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Introduction

In LP model, the input data (also known as parameters) are assumed constant and known with certainty during a planning period. These parameters are such as;

(i) Profit (cost) contribution (Cj) per unit of decision variable.

(ii) Availability of resources (bi).

(iii) Consumption of resources per unit of decision variable (aij).

However, in real-world situations some data may change over time because of the dynamic nature of the business such changes may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision-maker in such situation would like to know, how sensitive the optimal solution is to the changes in the original input data values.

Sensitivity Analysis

Sensitivity analysis is the study of sensitivity of the optimal solution of an LPP due to discrete variation (changes) in its parameters. The degree of sensitivity of the solution due to these variations can range from no change at all to a substantial change in the optimal solution of the given LPP. Thus, in sensitivity analysis we determine the range over which the LP model parameters can change without affecting the current optimal solution. The process of studying the sensitivity of the optimal solution of an LPP is also called post-optimality analysis because it is done after an optimal solution, assuming a given set of parameters, has been obtained for the model. Different categories of parameter changes in the original LP model includes the following;

Change in Objective Function Coefficient (Cj)

The question is; what happens to the optimal solution and the objective function when this coefficient is changed? There are three cases in this change;

Case I: Change in the coefficient of a non-basic variable

– The current optimal solution for a maximization LPP will remain optimal as long as all $Cj - Zj \leq 0$ for all j.

– Let Ck be the coefficient of non-basic variable xk in the objective function.

– Ck does not affect any of the Cj values listed in the CB column. Calculation of Zj = CBB−1aj values do not involve Cj , therefore change in Cj does not alter Zj values and hence Cj −Zj values.remain unchanged except Ck − Zk do to change in Ck.

– To retain optimality of the current optimal solution for a change _Ck in Ck we must have;

$$(Ck + \_Ck) - Zk \leq 0 \text{ or}$$

$$(Ck - \_Ck) - Zk \leq 0$$

– Hence for a maximization LPP, the value of Ck may be increased up to the value of Zk and decrease to $-\infty$ without affecting the optimal solution.

Case II: Change in the coefficient of a basic variable

– In the maximization LPP the change in the coefficient say C_k of a basic variable x_k affects the Cj − Zj values corresponding to all non-basic variables in the simplex table. It is because the C_k is listed in the CB column of the simplex table and affects the calculation of Zj values.

– The sensitivity limits for the Ck of a basic variable are calculated as under;

Lower limit = (Original Value C_k) − (Lower absolute value of improvement ratio $-\infty$)

Upper limit = (Original Value C_k) + (Lower positive value of improvement ratio ∞)

Where

$$\text{Improvement ratio} = \frac{\text{Per unit Impr.Ratio}}{\text{Coefficient in the Variable row}} = \frac{Cj - Zj}{A_{kj}}$$

Note: While performing sensitivity analysis, the artificial variable columns in the simplex table are ignored because artificial variable has no economic interpretation.

Case III: Change in the coefficient of non-basic variable in cost min. problem

– The procedure for calculating sensitivity limits to a cost minimization LPP when the objective function coefficients are unit costs is identical to the Case I above.

– In this case the unit cost coefficient can be increased to any arbitrary level but it cannot be decreased by more than per unit improvement value without making it eligible so that a non-basic variable can be entered into the new solution mix.

– The sensitivity limits can be calculated as;

Lower limit = (Original V alue) − (Unit improvement value) Upper limit = Infinity (∞)

Example: Given the optimal solution Table 1;

Table 1:

		$C_j \longrightarrow$	4	6	2	0	0
C_B	B	$b(=x_B)$	x_1	x_2	x_3	S_1	S_2
4	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/5	1/3
$Z = 16$		Z_j	4	6	8	10/3	2/3
		$C_j - Z_j$	0	0	-6	-10/3	-2/3

(a) Effect of change in C_3 of non-basic variable x3 is shown in Table 2;

Table 2 :

		$C_j \longrightarrow$	4	6	$2+\Delta C_3$	0	0
C_B	B	$b(=x_B)$	x_1	x_2	x_3	S_1	S_2
4	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/5	1/3
$Z=16$		Z_j	4	6	8	10/3	2/3
		$C_j - Z_j$	0	0	$\Delta C_3 - 6$	-10/3	-2/3

- Then

$$\Delta C_3 - 6 \leq 0$$

$$\Delta C_3 \leq 6$$

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- Recall that

$$C_3 - 6 \leq 2 + \Delta C_3$$

$$C_3 \leq 2 + 6$$

$$C_3 \leq 8$$

$$\text{or } -\infty \leq C_3 \leq 8$$

(b) Effect of change in the coefficient of C_1 of basic variable x_1;

Table 3 :

		$C_j \longrightarrow$	4	6	2	0	0
C_B	B	$b(=x_B)$	x_1	x_2	x_3	S_1	S_2
$4+\Delta C_1$	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/5	1/3
$Z-12+(4+\Delta C_1)$		Z_j	$4+\Delta C_1$	6	$8-\Delta C_1$	$(10/3)-$ $4\Delta C_1/3$	$(2/3)-\Delta C_1/3$
		$C_j - Z_j$	0	0	$\Delta C_1 - 6$	$(-10/3)-$ $4\Delta C_1/3$	$(\Delta C_1/3)-2/3$

- For the solution to remain optimal we must have all $C_j - Z_j \le 0$. That is

$$\Delta C_1 - 6 \le 0 \quad \text{i.e} \quad \Delta C_1 \le 6$$

$$(-10/3) - 4\Delta C_1/3 \le 0 \quad \text{i.e} \quad \Delta C_1 \ge -5/2$$

$$(\Delta C_1/3) - 2/3 \le 0 \quad \text{i.e} \quad \Delta C_1 \le 2$$

- Thus, the range of values within which C1 may change without affecting the current optimal solution is;

$$-5/2 \le \Delta C1 \le 2 \text{ or}$$

$$4 - (5/2) \le C1 \le 4 + 2$$

$$\text{i.e } 3/2 \le C1 \le 6$$

Change in the Availability of Resources (bi)

Case I: While slack variable is not in the solution mix (i.e not in the basis)

– The procedure for finding the range for resource value within which current optimal remain unchanged is;

1. Calculate exchange ratio (minimum ratio) for every row for slack variables.

$$\text{Exchange Ratio} = \frac{Solution\ Value\ XB}{Coefficient\ in\ Slack\ V\ riable\ Column}$$

2. Find both lower and upper sensitivity limits

Lower limit = (Original Value) − (Smallest positiveratio or −∞)

Upper limit = (Original V alue) + (Smallest absolute negative ratio or ∞)

Example: Given an optimal solution in Table 4;

Table 4 :

		$C_j \longrightarrow$	4	6	2	0	0
C_B	B	$b(=x_B)$	x_1	x_2	x_3	S_1	S_2
4	x_1	1	1	0	-1	4/3	-1/3
6	x_2	2	0	1	2	-1/5	1/3
$Z = 16$		Z_j	4	6	8	10/3	2/3
		$C_j - Z_j$	0	0	-6	-10/3	-2/3

– No slack variable in the solution mix (B).

– Thus ratios are calculated as;

Table 5 :

B	X_B	Coef.in S_1 Column	Exchange ratio	Coef.in S_2 Column	Exchange ratio
x_1	1	4/3	1/(4/3)=3/4	-1/3	1/(-1/3)=-3
x_2	2	-1/3	2/(-1/3)=-6	1/3	2/(1/3)=6

– Lower and upper limits for resources for constraints are calculated as;
Suppose b1 = 3 and b2 = 9 for constraint 1 and 2 respectively;

Table 6 :

b_i	Solution Mix	Lower Limit	Upper Limit
3	x_1(Manpower)	3-(3/4)=9/4	3+3=6
9	x_2(Row material)	9-6=3	9+6=15

Case II: When slack variable is in the basis (Column CB)

– The procedure for finding the range of variation for corresponding right hand of the constraint is as follows;

$$LL = \text{Original V alue bi} - \text{Solution of Slack V ariable (XB)}$$

$$LU = \text{Infinity } (\infty)$$

Case III: Change in right hand side when constraints are mixed type

– When surplus is not in the basis (B)

$$LL = (\text{Original V alue}) - (\text{Smallest absolute value of} - \text{ve exchange ratio or } -\infty)$$

$$LU = (\text{Original V alue}) - (\text{Smallest + ve minimum ratio or } \infty)$$

– When surplus variable is in the basis

$$LL = \text{Minus infinity } (-\infty)$$

$$LU = \text{Original value + Solution value of surplus variable}$$

Change in the Input-Output Coefficient (a'_{ij}s)

Case I: When a non-basic column ak ∈ B Changed to a*k

– The only effect of such change will be on optimality condition. Thus the solution will remain optimal if;

$$C_k - Z_k^* = C_k - C_B B^{-1} a_k^* \leq 0$$

otherwise the simplex method is continued, after column k of the simplex table is updated, by introducing the non-basic variable xk into the basis.

– The range of discrete change _aij in the coefficient of non-basic variable x_j in the constr aint i can be determined as follows;

$$Max = \{\frac{C_j - Z_j}{C_B\beta_i > 0}\} \leq a_{ij} \leq Min\{\frac{C_j - Z_j}{C_B\beta_i < 0}\}$$

– β_i is the i^{th} column of B^{-1}. If $C_B\beta_i = 0$ then Δa_{ij} is unrestricted in sign. B^{-1} is matrix of coefficients corresponding to slack variables in the optimal solution table.

Case II: When a basic variable column $a_k \in B$ is changed to $a*_k$

– The condition to maintain both feasibility and optimality of the current optimal solution are;

$$(a) \quad Max_{k=p}\{\frac{-x_{Bk}}{x_{Bk}\beta_{pi} - x_{Bp}\beta_{ki} > 0}\} \leq \triangle a_{ij} \leq Min_{k=p}\{\frac{-x_{Bk}}{x_{Bk}\beta_{pi} - x_{Bp}\beta_{ki} < 0}\}$$

$$(b) \quad Max\{\frac{C_j - Z_j}{(C_j - Z_j)\beta_{pi} - y_{pj}C_B\beta_i > 0}\} \leq \triangle a_{ij} \leq Min\{\frac{C_j - Z_j}{(C_j - Z_j)\beta_{pi} - y_{pj}C_B\beta_i < 0}\}$$

Addition of a New Variable (Column)

– Let an extra variable x_{n+1} with coefficient C_{n+1} be added in the system of original constraint AX = B, X ≥ 0. Thus, it creates an extra column a_{n+1} in the matrix A of coefficients.

– To see the impact of this addition on the current optimal solution, we compute;

$$y_{n+1} = B^{-1}a_{n+1} \quad and$$
$$C_{n+1} - Z_{n+1} = C_{n+1} - C_B y_{n+1}$$

– Two cases of the maximization LP model may arise;

1. If Cn+1 −Zn+1 ≤ 0, then XB = 0 and hence current solution remain optimal.
2. If Cn+1 − Zn+1 ≥ 0, then the current optimal solution can be improved by introducing a new column an+1 into the basis to find a new optimal solution.

Example: Given the optimal solution in Table 7, discuss the effect on optimality by adding a new variable with column coefficients (3,3,3) and coefficient 5 in the objective function (Minimization)

Table 7 :

		$C_j \longrightarrow$	3	8	0	0	M
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	A_1
0	S_2	60	0	0	-1	1	1
3	x_1	80	1	0	1	0	0
8	x_2	120	0	1	-1	0	1
$Z = 1,200$		Z_j	3	8	-5	0	8
		$C_j - Z_j$	0	0	5	0	M-8

Solution: To see the changes in the optimal solution, we were given $C_7 = 5$, $a_7 = (3, 3, 3)^T$ then we calculate;

$$C_7 - Z_7 = C_7 - C_B B^{-1} a_7$$

$$= 5 - (0,3,8) \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -4$$

$$a_7^* = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

New column with variable x7 added as shown in Table 6.6

Table 8 :

		$C_j \longrightarrow$	3	8	0	0	M	5	
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	A_1	x_7	Min.Ratio
0	S_2	60	0	0	-1	1	1	3	$60/3 \rightarrow$
3	x_1	80	1	0	1	0	0	3	80/3
8	x_2	120	0	1	-1	0	1	0	-
$Z = 1,200$		Z_j	3	8	-5	0	8	9	
		$C_j - Z_j$	0	0	5	0	M-8	-4	

From the above table, variable x_7 must enter into the solution and S_2 should leave it. The new solution is shown in Table 9;

Table 9 :

		$C_j \longrightarrow$	3	8	0	0	M	5
C_B	B	$b(=x_B)$	x_1	x_2	S_1	S_2	A_1	x_7
5	x_7	20	0	0	-1/3	1/3	1/3	1
3	x_1	20	1	0	2	-1	-1	0
8	x_2	120	0	1	-1	0	1	0
$Z = 1,120$		Z_j	3	8	-11/3	-4/3	-4/3	5
		$C_j - Z_j$	0	0	11/3	4/3	M+4/3	0

Since all $C_j - Z_j \geq 0$, optimal solution $x_1 = 20$, $x_2 = 120$, $x_7 = 20$ and Min Z=1,120

Solving LPP using LINDO

Introduction

Computer programs designed to solve LP problems are now widely available. Most large LP problems can be solved with just a few minutes of computer time. Small LP problems usually require only a few seconds. We will use LINDO to solve our LP problems.

Using LINDO to Solve LPP

You can Download Free LINDO Software and follow the instructions which explains how to enter (and solve) a program in LINDO.

Interpretation of LINDO Output

We will discuss the following output:

- Objective function value
- Values of the decision variables
- Reduced costs
- Slack/surplus

Reduced Cost

The reduced cost for a decision variable whose value is 0 in the optimal solution is the amount the variable's objective function coefficient would have to improve (increase for maximization problems, decrease for minimization problems) before this variable could assume a positive value. The reduced cost for a decision variable with a positive value is 0.

Example

Consider the following objective function: Min $2x_1 + 5x_2 + 4x_3$ (6.1) Suppose the optimal value ofx_1 is zero, with a reduced cost of 1.2. Since this is a minimization problem, this tells us that the current coefficient of x_1 , which is 2, must be decreased by 1.2 in order for the optimal value of x_1 to be non zero. Thus if the objective function coefficient of x_1 was 0.8 (or less), resolving the LP would yield a non zero value of x_1.

Slack/Surplus

The slack for "less than or equal to" constraints is the difference between the right hand side of an equation and the value of the left hand side after substituting the optimal values of the decision variables. The slack represents the amount of unused units of the right hand side resources. The surplus for "greater than or equal to]] constraints is the difference between the right hand side of an equation and the value of the left hand side after substituting the optimal values of the decision variables. The surplus represents the number of units in which the optimal solution causes the constraint to exceed the right hand side lower limit.

Conclusion

However, in real-world situations some data may change over time because of the dynamic nature of the business such changes may raise doubt on the validity of the optimal solution of the given LP model. Thus, a decision-maker in such situation would like to know, how sensitive the optimal solution is to the changes in the original input data values.

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