GSJ: Volume 9, Issue 8, August 2021, Online: ISSN 2320-9186
www.globalscientificjournal.com

# SIX-POINTS COSINE RUNGE KUTTA METHODS FOR SOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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## ABSTRACT

High order implicit Runge-Kutta method for solving first order differential equations is constructed, Cosine functions are used to obtain special points which are used to construct the high order implicit Runge-Kutta method. This method is an improvement of Gauss-quadrature Legendre methods. The nodes of this method is simpler than Gauss methods whose roots are too complicated to use in practice. Collocation approach at these special points are used to generate continuous schemes for the generation of discrete schemes also developed.

KEYWORDS

*Runge – Kutta Method, Ordinary Differential Equation (ODE), implicit method, collocation method, continuous scheme, cosine function, interpolation.*

## 1.0 INTRODUCTION

Numerical methods is one of the most important areas of mathematics use in solving real-life problems. The family of explicit Runge-kutta methods is quite rich, they may be ineffective for some problems. Indeed, we see that no explicit method is suitable for so called stiff problems, which frequently arise in practice, in particular from the spatial discretization of time dependent partial differential equations.

Runge-kutta methods are useful for solving non-linear stiff and oscillatory differential equation. A number of numerical methods for oscillatory problems have not fully been developed. We use generalized collocation techniques based on fitting the special points of cosine function by transforming the points to implicits Runge-Kutta methods for solving ordinary differential equations problems. The coefficients of the methods are functions of the frequency and the step-size.

### 1.1 Objective of the study

The objectives of this paper are:

i. To obtain the six special points of cosine function in $[0, \pi]$.

ii. To use the six points to generate nodes in $[0, 1]$ which are used to get the continuous scheme.

iii. To obtain Runge-Kutta method in solving general and oscillatory first order differential equations.

## 2. METHODOLOGY

### 2.1 Collocation method

This is the method which involves the determination of an approximate solution in a suitable set of functions called trial or basis functions. The approximate solution is required to satisfy the differential equation and its supplementary condition of certain points in the range of interest, called the collocation points.

The collocation methods by their very nature yield continuous solution and the principle behind multistep collocation is allowing the collocation polynomial to use information from previous points in the integration.

A collocation method is defined in the interval $[x_{k-1}, x_k]$ by a continuous scheme, from the initial value problem of ordinary differential equation.

$$y' = f(x,y), y(x_0) = y_0 \quad a \leq x \leq b \qquad 2.1$$

We assume our approximate solution to be in form of

$$y(x) = \sum_{j=0}^{t-1} d_j(x) y_{n+1} + h \sum_{j=0}^{m-1} v_j(x) f(\bar{x}_j, y(\bar{x}_j)) \qquad 2.2$$

Where $t$ denote the number of interpolation points $x_{n+j}$, $(j = 0,1, \dots t-1)$ and m denotes the distinct collocation points $\bar{x}_j (j = 0, \dots m-1), f(x,y)$ is continuous and differentiable. The numerical coefficients $d_j (j = 0,1, \dots k-1) and\ hv_j (j = o, 1, \dots, m-1)$ are elements of the $(t+m) \times (t+m)$ square matrix A.

The $d_j(x) and\ v_j(x)$ in (1.2) can be represented by polynomial of the form

$$d_j(x) = \sum_{j=0}^{t+m-1} d_{j.i+1} x^i, (j = 0,1, \dots, t-1) \qquad 2.3$$

$$hv_j(x) = \sum_{j=0}^{t+m-1} hv_{j.i+1} x^i, (j = 0,1, \dots, m-1) \qquad 2.4$$

With the constant coefficients $d_{j.i+1}\ and\ hv_{j.i+1}$ to be determined. Substituting (2.3) and (2.4) into (2.2) we have

$$y(x) = \sum_{j=0}^{t-1} \sum_{j=0}^{t+m-1} d_{j.i+1} x^i y_{n+j} + h \sum_{j=0}^{m-1} \sum_{j=0}^{t+m-1} v_{j.i+1} x^i f_{n+j}$$

$$y(x) = \sum_{j=0}^{t+m-1}\{\sum_{i=0}^{t-1} d_{j.i+1}y_{n+j} + \sum_{j=0}^{m-1} hv_{j.i+1}f_{n+j}\} x^i \quad 2.5$$

And let

$$a_j = (\sum_{i=0}^{t-1} d_{j.i+1}y_{n+j} + \sum_{i=0}^{m-1} hv_{j.i+1}f_{n+j}) \quad a_j \in R^i i = 0,1, \dots t+m-1 \quad 2.6$$

Such that (2.5) reduces to the power series of a single variable $x$ in the form

$$p(x) = \sum_{j=0}^{\infty} a_j x^i \quad 2.7$$

And (2.7) is used as the basis or trial function to produce an approximate solution to IVP as

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i \quad 2.8$$

Where $a_j \in R^j j = 0,1, \dots t+m-1$, and $U \in c^m(a,b) \mathrm{c} p(x)$

Thus equation (2.5) can be expressed explicitly in matrix form as follows

$$y(x) = (y_n, y_{n+1}, \dots, y_{n+k-1}, f_n, f_{n+1}, \dots, f_{n+m-1})A^T B$$

$$A=\begin{pmatrix} d_{0,1} & d_{1,1} & \cdots & d_{t-1,1} & hv_{0,1} & hv_{1,1} & \cdots & hv_{m-1,1} \\ d_{0,2} & d_{1,2} & \cdots & d_{t-1,2} & hv_{0,2} & hv_{1,2} & \cdots & hv_{m-1,2} \\ d_{0,3} & d_{1,3} & \cdots & d_{t-1,3} & hv_{0,3} & hv_{1,3} & \cdots & hv_{m-1,3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{0,t+m} & d_{1,t+m} & \cdots & d_{t-1,t+m} & hv_{0,t+m} & hv_{1,t+m} & \cdots & hv_{m-1,t+m} \end{pmatrix} \quad 2.9$$

With B defined as

$$B=\begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 0 & 1 & 2x_{n+c_1} & \cdots & (t+m-1)x_{n+c_1}^{t+m-2} \\ 0 & 1 & 2x_{n+c_2} & \cdots & (t+m-1)x_{n+c_2}^{t+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+c_{m-1}} & \cdots & (t+m-1)x_{n+c_{m-1}}^{t+m-2} \end{pmatrix} \quad 2.10$$

Where $c_j\,(j = 1,2, \ldots m-1)$ are collocation points chosen from the special points. Matrix B is called the multistep collocation matrix which has a very simple structure and of dimension $(t+m) \times (t+m)$ are the constant coefficients of the polynomials given in (2.6)

**Derivation of six points cosine function in [0,$\pi$]**

Points of cosine functions are special values of the $cos\theta$ in $[0,\pi]$ where $\theta$ are: 0, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$ ,$\frac{3\pi}{4}$ and $\pi$, such that

$$\cos(0) = 1$$

$$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos(\frac{\pi}{3}) = 1/2$$

$$\cos(\frac{2\pi}{3}) = -1/2$$

$$\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\cos(\pi) = -1$$

**Special Points**: special points are the transformed point of a cosine function form [-1, 1] onto the interval [0, 1] by a linear transformation.

$$T(x_i) = \frac{1}{2}(1 \pm x_i) \quad [\,\textit{Agam}\ (2015)]$$

Where $x_i$ are the values of cosine functions, given by

$$\cos(0) = 1,\ \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}, \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}, \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \cos(\pi) = -1,$$

Where $x_1 = 1, x_2 = \frac{\sqrt{2}}{2}, x_3 = \frac{1}{2}, x_4 = -\frac{1}{2}, x_5 = -\frac{\sqrt{2}}{2}, x_6 = -1$

Transformation:

$T(0) = \frac{1}{2}(1+0) = \frac{1}{2}$ $, T(\frac{\pi}{4}) = \frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right) = \frac{1}{2}+\frac{\sqrt{2}}{4}, T\left(\frac{\pi}{3}\right) = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$

$T\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}, T\left(\frac{3\pi}{4}\right) = \frac{1}{2}\left(1-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}-\frac{\sqrt{2}}{4}, T(\pi) = \frac{1}{2}(1-1) = 0$

The new special points after arranging it in ascending order are now

$p_1 = 0\,,\ \ p_2 = \frac{1}{2}-\frac{\sqrt{2}}{4}\,,\ p_3 = \frac{1}{4}\,,\ p_4 = \frac{3}{4}\,, p_5 = \frac{1}{2}+\frac{\sqrt{2}}{4}\,,\ p_6 = 1$

We assume a power series solution of six points of the form

$y(x) = \sum_{j=0}^{6} d_j x^i,\ y'(x) = \sum_{j=0}^{6} i d_j x^{i-1}$

Interpolation at $x_n$ and collocate at $x_n = x_n \ +p_i$, where $i = 0,1,\ldots,6$, yield a system of simultaneous equation of the form

$y_n = d_0 + d_1 x_n + d_2 x_n^2 + d_3 x_n^3 + d_4 x_n^4 + d_5 x_n^5 + d_6 x_n^6$

$y'_{n+p_1} = f_{n+p_1} = 0 + d_1 + 2d_2 x_{n+p_1} + 3d_3 x_{n+p_1}^2 + 4d_4 x_{n+p_1}^3 + 5d_5 x_{n+p_1}^4 + 6d_6 x_{n+p_1}^5$

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$y'_{n+p_6} = f_{n+p_i} = 0 + d_1 + 2d_2 x_{n+p_6} + 3d_3 x_{n+p_6}^2 + 4d_4 x_{n+p_6}^3 + 5d_5 x_{n+p_6}^4 + 6d_6 x_{n+p_6}^5$

(2.11))

where $d_j$ are to be determined.

Thus, (2.11) can be rewritten in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & 1 & 2x_{n+p_1} & 3x_{n+p_1}^2 & 4x_{n+p_1}^3 & 5x_{n+p_1}^4 & 6x_{n+p_1}^5 \\ 0 & 1 & 2x_{n+p_2} & 3x_{n+p_3}^2 & 4x_{n+p_2}^3 & 5x_{n+p_2}^4 & 6x_{n+p_2}^5 \\ 0 & 1 & 2x_{n+p_3} & 3x_{n+p_3}^2 & 4x_{n+p_3}^3 & 5x_{n+p_3}^4 & 6x_{n+p_3}^5 \\ 0 & 1 & 2x_{n+p_4} & 3x_{n+p_4}^2 & 4x_{n+p_4}^3 & 5x_{n+p_4}^4 & 6x_{n+p_4}^5 \\ 0 & 1 & 2x_{n+p_5} & 3x_{n+p_5}^2 & 4x_{n+p_5}^3 & 5x_{n+p_5}^4 & 6x_{n+p_5}^5 \\ 0 & 1 & 2x_{n+p_6} & 3x_{n+p_6}^2 & 4x_{n+p_6}^3 & 5x_{n+p_6}^4 & 6x_{n+p_6}^5 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} y_n \\ f_{n+p_1} \\ f_{n+p_2} \\ f_{n+p_3} \\ f_{n+p_4} \\ f_{n+p_5} \\ f_{n+p_6} \end{pmatrix}$$

i.e

BA=Y, where

$$B=\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 0 & 1 & 2x_{n+p_1} & 3x_{n+p_1}^2 & 4x_{n+p_1}^3 & 5x_{n+p_1}^4 & 6x_{n+p_1}^5 \\ 0 & 1 & 2x_{n+p_2} & 3x_{n+p_3}^2 & 4x_{n+p_2}^3 & 5x_{n+p_2}^4 & 6x_{n+p_2}^5 \\ 0 & 1 & 2x_{n+p_3} & 3x_{n+p_3}^2 & 4x_{n+p_3}^3 & 5x_{n+p_3}^4 & 6x_{n+p_3}^5 \\ 0 & 1 & 2x_{n+p_4} & 3x_{n+p_4}^2 & 4x_{n+p_4}^3 & 5x_{n+p_4}^4 & 6x_{n+p_4}^5 \\ 0 & 1 & 2x_{n+p_5} & 3x_{n+p_5}^2 & 4x_{n+p_5}^3 & 5x_{n+p_5}^4 & 6x_{n+p_5}^5 \\ 0 & 1 & 2x_{n+p_6} & 3x_{n+p_6}^2 & 4x_{n+p_6}^3 & 5x_{n+p_6}^4 & 6x_{n+p_6}^5 \end{pmatrix}$$



$A = (d_0, d_1 d_2, d_3, d_4, d_5, d_6)^T$,

$$Y = \ (y_n, f_{n+p_1}\ , f_{n+p_2}\ \ , f_{n+p_3}\ \ , f_{n+p_4}\ \ , f_{n+p_5}, f_{n+p_6})^T$$

Using MAPLE mathematical software we obtain the continuous scheme of the form

$$y(x) = y_n + h(f_{n+p_1} + f_{n+p_2} + f_{n+p_3} + f_{n+p_4} + f_{n+p_5} + f_{n+p_6})$$

Now evaluating the continuous scheme at $p_1$= (0), $p_2$= $(\frac{1}{2} - \frac{\sqrt{2}}{4})$, $p_3$ =$\frac{1}{4}$, $p_4$ =$\frac{3}{4}$, $p_5$= $(\frac{1}{2} + \frac{\sqrt{2}}{4})$,

$p_6$=1

We obtain the discrete schemes: by putting $x_n = 0$ in the continuous scheme, we give

$$y_{n+p_1} = y_n$$

$$y_{n+p_2} = y_n + \left(\frac{107}{1440} - \frac{\sqrt{2}}{60}\right) hf_{n+p_1} + \left(-\frac{1}{15} + \frac{23\sqrt{2}}{160}\right) hf_{n+p_2} + \left(\frac{43}{180} - \frac{\sqrt{2}}{5}\right) hf_{n+p_3}$$
$$+ \left(\frac{53}{180} - \frac{\sqrt{2}}{5}\right) hf_{n+p_4} + \left(-\frac{1}{15} + \frac{19\sqrt{2}}{480}\right) hf_{n+p_5} + \left(\frac{37}{1440} - \frac{\sqrt{2}}{60}\right) hf_{n+p_6}$$

$$y_{n+p_3} = y_n + \left(\frac{31}{640}\right) hf_{n+p_1} + \left(\frac{11}{120} + \frac{9\sqrt{2}}{128}\right) hf_{n+p_2} + \left(\frac{1}{120}\right) hf_{n+p_3} + \left(\frac{1}{120}\right) hf_{n+p_4}$$
$$+ \left(\frac{11}{120} - \frac{9\sqrt{2}}{128}\right) hf_{n+p_5} + \left(\frac{1}{640}\right) hf_{n+6}$$

$$y_{n+p_4} = y_n + \left(\frac{63}{640}\right) hf_{n+p_1} + \left(-\frac{9}{40} + \frac{9\sqrt{2}}{128}\right) hf_{n+p_2} + \left(\frac{21}{40}\right) hf_{n+p_3} + \left(\frac{21}{40}\right) hf_{n+p_4}$$
$$+ \left(-\frac{9}{40} - \frac{9\sqrt{2}}{128}\right) hf_{n+p_5} + \left(\frac{33}{640}\right) hf_{n+p_6}$$

$$y_{n+p_5} = y_n + \left(\frac{107}{1440} + \frac{\sqrt{2}}{60}\right) hf_{n+p_1} + \left(-\frac{1}{15} - \frac{19\sqrt{2}}{480}\right) hf_{n+p_2} + \left(\frac{43}{180} + \frac{\sqrt{2}}{5}\right) hf_{n+p_3}$$
$$+ \left(\frac{53}{180} + \frac{\sqrt{2}}{5}\right) hf_{n+p_4} + \left(-\frac{1}{15} - \frac{23\sqrt{2}}{160}\right) hf_{n+p_5} + \left(\frac{37}{1440} + \frac{\sqrt{2}}{60}\right) hf_{n+p_6}$$

$$y_{n+p_6} = y_n + \left(\frac{1}{10}\right) hf_{n+p_1} + \left(-\frac{2}{15}\right) hf_{n+p_2} + \left(\frac{8}{15}\right) hf_{n+p_3} + \left(\frac{8}{15}\right) hf_{n+p_4} + \left(-\frac{2}{15}\right) hf_{n+5} +$$
$$\left(\frac{1}{10}\right) hf_{n+p_6} \qquad 3.12$$

The discrete schemes (3.12) must satisfy the (1.3), to change to Runge-Kutta formula, Hence

$y'_{n+p_1} = f(x_{n+p_1}, y_{n+p_1})$ , if $p_1 = 0$

$y'_{n+p_1} = f(x_{n+p_1}, y_n + 0) = f(x_{n+p_1}, y_n)$

$$y'_{n+p_2} = f(x_{n+p_2}, y_{n+p_2})$$

$$y'_{n+p_2} = f\left(x_{n+p_2}, y_n + \left(\frac{107}{1440} - \frac{\sqrt{2}}{60}\right)hf_{n+p_1} + \left(-\frac{1}{15} + \frac{23\sqrt{2}}{160}\right)hf_{n+p_2} + \left(\frac{43}{180} - \frac{\sqrt{2}}{5}\right)hf_{n+p_3} + \left(\frac{53}{180} - \frac{\sqrt{2}}{5}\right)hf_{n+p_4} + \left(-\frac{1}{15} + \frac{19\sqrt{2}}{480}\right)hf_{n+p_5} + \left(\frac{37}{1440} - \frac{\sqrt{2}}{60}\right)hf_{n+p_6}\right)$$

$$y'_{n+p_3} = f(x_{n+p_3}, y_{n+p_3})$$

$$y'_{n+p_3} = f\left(x_{n+p_3}, y_n + \left(\frac{31}{640}\right)hf_{n+p_1} + \left(\frac{11}{120} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(\frac{1}{120}\right)hf_{n+p_3} + \left(\frac{1}{120}\right)hf_{n+p_4} + \left(\frac{11}{120} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_5} + \left(\frac{1}{640}\right)hf_{n+p_6}\right)$$

$$y'_{n+p_4} = f(x_{n+p_4}, y_{n+p_4})$$

$$y'_{n+p_4} = f\left(x_{n+p_4}, y_n + \left(\frac{63}{640}\right)hf_{n+p_1} + \left(-\frac{9}{40} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(\frac{21}{40}\right)hf_{n+p_3} + \left(\frac{21}{40}\right)hf_{n+p_4} + \left(-\frac{9}{40} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_5} + \left(\frac{33}{640}\right)hf_{n+p_6}\right)$$

$$y'_{n+p_5} = f(x_{n+p_5}, y_{n+p_5})$$

$$y'_{n+p_5} = f\left(x_{n+p_5}, y_n + \left(\frac{107}{1440} + \frac{\sqrt{2}}{60}\right)hf_{n+p_1} + \left(-\frac{1}{15} - \frac{19\sqrt{2}}{480}\right)hf_{n+p_2}\left(\frac{43}{180} + \frac{\sqrt{2}}{5}\right)hf_{n+p_3} + \left(\frac{53}{180} + \frac{\sqrt{2}}{5}\right)hf_{n+p_4} + \left(-\frac{1}{15} - \frac{23\sqrt{2}}{160}\right)hf_{n+p_5} + \left(\frac{37}{1440} + \frac{\sqrt{2}}{60}\right)hf_{n+p_6}\right)$$

$$y'_{n+p_6} = f(x_{n+p_6}, y_{n+p_6})$$

$$y'_{n+p_6} = f\left(x_{n+p_6}, y_n + \left(\frac{1}{10}\right)hf_{n+p_1} + \left(-\frac{2}{15}\right)hf_{n+p_2} + \left(\frac{8}{15}\right)hf_{n+p_3} + \left(\frac{8}{15}\right)hf_{n+p_4} + \left(-\frac{2}{15}\right)hf_{n+5} + \left(\frac{1}{10}\right)hf_{n+p_6}\right)$$

Putting

$$y'_{n+p_1} = f(x_{n+p_1}, y_{n+p_1}) = f_{n+p_1} = k_1 \quad y'_{n+p_2} = f(x_{n+p_2}, y_{n+p_2}) = f_{n+p_2} = k_2,$$

$$y'_{n+p_3} = f(x_{n+p_3}, y_{n+p_3}) = f_{n+p_3} = k_3, \quad y'_{n+p_4} = f(x_{n+p_4}, y_{n+p_4}) = f_{n+p_4} = k_4,$$

$$y'_{n+p_5} = f(x_{n+p_5}, y_{n+p_5}) = f_{n+p_5} = k_5, \quad y'_{n+p_6} = f(x_{n+p_6}, y_{n+p_6}) = f_{n+p_6} = k_6$$

We obtain the function evaluation as

$$k_1 = 0$$

$$k_2 = f\left(x_n + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)h, y_n + \left(\frac{107}{1440} - \frac{\sqrt{2}}{60}\right)hf_{n+p_1} + \left(-\frac{1}{15} + \frac{23\sqrt{2}}{160}\right)hf_{n+p_2}\left(\frac{43}{180} - \frac{\sqrt{2}}{5}\right)hf_{n+p_3} + \left(\frac{53}{180} - \frac{\sqrt{2}}{5}\right)hf_{n+p_4} + \left(-\frac{1}{15} + \frac{19\sqrt{2}}{480}\right)hf_{n+p_5} + \left(\frac{37}{1440} - \frac{\sqrt{2}}{60}\right)hf_{n+p_6}\right)$$

$$k_3 = f\left(x_n + \left(\frac{1}{4}\right)h, y_n + \left(\frac{31}{640}\right)hf_{n+p_1} + \left(\frac{11}{120} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(\frac{1}{120}\right)hf_{n+p_3} + \left(\frac{1}{120}\right)hf_{n+p_4} + \left(\frac{11}{120} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_5} + \left(\frac{1}{640}\right)hf_{n+p_6}\right)$$

$$k_4 = f\left(x_n + \left(\frac{3}{4}\right)h, y_n + \left(\frac{63}{640}\right)hf_{n+p_1} + \left(-\frac{9}{40} + \frac{9\sqrt{2}}{128}\right)hf_{n+p_2} + \left(\frac{21}{40}\right)hf_{n+p_3} + \left(\frac{21}{40}\right)hf_{n+p_4} + \left(-\frac{9}{40} - \frac{9\sqrt{2}}{128}\right)hf_{n+p_5} + \left(\frac{33}{640}\right)hf_{n+p_6}\right)$$

$$k_5 = f\left(x_n + \left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right)h, y_n + \left(\frac{107}{1440} + \frac{\sqrt{2}}{60}\right)hf_{n+p_1} + \left(-\frac{1}{15} - \frac{19\sqrt{2}}{480}\right)hf_{n+p_2} + \left(\frac{43}{180} + \frac{\sqrt{2}}{5}\right)hf_{n+p_3} + \left(\frac{53}{180} + \frac{\sqrt{2}}{5}\right)hf_{n+p_4} + \left(-\frac{1}{15} - \frac{23\sqrt{2}}{160}\right)hf_{n+p_5} + \left(\frac{37}{1440} + \frac{\sqrt{2}}{60}\right)hf_{n+p_6}\right)$$

$$k_6 = f\left(x_n + (1)h, y_n + \left(\frac{1}{10}\right)hf_{n+p_1} + \left(-\frac{2}{15}\right)hf_{n+p_2} + \left(\frac{8}{15}\right)hf_{n+p_3} + \left(\frac{8}{15}\right)hf_{n+p_4} + \left(-\frac{2}{15}\right)hf_{n+5} + \left(\frac{1}{10}\right)hf_{n+p_6}\right)$$

The weight $b = (b_1, b_2, b_3, b_4, b_5, b_6)$, evaluating the continuous scheme at $x = x_n + h$, we obtain $b = \left(\frac{3}{30}, -\frac{4}{30}, \frac{16}{30}, \frac{16}{30}, -\frac{4}{30}, \frac{3}{30}\right)$

The six points general formula for Runge-kutta method is defined as

$$y_{n+1} = y_n + h\sum_{j=1}^{6} b_j k_j = y_n + \frac{3}{30}h(k_1 + k_6) - \frac{4}{30}h(k_2 + k_5) + \frac{16}{30}h(k_3 + k_4)$$

Where $k_j, j = 1,2, \dots 6$ are given by (2.2), can be summarized in table below

Table 2.0 Butcher table summary

| | | |
|---|---|---|
| C | A | U |
| | B | V |

Where $C = (c_1, c_2 \dots c_6)^T$, $A = a_{ij}, ij = 1,2 \dots 6$, $U = (1,1,1,1,1,1)^T$, $V = (1)$,

$B = (b_1, b_2 \dots b_6)^T$

We can represent the Butchers' tableau

Table 2.1 Butcher table summary of six point cosine function Runge-Kutta method

| $C$ | $A = (a_{ij}), i, j = 1,2, \dots 6$<br>$A = (a_{ij})$ | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{1}{2} - \frac{\sqrt{2}}{4}$ | $\left(\frac{107}{1440} - \frac{\sqrt{2}}{60}\right)$ | $\left(-\frac{1}{15} + \frac{23\sqrt{2}}{160}\right)$ | $\left(\frac{43}{180} - \frac{\sqrt{2}}{5}\right)$ | $\left(\frac{53}{180} - \frac{\sqrt{2}}{5}\right)$ | $\left(-\frac{1}{15} + \frac{19\sqrt{2}}{480}\right)$ | $\left(\frac{37}{1440} - \frac{\sqrt{2}}{60}\right)$ |
| $\frac{1}{4}$ | $\left(\frac{31}{640}\right)$ | $\left(\frac{11}{120} + \frac{9\sqrt{2}}{128}\right)$ | $\left(\frac{1}{120}\right)$ | $\left(\frac{1}{120}\right)$ | $\left(\frac{11}{120} - \frac{9\sqrt{2}}{128}\right)$ | $\left(\frac{1}{640}\right)$ |
| $\frac{3}{4}$ | $\left(\frac{63}{640}\right)$ | $\left(-\frac{9}{40} + \frac{9\sqrt{2}}{128}\right)$ | $\left(\frac{21}{40}\right)$ | $\left(\frac{21}{40}\right)$ | $\left(-\frac{9}{40} - \frac{9\sqrt{2}}{128}\right)$ | $\left(\frac{33}{640}\right)$ |
| $\frac{1}{2} + \frac{\sqrt{2}}{4}$ | $\left(\frac{107}{1440} + \frac{\sqrt{2}}{60}\right)$ | $\left(-\frac{1}{15} - \frac{19\sqrt{2}}{480}\right)$ | $\left(\frac{43}{180} + \frac{\sqrt{2}}{5}\right)$ | $\left(\frac{53}{180} + \frac{\sqrt{2}}{5}\right)$ | $\left(-\frac{1}{15} - \frac{23\sqrt{2}}{160}\right)$ | $\left(\frac{37}{1440} + \frac{\sqrt{2}}{60}\right)$ |
| 1 | $\left(\frac{1}{10}\right)$ | $\left(-\frac{2}{15}\right)$ | $\left(\frac{8}{15}\right)$ | $\left(\frac{8}{15}\right)$ | $\left(-\frac{2}{15}\right)$ | $\left(\frac{1}{10}\right)$ |
| $b$ | $\frac{3}{30}$ | $-\frac{4}{30}$ | $\frac{16}{30}$ | $\frac{16}{30}$ | $-\frac{4}{30}$ | $\frac{3}{30}$ |

Where $a_{ij}{}'s$ are coefficients

### 3.0 Analysis of Results

1. Consistency: the Runge-Kutta method is consistent since it satisfy the consistency condition

$\sum_{j=1}^{6} a_{ij} = c_i$,$\sum_{j=1}^{6} b_i = 1$, (see Table3.1)

2. Stability: the stability of the method is investigated by considering the linear test equation.

$$y' = \lambda y, \quad \lambda \in \mathbb{C}$$

Putting $Z = \lambda h, \quad h \in (0,1)$

The stability function is $R(Z)$

$$R(Z) = I + Zb^T(I - ZA)^{-1}e$$

Where I is the identity matrix, $b = (b_1, b_2, b_3, b_4, b_5, b_6)$ is the weight, e $= (1,1,1,1,1,1)$, A is the Runge –Kutta matrix of the coefficient of the butcher .

The Runge-Kutta method is generally represented by a butcher's tableau as

Table 2.4 Butcher table summary

| | |
|---|---|
| C | A |
| | b |

Where $C = (c_1, c_2 \ldots c_6)^T$is the abscissae or Gausian nodes, the transformed zeros of special points on the interval$[0,1]$, $b = (b_1, b_2, b_3, b_4, b_5, b_6)$, is the weight of the method 1, $A = (a_{ij}), i, j = 1,2, \ldots 6$, is the coefficient matrix for the method.

The stability domain, $R(Z)$ is defined as $R(Z) = \{Z \in \mathbb{C} / R(z) / \leq 1\}$ , R (Z) is a rational polynomial defined by

$R(z) = \frac{\det(I - ZA + Zeb^T)}{\det(I - ZA)}$, where e $= (1,1,1,1,1,1), Z = \lambda h.$ 3.24

$\lambda \in \mathbb{C}$**,** h is the step-size, $\mathbb{C}$ is the set of complex numbers.

For the six point cosine function, with $b = \left(\frac{3}{30}, -\frac{4}{30}, \frac{16}{30}, \frac{16}{30}, -\frac{4}{30}, \frac{3}{30}\right)$,

$A = (a_{ij}), i, j = 1,2, \dots 6,$

$a_{11} = 0 \quad - \quad - \quad - \quad - \quad - \quad a_{16} = 0$

$a_{21} = \frac{107}{1440} - \frac{\sqrt{2}}{60}, a_{22} = -\frac{1}{15} + \frac{23\sqrt{2}}{160}, a_{23} = \frac{43}{180} - \frac{\sqrt{2}}{5}, a_{24} = \frac{53}{180} - \frac{\sqrt{2}}{5}, a_{25} = -\frac{1}{15} + \frac{19\sqrt{2}}{480},$

$a_{26} = \frac{37}{1440} - \frac{\sqrt{2}}{60}.$

- - - - - -

$a_{61} = \left(\frac{1}{10}\right), a_{62} = \left(-\frac{2}{15}\right), a_{63} = \left(\frac{8}{15}\right), a_{64} = \left(\frac{8}{15}\right), a_{65} = \left(-\frac{2}{15}\right), a_{66} = \left(\frac{1}{10}\right)$

Simplifying (3.24) using MAPLE or MATLAB, and substituting the values of $a_{ij}$, and b for s=6. We obtain the rational function.

$$R(Z) = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + - - - + \frac{Z^6}{6!}$$

Since $R(Z)$ is precisely the truncated exponential series

$$R(Z) = 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + - - - + \frac{Z^s}{s!}$$

Which is bounded, and stability functions are always bounded. Thus our method is stable.

## 4.0 CONCLUSION

In this paper, we made use of points of cosine functions in generating special points which we used to develop our continuous scheme of six point points of special-cosine function Runge-Kutta methods. This method is used to solve general and oscillatory problems in first order ODE.

Yakubu & Babu (2011) on Runge-kutta collocation method of six stages and order six for solving first order differential equations is studied and improved.

Therefore, We have derived new more efficient method, with simple collocation points, which is less tedious, less complicated than most methods in literature so far.

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