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Non deranged permutations as functions, zeros and poles

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Abstract

In analysis we solve problems by finding a sequence of approximate solutions whose limit is an actual solution. We need the limit to exist in our space. That is why we do calculus in the space of real or complex numbers which are complete vector spaces. In this work, we embed a set of non deranged permutations of the first p(prime) natural numbers into a complete vector space C^n with p = n. we show that the non deranged permutations are functions, solutions and poles of functions. We use Borel Caratheodory theorem to find the highest value of complex functions whose real parts are non deranged permutations. we use nevanlinna theorem to measures the growth of the number of poles in the disks $\mid z \mid \leq$ r as r and p increases. We use the symmetric version of Rouche's theorem to show that two functions f and g whose zeros are 2 different permutations of the same first p numbers must both be greater than 0. We also apply Jensen's formula and Rouche's theorem to analyse non deranged permutations as poles and zeros respectively. We apply Maximum modulus theorem and Liouville's theorem to analyse the special embedded cycle of our set of non deranged permutations. Finally, Picard's little theorem shows that 0 is the only number the function cannot attain.

Introduction

Several papers have been written on the algebraic structure G'_p. First Garba and Abubakar (2015) used modular arithmetic to construct an algebraic structure $G_p = \{w_1, w_2 \cdots w_{p-1}\}$, where each w_i=$\left((1)\,(1+i)_{mp}\,(1+2i)_{mp}\ldots(1+(p-1)\,i)_{mp}\right)$.

Each w_i is called a cycle and the elements in each w_i are distinct and called Successors. A special Cycle $w_p = \{(p)\,(p)\,(p)\ldots(p)\}$ was embedded into the algebraic structure G_p, where the special Cycle is of length 1 and the Successors are not distinct. The resulting structure was defined as

$$G'_p = G_p \cup w_p$$

A concatenation map was defined as follows:

$$\varphi_{i,j} : G'_p \times G'_p \to G'_p$$

where $1 \leq i, j \leq p, p \geq 5$ is prime. Then $\left(G'_p, \varphi\right)$ is an abelian group. In this work, we embed the algebraic structure G'_p into C^n by taking the elements of G'_p to be the real parts of some complex numbers in C^n where p = n. We show identity mapping from C^n to C^n.

We show some functions whose zeros and poles are all elements of each non deranged permutations of the first p natural numbers. We also show that non deranged permutations are functions on the complex numbers.

Main results

2.1 Permutations as functions:

In this section, we study non deranged permutations as complex functions. We apply Borel Caratheodory theorem which states that an analytic function in a closed disk of radius R centered at the origin is bounded by it's real part. Consider the mapping f:$C^n \to C$ such that

$$f\,(z_1, z_2, ..., z_n) = f\,(z_i) = z_i$$

where each z_i given by

$$z_i = (ix+1)_{modp} + y_j$$

for $1 \leq i \leq p$, $1 \leq x \leq p-1$ and p = n.

By Borel caratheodory theorem,

$$\parallel f \parallel \leq \frac{2r}{R-r}.\text{Sup Re f(z)} + \frac{R+r}{R-r} \mid f(0) \mid$$

Sup Re f(z) = p and $\mid f(0) \mid = 0$. If r = $\frac{R}{3}$, then $\parallel f \parallel_r \leq$ p.

2.2 Permutations as poles:

Let f be different complex functions with poles at $(ix+1)_{modp}$. We use the nevalinna characteristic which measures the growth of the poles as the radius of the disk increases.

$$N(r,f) = \sum_{k=1}^{n} log\left(\frac{r}{(ix+1)_{kmodp}}\right) + n(0,f)\text{log r}.$$

Solving, we get p.log r - log(1 x 2 x 3 x ...x p) where there is no pole at the origin. Therefore the quantity

$$\log\left(\frac{r^p}{1 \times 2 \times ... \times p}\right)$$

measures the growth of the number of poles in the discs $\mid z \mid \leq$ r as r increases.

Jensen's formula: Suppose that f is an analytic function in the complex plane which contains the closed disk D_r of radius r > 0 about the origin, a_1, a_2, ... a_n are the zeros of f in the interior of D_r, repeated according to their multiplicity, and that $f(0) \neq 0$, then

$$\log\mid f(0) \mid = \Sigma_{k=1}^{N}\log\left(\frac{\mid a_k \mid}{r}\right) + \frac{1}{2\pi}\int_0^{2\pi}\log\mid f\left(re^{i\theta}\right) \mid d\theta$$

If the first p natural numbers are the zeros of a complex function f, then Jensen's formula gives

$$\log\frac{\mid a_1 \mid}{r} + \log\frac{\mid a_2 \mid}{r} + ... + \log\frac{\mid a_p \mid}{r} + \frac{1}{2\pi}\int_0^{2\pi}\log\mid f\left(re^{i\theta}\right) \mid d\theta$$

which gives

$$\frac{1}{r^p}\left[log\left[1X2X...Xp\right]\right] + \frac{1}{2\pi}\int_0^{2\pi}\log\mid f\left(re^{i\theta}\right) \mid d\theta$$

Jensen's formula estimates the magnitude of the function whose zeros are the first p natural numbers. Irrespective of the ordering of the numbers in the non derannged permutation which fixes 1 as the first number, the magnitude remains the same.

2.3 Permutations as zeros:

Rouche's Theorem(Symmetric version): Let K $\subset$ G be a bounded region with a continuous boundary. Two holomorphic functions f, g $\in H(G)$ have the same number of roots(counting multiplicity) in K if the strict inequality

$$|\ f(z) - g(z)\ | < |\ f(z)\ | + |\ g(z)\ |$$

on the boundary of K.

Theorem: Two functions f and g that generate 2 different permutations of the same first p numbers must both be positive or negative.

Proof: $|\ f(z) - g(z)\ | < |\ f(z)\ | + |\ g(z)\ |$ on the boundary of K.

$$-|\ f(z)\ | - |\ g(z)\ | < f(z) - g(z) < |\ f(z)\ | + |\ g(z)\ |$$

-$|\ f(z)\ |$ - f(z) < - g(z) + $|\ g(z)\ |$, -2$|\ f(z)\ | < 0$, $|\ f(z)\ | > 0$.

f(z) - $|\ f(z)\ | < |\ g(z)\ |$ + g(z), 0 < 2g(z), g(z) > 0.

2.4 Constant functions

Maximum modulus principle: Let f be a holomorphic function on some connected open subset D of the complex plane and taking complex values. If z_0 is a point in D such that $|\ f(z_0)\ | \geq |\ f(z)\ |$ for all z in the neighborhood of z_0, then f is constant on D.

Liouville's Theorem: Every bounded entire function is constant.

Consider a function $z_i = (ix+1)_{modp} + y_j$ whose real parts are $w_p = \{(p)\,(p)\,(p)\ldots(p)\}$. If the y_j are also constant, then the function must be bounded and entire according to the Liouville's theorem and it could have attain it's maximum value within some connected open subset of the complex plane according to the maximum modulus principle.

Casorati Weierstrass Theorem: Let f be holomorphic in $U \setminus \{z_0\}$ but has an essential singularity at z_0. If V is any neighborhood of z_0 contained in U, then $f(V \setminus \{z_0\})$ is dense in C.

Picard's little Theorem: A non constant entire function omits at most one complex value.

Consider the mapping from $G'_p \subset C^n$ to $G'_p \subset C$ which gives the complex number $z_i = (ix+1)_{modp} + y_j$. $z_i = (ix+1)_{modp} + y_j$ cannot have a real part which equals 0 for $1 \leq i \leq p$, $1 \leq x \leq p-1$ and p = n.

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