Global Scientific JOURNALS

GSJ: Volume 8, Issue 5, April 2020, Online: ISSN 2320-9186
www.globalscientificjournal.com

Simulation of Modified Kicked Rotor Hamiltonian System

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ABSTRACT-In this paper, numerical simulation of Modified Kicked Rotor Hamiltonian System have been investigated and presented. The onset of chaos in both the unmodified and modified system is also investigated through numerical simulation. The results for the modified standard map are compared with that for the unmodified standard map. The shift in the onset of chaos and the existence of resonance trajectories in the modified standard map for large values of the kick strength are observed and discussed. We observed that the kicked rotor mapping is an interesting standard map from a fundamental point of view in physics because it's a model of a conservative system that displays Hamiltonian chaos and many physical systems can be approximated by this model. The physical realization of the modified kicked rotor is also discussed.

Keywords: Chaos, Hamiltonian, Kicked Rotor, Phase-space, Resonance, Standard map, Trajectories.

I. Introduction

The kicked rotor is one of the classical examples of non-integrable Hamiltonian system with regular and chaotic behavior and a prototype model for quantum chaos studies in Physics [1]. It describes a particle that is constrained to move in plane horizontal circular motion (e.g., pendulum on a rotating stick). The particle is kicked periodically by homogeneous field (equivalently: gravitational field which is switched on periodically in short pulses). When the field is switched on, it creates an imposing force which acts on the particle [2],[3].

The kicked rotor approximates systems studied in the fields of mechanics of particles (an electric dipole with fixed center rotating in a homogeneous electric field), electro-acoustic music, accelerator physics [4], plasma physics (neutral beam injection in a rotating plasma), theoretical solid-state physics and condensed matter physics. For example, circular particle accelerators accelerate particles by applying periodic kicks, as they circulate in the beam tube [5]. Thus, the structure of the beam can be approximated by the kicked rotor.

The kicked rotor dynamic system provides a model with a simple mathematical structure. Despite its simplicity, the kicked rotor is not merely a theoretical toy system but can be used to introduce graphical methods for studying mappings and recursion equations. In Physics, it helps to introduce the concept of a phase space and to demonstrate its usefulness and many general aspects of chaotic dynamics can be investigated [6]. It is worth to mention that this kicked rotor system can be quantized and this provides a basis for recent experimental research in quantum dynamics investigation [7],[8],[9]. The corresponding kicked quantum rotor has many physical realizations, including particle beams in an accelerator, atoms or molecules excited by microwaves [10], as well as ultracold atoms subjected to a pulsed standing wave of near resonant light [11].

Several modifications of the kicked rotor mappings have been investigated and the display of chaotic behavior in these models has been reported [12],[13],[14]. Literature however has it that at a critical value of $k_c \approx 0.971635$ (the Golden KAM² curve), the trajectories splits the phase space into disjoint manifolds and the phase space becomes predominantly chaotic with the probability of finding a regular island in the chaotic sea dropping exponentially with k above this critical value [15]. In fact for values of $k > 6$ the entire phase space becomes fully chaotic with no regular motion [16],[17]. This has limited the study of the system's behaviour at higher k values. In this paper, consideration is given to this limitation.

This paper is organized as follows: the Hamiltonian system and the Kicked Rotor Hamiltonian Model are presented in section II. The Modified Kicked Rotor Hamiltonian Models is also presented in this section. Numerical results and detailed discussions are given in section III. Summary and conclusion are presented in section IV.

II. Theoretical Considerations

A. The Hamiltonian Method

A Hamiltonian system is a dynamical system completely described by the scalar function $H(q,p,t)$, which is the Hamiltonian often corresponds to the total energy of the system [17].

$$H(q,p,t) = H(q,p) \tag{1}$$

The state of the system r, is described by the generalized coordinates momentum p and position q where both p and q are vectors with the same dimension N. So, the system is completely described by the $2N$ dimensional vector $r = (q,p)$ and the time evolution of the system is uniquely defined by the Hamilton's equations [17]:

$$\dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{2}$$

The Hamiltonian equations are then integrated as;

$$q(t) = \int \left(\frac{\partial H}{\partial p}\right) dt \quad , \quad p(t) = -\int \left(\frac{\partial H}{\partial q}\right) dt \tag{3}$$

The solution to these equations of motion $q(t)$ and $p(t)$ are then used to construct the iterated maps and simulated using the developed program codes.

B. The Kicked Rotor Hamiltonian Model

Considering the Kicked Rotor Hamiltonian system, the field is on only for a very brief periods of time, and the force can be approximated by the Dirac delta function such that in the limit of arbitrarily short pulses, the model is described by the Hamiltonian [3];

$$H(q,p,t) = \frac{p^2}{2} + k\cos\theta \sum_n \delta\,(t - n\tau) \tag{4}$$

where $q \equiv \theta$ is the angular position, p is the momentum, k is the kicking strength, τ is the period, $\delta(t)$ Dirac delta function and this limit of the problem is commonly termed the δ-kicked rotor. This limit is particularly convenient because the equations of motion can be reduced to a simple and accessible two-dimensional area preserving discrete map in phase space.

The kicked Rotor Model as reviewed in "equation (4) is described by the Hamiltonian,":

$$H(q,p,t) = \frac{p^2}{2} + k\cos q \sum_n \delta\,(t - n\tau) \tag{5}$$

The Hamiltonian "equations (5)," describes a free particle or kicked pendulum rotating in free field (more precisely a free rotor, because of the 2π periodicity of the position) which, when the time t is equal to an integer n, receives a kick [18]. From the Hamiltonian it is obvious that during the kick, the potential term dominates the kinetic term. Between kicks, the potential term is zero, and the motion is that of a free rotor. Using "(5)," we obtain the partial derivative with respect to p and q given by Hamilton's equations of motion as:

$$\left.\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial q} = k\sin q \textstyle\sum_n \delta\,(t - n\tau) \end{aligned}\right\} \tag{6}$$

To construct a standard map for the Kicked Rotor Hamiltonian Model, for q and p just before the n[th] kick, we integrate the "equation (6)" by letting β to be a small positive number. Integrating for p we have;

$$\int_{t_n-\beta}^{t_{n+1}-\beta} \partial p(t)\,dt = \int_{t_n-\beta}^{t_{n+1}-\beta} k\sin q \textstyle\sum_n \delta\,(t-n)dt \tag{7}$$

where $t_n = n$ is the time of the n[th] kick which yields $\delta(t-n) = 1$. "Equation (7) becomes."

$$p(t_{n+1} - \beta) - p(t_n - \beta) = k\sin q \tag{8}$$

Similarly, integrating for q we have;

$$\int_{t_n-\beta}^{t_{n+1}-\beta} \partial q(t)\,dt = \int_{t_n-\beta}^{t_{n+1}-\beta} p\,dt \tag{9}$$

"Equation (9) becomes."

$$q(t_{n+1} - \beta) - q(t_n - \beta) = \beta p(t_n - \beta) + (1-\beta)p(t_{n+1} - \beta) \tag{10}$$

Then letting β to be zero and defining q_n and p_n to be the values of q and p just before the n[th] kick, we obtain from "(8)," and "(10)," the mapping;

$$\left.\begin{aligned} p_{n+1} &= p_n + k\sin q_n \\ q_{n+1} &= q_n + p_{n+1} \end{aligned}\right\} \tag{11}$$

"Equation (11) is the standard map for the Kicked Rotor Hamiltonian Model." This mapping which depends only on a single parameter k (Stochasticity parameter) known as the standard map or Chirikov map [15], is so named because of its broad importance in the study of Hamiltonian Chaos. One important feature of the Standard map is the periodicity of the coordinate parameter q.

C. Modified Kicked Rotor Hamiltonian Model

To modify the standard map for the Kicked Rotor Hamiltonian Model, we define the domain $0 \le D \ge 1$ such that the modified standard map can be described by;

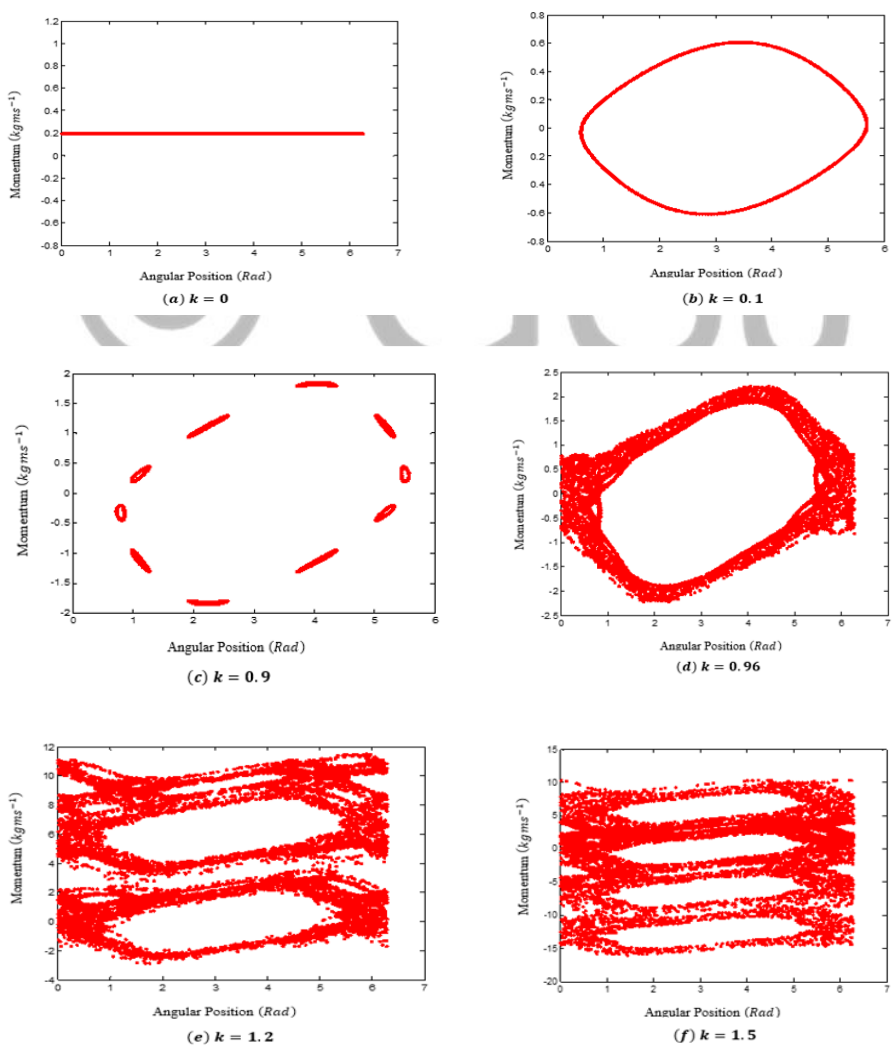
$$\left.\begin{aligned} p_{n+1} &= p_n + [Dk \sin q_n \pm (1-D)\sin q_n] \\ q_{n+1} &= q_n + p_{n+1} \end{aligned}\right\} \quad (12)$$

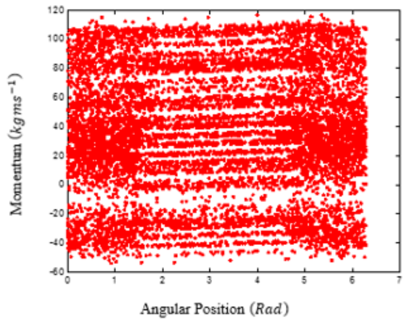
where the angle q_{n+1} is taken q_n modulus 2π and $q_{n+1} \in (0\,, 2\pi)$.

"Equation (12) represents the Modified Kicked Rotor Standard Map," where the D coupling is a perturbation added to the original system and serves as an ordering parameter of the system. These two equations "(11)," and "(12)," were implemented in MATLAB with executable program codes performing up to ten thousand time-step iterations.

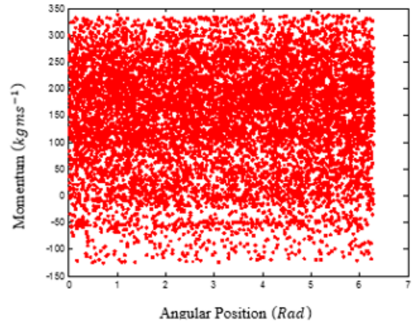
III. Results

The results obtained from the simulation are presented below;



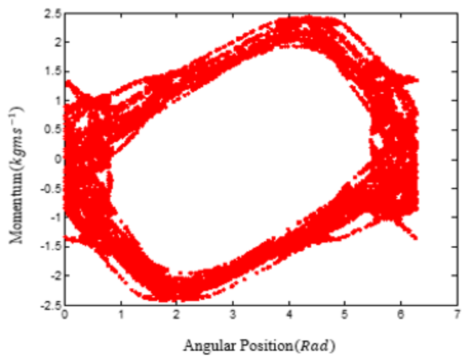


$(g)\ k = 2.5$

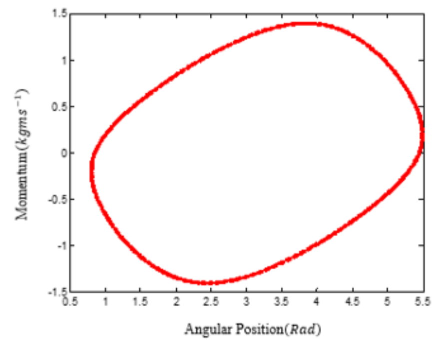


$(h)\ k = 6.0$

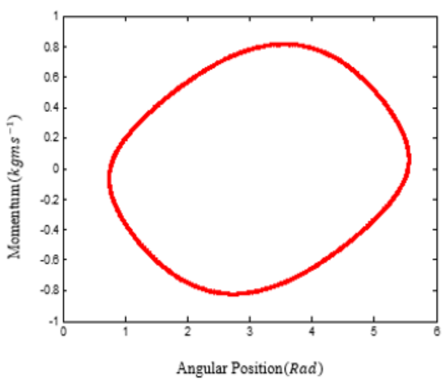
FIGURE 1(a-h). Phase Space for the Unmodified Kicked Rotor Standard Map.



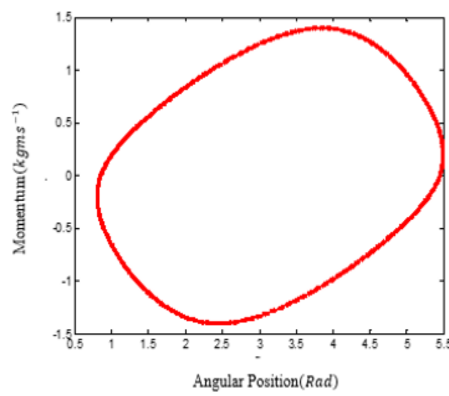
(a) k= 0.1, D = 0



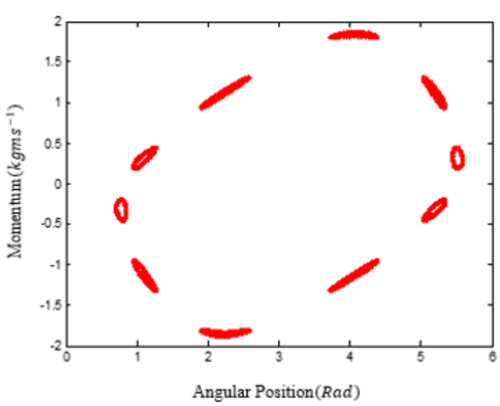
(b) k= 0.1, D= 0.5



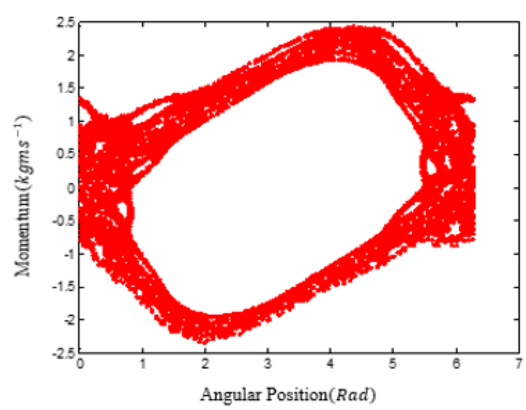
(c) k= 0.1, D = 0.9



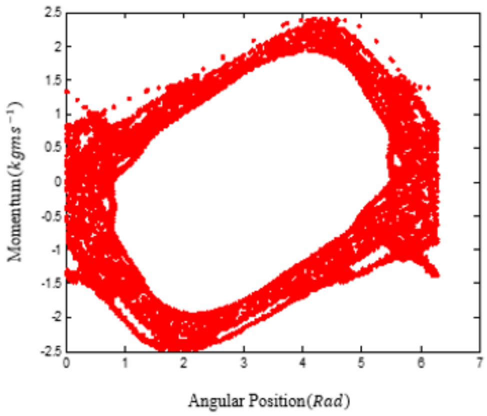
(d) k= 0.5, D= 0.9



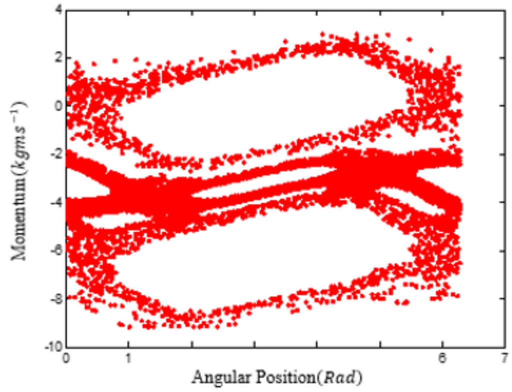
(e) k= 0.9, D = 0.9



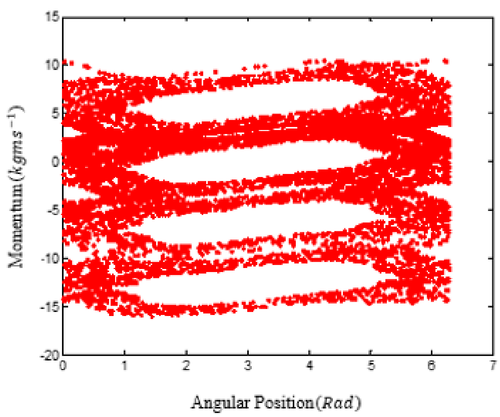
(f) k= 1, D= 0.1



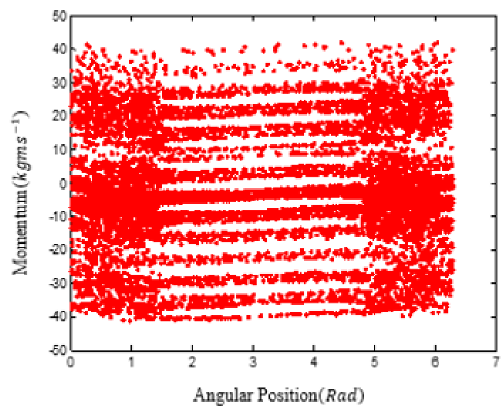
(g) k= 1, D = 0.9



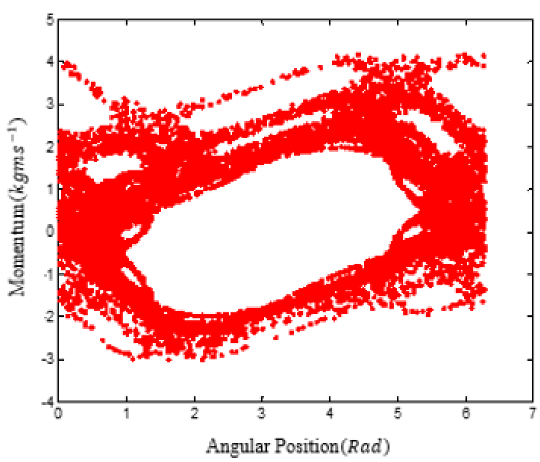
(h) k= 2, D= 0.2



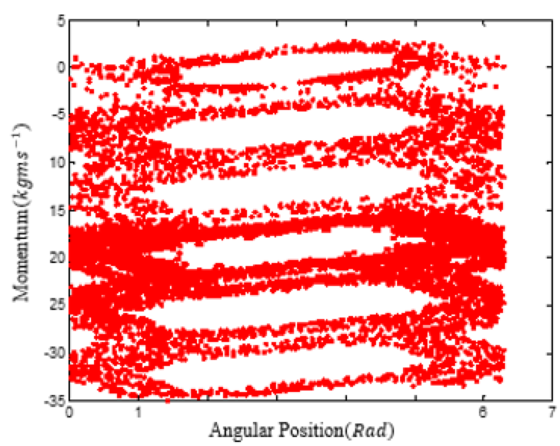
(i) k= 2, D = 0.5



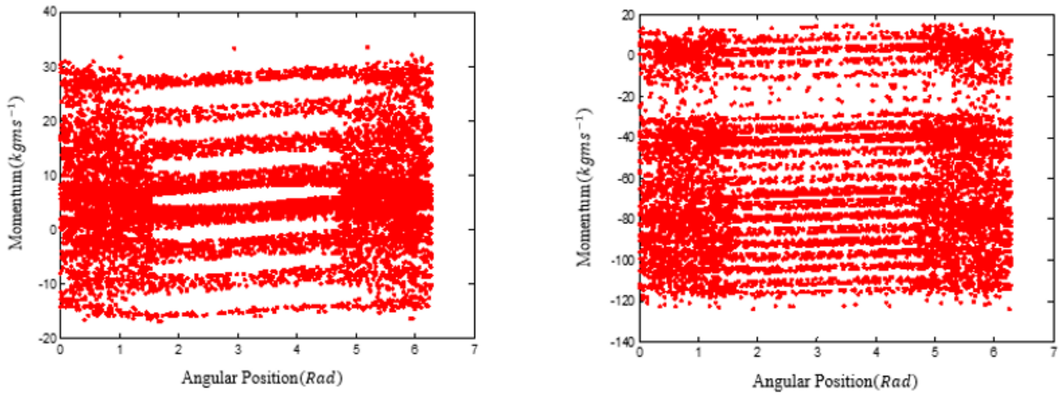
(j) k= 2, D= 0.9



(k) k= 3, D = 0.1

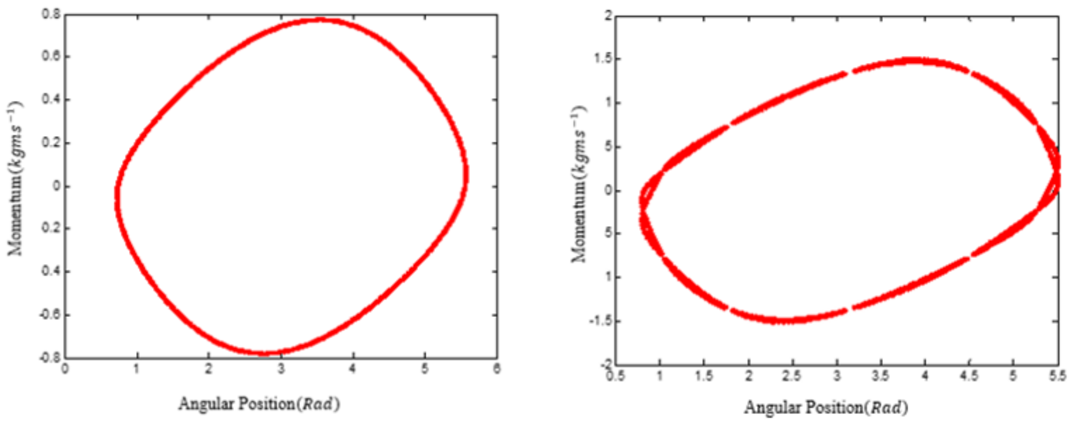


(l) k= 3, D= 0.3

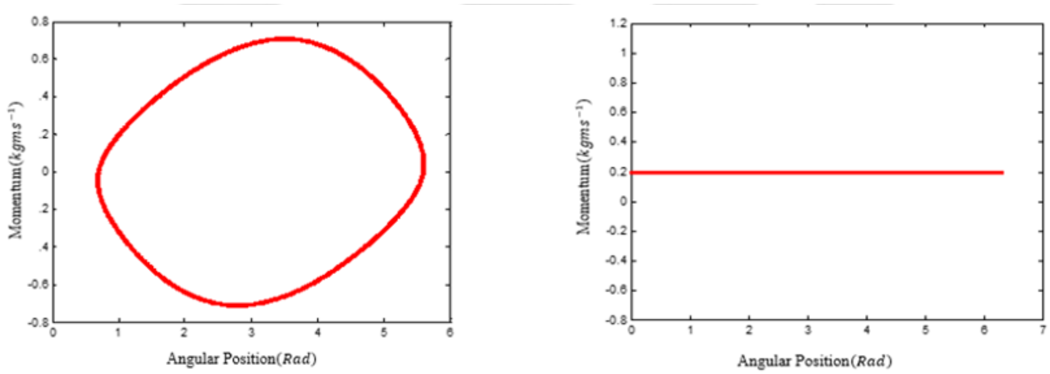


(m) k= 8, D = 0.1 (n) k= 12, D= 0.1

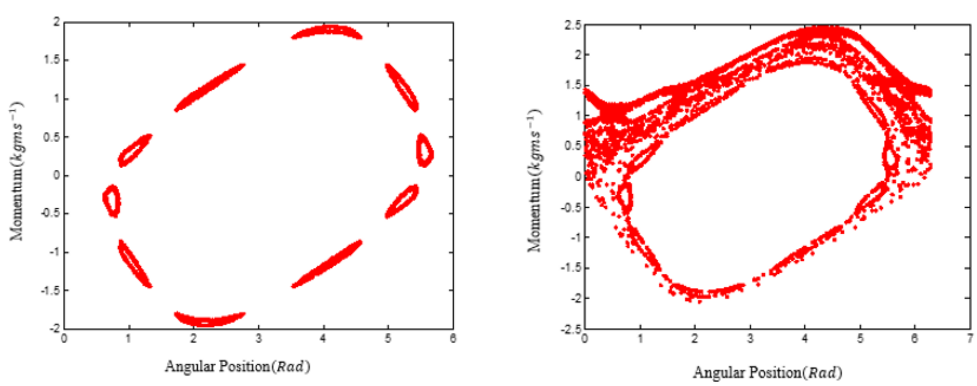
FIGURE 2 (a-n). Phase Space for the Modified Kicked Rotor with Additive Term.



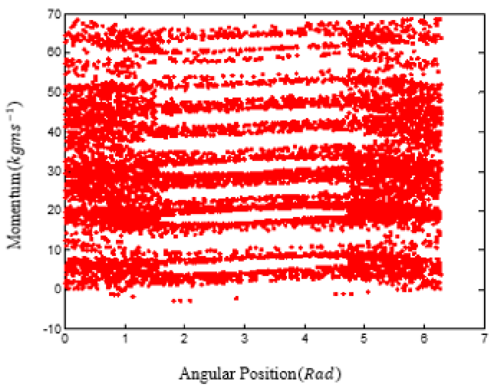
(a) k=0.3, D=0.9 (b) k=0.8, D=0.9



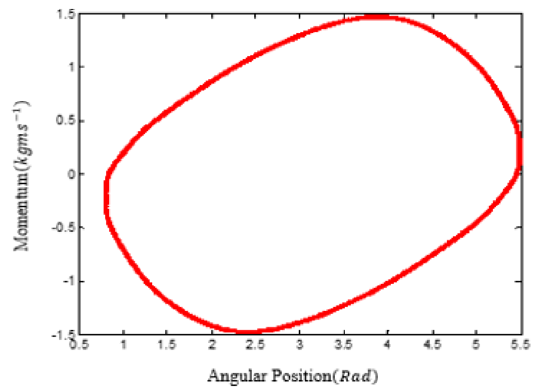
(c) k=0.9, D=0.6 (d) k=1, D=0.5



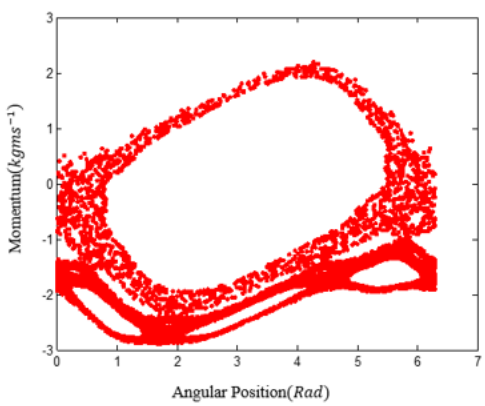
(e) k = 2, D = 0.65 (f) k = 2, D= 0.651



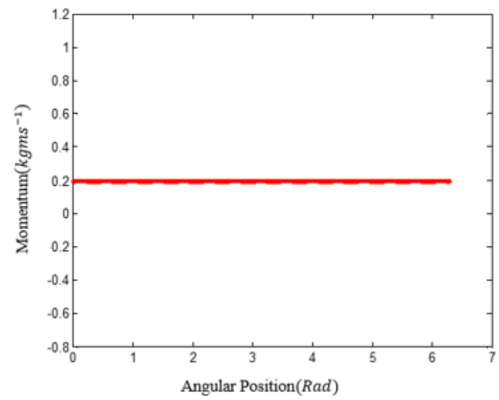
***(g)* k = 2, D = 0.9**



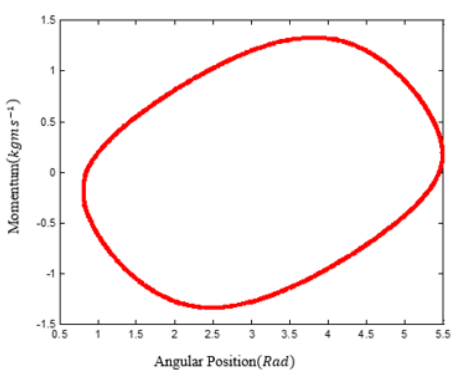
***(h)* k = 3, D= 0.4**



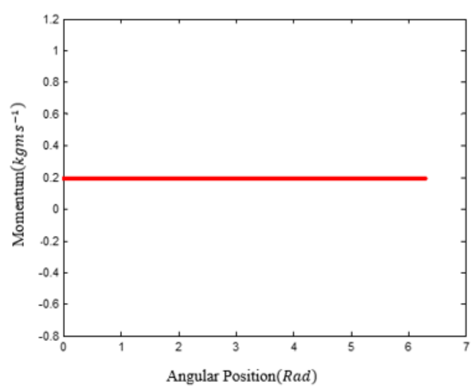
***(i)* k = 3, D = 0.5**



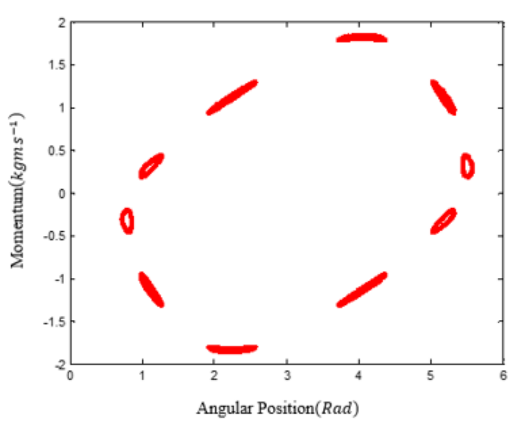
***(j)* k = 4, D= 0.2**



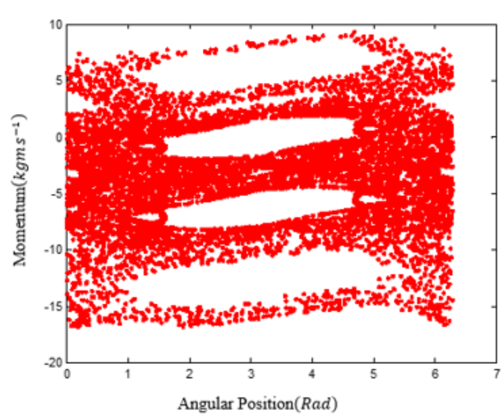
***(k)* k = 4, D = 0.3**



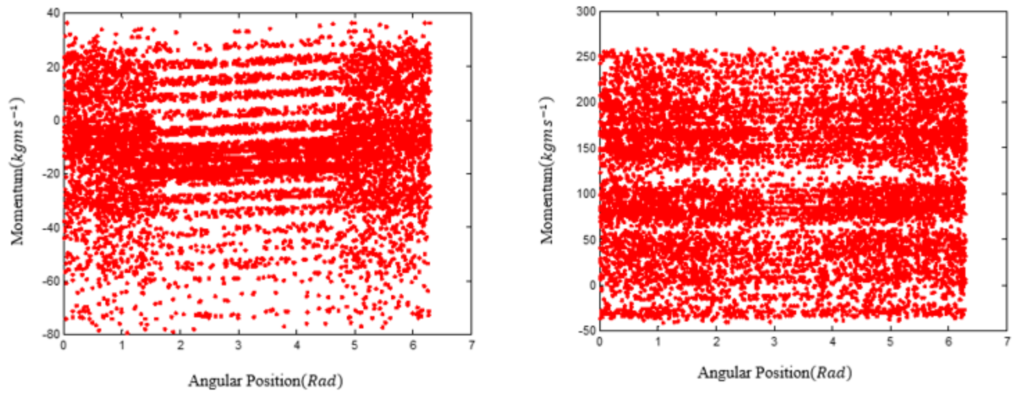
***(l)* k = 9, D= 0.1**



***(m)* k = 18, D = 0.1**



***(n)* k = 25, D= 0.1**



(*o*) $k = 33$, $D = 0.1$ (*p*) $k = 40$, $D = 0.1$

FIGURE 3 (a-p). Phase Space for the Modified Kicked Rotor with Subtractive Term.

IV. Discussion

The results of phase space for the Unmodified Kicked Rotor Standard Map are presented Figure 1. The phase space trajectory of the unmodified kicked rotor standard map is defined by the initial conditions p_0=0.2, q_0=1.0 and the kicking strength parameter k. Different trajectories corresponding to several values of k are plotted in Figure 1 (*a*) to (*h*). It should be noted that the phase space of the standard map corresponds to the motion of a periodically kicked rotor. If the kicking strength vanishes (k=0), the system would rotate freely, which corresponds to the uniform increment of its phase q with a constant momentum p (Figure 1*a*) and the standard map is integrable.

At each kick, the momentum changes by a quantity $k \sin q_n$ which can be either positive or negative. As we increase the kick strength to k=0.1 (Figure 1*b*), for the given initial conditions above, the orbits remains a rotational invariant circles (elliptical orbits) that seem to be very regular which implies that there are non-chaotic orbits for low values of the kick strength up to $k = 0.8$. On varying the kick strength to k=0.9 (Figure 1*c*) the elliptical orbits appear to be deformed into doughnut and banana shapes along the torus. This transition from periodic to aperiodic behavior of the system is a signature for onset of chaos. Consequently at k=0.96 (Figure 1*d*), there is onset of chaos with emergence of two chaotic regions in the phase space where $p \approx 0$ and q is near to 0 and 2π which is close to the critical value of $k_c \approx 0.971635$, literally implying that the map becomes fully chaotic above this value of k. This however doesn't mean that above this value there is no regular motion in the phase space, but the probability of finding a regular island in the chaotic sea drops exponentially with k above this critical value.

As the parameter k becomes very large, each kick is so strong that it gives rise to a diffusive-like behavior in the phase space. It can be observed that inside the chaotic zones, there exist small islands of non-chaos and some adjacent to the major orbit in the phase space. As we progress to k= 1.2 (Figure 1*e*), there are visibly two chaotic regions where p is near to 0 or 2π and q is near to 0 or 2π with two major islands inside which appears to be a Period-doubling bifurcation of the single major orbit with some minor island lying side by side and at the corners. The centers of these major island are stable fixed points which means that initially close orbits will stay in their neighborhood for all times. There are four such stable fixed points as the value of k is increased to k=1.5 (Figure 1*f*).

These regular orbits are observed to diminish from k= 2.5 (Figure 1*g*) and above where the entire phase space become fully chaotic with no visible island within the chaotic sea. Subsequent increment in the value of k rather gives rise to a denser chaotic sea in the phase space with no single visible regular orbit (Figure 1*h*). This result is consistent with the work done by other authors [6],[12],[15],[17].

In attempt to seeking order out of this disorderliness, we will analyze the phase space for the modified kicked rotor standard map first applying the additive term with the same initial conditions used for the unmodified rotor. The results of phase space for the Modified Kicked Rotor with Additive Term are presented in Figure 2. Whenever our ordering parameter D=0 (Figure 2*a*) we obtain a phase space for the well-known unmodified kicked rotor mapping for k =1 which appears Chaotic. Similarly if we set D=1 for any value of k we obtain the phase space trajectory for that corresponding value of k in the unmodified standard map. Hence the choice of setting D=0 or D=1 does not alter the Physics of the unmodified kicked rotor.

For k=0.1 when we set D=0.1, it is observed that the phase space trajectory is identical to the unmodified kicked rotor mapping for k=0.9 which follows immediately that at this low value of k when the ordering parameter is below D=0.5, the system behaves aperiodically. This requires that we vary the ordering parameter from D=0.5 and above in order to obtain an invariant rotational orbit as depicted in figure 2*b*. This however is not a computational error since similar effect can be observed for k=0.5 with D=0.1 corresponding to the phase space trajectory of the unmodified map at k<0.8. On setting D=0.5 to 0.9 we obtain elliptical orbit (Figure 2*c-d*). Going by these observations, it is sufficient to state that for values of k below 0.9 one require to set D above 0.5 to observe a regular motion in the phase space. Dramatical deformation of the rotational orbits is observed at k=0.9, D=0.9 (Figure 2*e*), consequently chaotic behavior is manifested at k=1 with D=0.1 to 0.9 (Figure 2*f*-2*g*) contrary to the case with the unmodified rotor.

Furthermore we explore the effect of the ordering parameter at k=2. We can observe that with D=0.2 (Figure 2*h*) the phase space is not chaotically filled as opposed to the unmodified map. The webs of islands embedded in an extended chaotic sea are possibly organized by the two major stable orbits found in their centers. Consequently what is obtainable only at k= 1.2 (Figure 1*e*) for the unmodified map is now achieved at k=2 with D=0.2. Without argument we can agree that k=2 with D=0.5 (Figure 2*i*) corresponds to the phase space for k= 1.5 in the unmodified map with four major regular stable orbits (Figure 1*f*). This reveals that the ordering parameter plays a vital role of organizing regular motion in the chaotic region. But at k=2 with D=0.9 the phase space becomes smeared and the regular orbits are seen to diminish (Figure 2*j*). This chaotic behavior is being suppressed by the ordering parameter at k=3 with D=0.1 (Figure 2*k*) as oppose to the case in the unmodified map. However the limit to which regular orbits can be obtained for k=3 lies between D=0.1 to D=0.5 and above this value the system retains its full chaotic behavior as it is with the unmodified map.

Furthermore, it is only possible to obtain regular orbits for k=4 between D=0.1 to D=0.3 above which the phase space is filled with chaos. This same limit is observed for k=5 but for k=6 the limit lies between D=0.1 to D=0.2 only. It is thus reasonable to state that as the kicked strength increases the value of the ordering parameter D required for regular orbits must be low. Consequently from k=8 (Figure 2*m*) upwards, regular orbits are only visible with D=0.1 and appear to be diminishing at k-12 (Figure 2*n*). By implication, for k>12 there is no possibility of obtaining regular orbits even with D=0.1 in the phase space since the phase space is entirely occupied by chaotic behavior. The interesting point is that the ordering parameter allows us at least to theoretically observe regular motion for values of k>5 which is not possible with the unmodified map.

The results of phase space for the modified kicked rotor with subtractive term are presented in Figure 3. The application of the subtractive term in the modified kicked rotor standard map yield interesting results that differ significantly from the additive term with the same initial conditions. It was observed that rotational orbits can only be seen for k=0.2 with D=0.9 below which only hyperbolic and sinusoidal curves can be seen in the phase space. This is possibly due to the weak kick strength as suppressed by the ordering parameter. This requires that D should be high in order to obtain rotational orbits for low values of k. As can be seen in Figure 3*(a)* for k=0.3 when D=0.9 the motion follows an elliptical orbit which is our point of interest. This regular behavior persist up to k=0.8 with D=0.6 which agrees with the results obtained for the unmodified map. Dramatically for k=0.8 with D=0.9 the single elliptical orbit becomes unstable and split into two intertwine orbits seen in Figure 3*(b)* which signifies the onset of chaos.

As an evidence, the effect of the subtractive ordering parameter can be observed even as we increase the kick strength to k=0.9 with D=0.6 to D=0.9 (Figure 3*c*) the stable elliptical orbit resurfaced contrary to the additive ordering parameter. The subtractive term obviously yields a better result for this modification. To further Buttress this point, the unmodified map have only one single value of k=0 for which the phase space displays a constant momentum whereas for the modified map, the subtractive term there are three such set of values; (k=1, D=0.5), (k=4, D=0.2) and (k=9, D=0.1) for which one can obtain an invariant momentum in phase space (Figures 3*d*, 3*j* and 3*l*). That is the system follows the same trajectory for different set of k with D values. Theoretically this implies that for the subtractive term, the system possess resonant trajectories/orbits for some given values of k with respect to the ordering parameter D.

Our main aim is to ascertain whether the ordering parameter can subdue the chaotic behavior in this system. The result gotten so far has confirmed that Chaos can actually be suppressed as the aperiodic behavior observed for $k = 0.9$ (Figure 1*c*) in the unmodified map turn out to occur in the modified map for k=2 with D=0.65 (Figure 3*e*). With just a slight shift of D=0.651 we can see that the regular orbit splits into a disjoint unstable trajectory leaving the upper region of the phase space chaotic with some visible islands (Figure 3*f*). As we increase D the phase space becomes more chaotic (Fig 3*g*).

The possibility of having regular behavior in the system for higher values of k depends on low values of D and it was observed that even for k=17 with D=0.1 the system appear to be integrable. Any small increment above D=0.1 destroys the integrability of the system leading to a chaotic behavior. However for k=18 even with D=0.1 (Fig. 3*m*) the rotational stable orbit appears to be deformed and above this value there exists no such rotational stable orbit in phase space. That notwithstanding, this modification presents stable orbits in the phase space over a long range of kick strength unlike in the unmodified map as observed for k=25 with D=0.1 (Figure 3*n*) which gradually diminishes as we further increase k above this value (Figures 3*o*-3*p*). In general the ordering parameter serves as chaos control in the modified kicked rotor Hamiltonian system.

V. Physical Interpretation

The kicked rotor standard map can be applied to electro-acoustic music as a chaotic oscillator for generating harmonic and noisy sounds. A possible way to build a sound source base on this chaotic map was described by [19]. In his work he describe a method to interpret this system as an oscillator by considering the phase of the rotor as the phase of a wave table oscillator in which the p_0 coordinate of the initial points corresponds to the angular frequency of the oscillator according to the formula;

$$p_0 = \frac{2f}{f_0}$$

where f_0 is the sample rate of the system.

This formula however connects the initial conditions of the system with the frequency of the generated sound for the case where k=0 only. There is no known formula for the case where $k \neq 0$.

When the kick strength k=0, one obtain a phase generator with a constant frequency but in the limit $k \to \infty$ when the entire phase space is chaotic, the oscillator serves as a phase generator of noise. Thus it can be employed as an interpolating oscillator between noisy and harmonic sounds.

The most fascinating aspect of this map as a sound generator is in the region when $k \approx k_c$ where there are three types of trajectories; those that split the phase space which corresponds to sounds with definite pitch with some small deviations in their frequencies, those that are restricted to a small area of the phase space (small circles) which corresponds to sounds with strongly modulated frequencies and lastly those in the chaotic sea which corresponds to noise. The first two types of trajectories are closed implying that the motion returns to the initial point in the phase space after a periodic time relevant to that trajectory, hence these trajectories are called periodic orbits. In contrary, the last types of trajectories are open and fill the entire phase chaotic sea. The mixing effect of these trajectories creates really nice, yet unpredictable sounds achieved by simply tuning the kicking strength parameter k of the model and the initial conditions (p_0, q_0) of the system.

Despite these interesting features, it is practically impossible for one to predict whether a set of initial conditions (k, p_0, q_0) would result in a periodic orbit or an open chaotic one. It is even harder to guess whether an initial condition describing a periodic orbit corresponds to a trajectory that splits the entire phase space or it corresponds to a small circle. Base on these limitations, the chaotic map can only be employed as a low-frequency oscillator since it generates almost-constant values with small and unpredictable deviations in the low range of the parameter k.

Our modification presents the possibilities for the map to be employed as a high-frequency oscillator since it generates such constant frequencies even at high range of the kicking strength k. However, this does not simply require the tunning of the k-parameter but also carefully setting the ordering parameter D. Our modified model also provides the basis for generating sounds with resonant frequencies for specific values of the initial conditions (k, p_0, q_0) with appropriate D-control. Consequently for some specific values of the initial conditions with D-control, we can generate beautiful and unique harmonic sounds.

VI. Conclusions

We have investigated and presented the Modified Kicked Rotor Hamiltonian System by numerical simulation. The standard map for the kicked rotor has been studied and indeed it displays interesting behaviours (regular and chaotic). Most interestingly, the modified map presents us with an extension of the application of the kicked rotor even in the chaotic regime where it was initially considered undeterministic. We can conclude that our results are interesting not only for pure theoretical physicist but also for experimental physicist, because they are numerical and more general.

Acknowledgment

The authors wish to thank Prof. I. M. Echi who conceived of the presented idea and his supervisory advices in developing the research work. I appreciate Mr. Otor Daniel for his contribution in organizing the paper. I also acknowledge the financial support from my parents Mr. & Mrs. Washima Mue in the course of this work. Finally my heartfelt gratitude goes to my loving wife Juliet for her love and moral support in my course of study.

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