for all $x, y \in S$.

Such a unary operations $*$ is sometimes called an involution, and $(S,\star,*)$ is sometimes called an involution semi group. Where a^* is called the ad joint of a.

Theorem.14: Let S be semi prime *-semi group. If A is a non zero (m,n) *- bi ideals of S, then A contains a non zero (m,n) $*$- bi deal of S which has the form $(a^*)^m S(a^*)^n$.

Proof: Let $a \in A$, and $a^* = a, a \neq 0 \Longrightarrow a^n \in A \Longrightarrow (a^*)^n \in A$. Since A is an (m,n) *- bi ideal of S, it follows that $(a^*)^m S(a^*)^n \subseteq A^m S A^n \subseteq A^* \subseteq A$. Now it is well known that the *-semi group S is semi prime if and only if $(x^*)^m S(x^*)^n = 0$with $(x^*)^n \in S \Longrightarrow (a^*)^n = 0$. Therefore by the assumption$(a^*)^m S(a^*)^n \neq 0$, more over as $(a^*)^m S(a^*)^n$ is an (m,n) *- bi-ideal of S, so the assertion is valid.

Theorem.15: Let A be a sub set of a semi group with involution S. the following statements are equivalents.

i. $A$is a $(0,2)$ generalized *-bi ideal of S. ii) $A$is a *- ideal of some $2-$left ideal of S.

Proof:$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$let A be a $(0,2)$ generalized *- bi ideal of S. That is $ASA \subseteq A$ with $SA^{*^2} \subseteq A$ where $A^* \subseteq A$and $A^* = \{a^* \in S: a \in A\}$. Then we have $A(A \cup SA) = A^2 \cup ASA \subseteq A$and $(A \cup SA)A^* = A^2 \cup SA^2 \subseteq A \subseteq A$. Therefore A is an *- ideal of some $m-$ left ideal $A \cup SA$of S.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})LetA$ be a *- ideal of some m-left ideal $L$of S. $A$is a $(0,2)$ *- ideal of S and also A is a *- bi ideal of S.

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