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THE INTERDEPENDENCE OF GEOMETRY AND SOME BRANCHES OF MATHEMATICS

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ABSTRACT

This paper is a new piece in a sequence of papers and books the author has been writing, with the purpose to analyze some strange and poorly explained results and certain open questions we face under current mathematics. Thinking "outside the box", he assumed the hypothesis that mistaken math fundamentals and premises could be behind these math oddities and granted himself the right to follow a completely free reasoning approach in order to question the so far unshakable math framework and identify why these oddities occur and what to do to overcome them. His approach is in line with the idea of joining apparently independent areas of knowledge to analyze math, a unified view of this science. In a book published in 2019, he stated that a circumference geometric property, mathematically expressed by the quadratic relationship, "$x^2 + y^2 = (2R)^2 =$ Constant", has a broader scope than currently accepted. He demonstrated his statement with the help of a theorem he enunciated in that book and named "Theorem of Infinite Right Triangles". In this paper, he emphasized the interdependence of geometry and some branches of math (number theory, algebra and trigonometry) by showing that, explicitly or not, a circumference law expressed by that magic relationship rules every open or closed polygonal inscribed in a circumference, as well as three famous math subjects: Pythagoras' Theorem, Fermat's Conjecture (Fermat's Last Theorem) and Beal's Conjecture. In the referred quadratic relationship, the constant value "$(2R)^2$" is the square of the circumference diameter, while "x" and "y" are circumference chords from any common point in the circumference contour line, tied to the ends of the chosen diameter, since the author refers to plane geometry. This unbreakable connection between geometry and mathematics shows that Pythagoras' Theorem and Beal's Conjecture represent right triangles and are particular cases of the referred Theorem of Infinite Right Triangles, while Fermat's Conjecture expresses the nonexistence of right triangles under certain conditions imposed on the numbers in Fermat's math expression, and is a corollary of the author's Theorem.

INTRODUCTION

Mathematics as a scientific language that represents and describes the subjects of studies of the other sciences, their properties and/or phenomena we find in nature. Differently from physics, astronomy and other fields of work, including geometry, which are sciences that deal with a pre-existing subject of studies, mathematics does not have a pre-existing subject of studies. Math deals with its own postulates, principles and methods. Some people even say that mathematics is not a true science[1].

Applied mathematics is the math segment that serves other sciences. It means that, whenever dealing with a matter pertaining to any science, mathematics must fully comply with the requirements ruling said matter, whatever the science it serves.

Pure mathematics is the other math segment, which aims to serve itself, but ultimately with the purpose to improve applied mathematics. Then, every development within this math segment will have to be consistent with the requirements of the science said development would eventually apply[2].

The use of the Cartesian system is the most obvious evidence of the interdependence of geometry and algebra, what implies the use of arithmetic and/or algebraic operations and rules in connection with geometric laws. When we algebraically represent a real geometric figure, as an ellipse, math must comply with the geometric law that rules the referred geometric figure. Nevertheless, going in the opposite direction, when we search for a geometric figure that correspond to a random algebraic expression we may end up with a non-existing geometric figure, as an elliptic curve[3], and we cannot verify that compliance, since – as an abstraction - an imaginary figure does not follow a geometric law and is beyond rebuttal arguments.

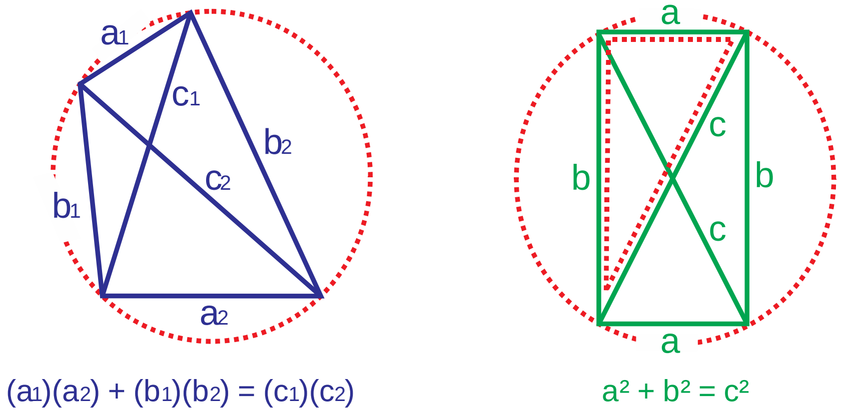
Keeping in mind this line of reasoning, this paper deals with three famous math subjects as seen under that referred interdependence of geometry and number theory, trigonometry and algebra: Pythagoras Theorem, Fermat's Conjecture (Fermat's Last Theorem) and Beal's Conjecture, since their respective math expressions represent right triangles.

Math books teach us that Ptolemy's Theorem is a property of the quadrilaterals on the circumference, and that Pythagoras's Theorem is a property of the right triangles. In a book published in 2019[4], I disagreed with these particular understandings by stating that Ptolemy's Theorem and Pythagoras's Theorem comply with a same circumference property, mathematically expressed by the quadratic relationship, "$x^2 + y^2 = (2R)^2 =$ Constant", in accordance with the Theorem of Infinite Right Triangles I enunciated in the referred book.

In my view, this geometric property has a broader scope than currently accepted, and encompasses these two theorems and other matters. I also emphasized that Pythagoras' Theorem, ($a^2 + b^2 = c^2$), is a particular case of Ptolemy's Theorem, ($a_1.a_2 + b_1.b_2 = c_1.c_2$), when the quadrilateral becomes a rectangle, as we see in Figure 1.

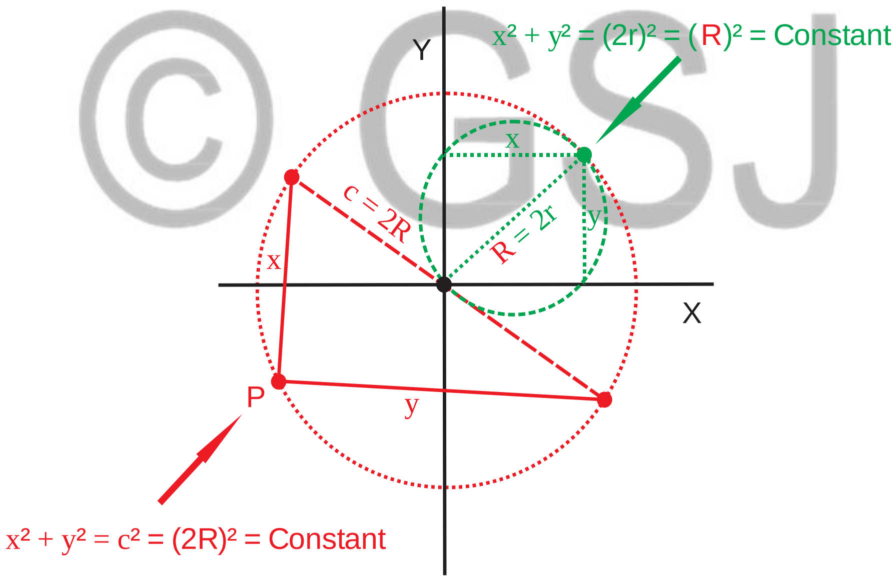
[1] To illustrate the statement, when Alfred Nobel laid out the Nobel Prize in his will, he did not include mathematics as a science entitled to it.
[2] As I understand, a concept not presently accepted, as if math were an independent and self-sufficient science, disconnected from the real world.
[3] This geometric figure does not exist in nature and its shape will depend on the representation method (presently, the Cartesian system).
[4] AMUI, Sandoval, *A Circunferência, Pitágoras e Fermat*, Editora Catalivros, Rio de Janeiro-RJ, 2019.

Figure 1: Pythagoras's Theorem as a particular case of Ptolemy's Theorem



As we see in Figure 2, it is relevant to keep in mind that I am in the field of plane geometry, and not dealing with the Cartesian method. In the referred quadratic relationship "$x^2 + y^2 = (2R)^2 =$ Constant", the constant value "$(2R)^2$" is the square of the circumference diameter, and "x" and "y" are circumference chords from a common point "P" in the circumference contour line, tied to the ends of the chosen diameter.

Figure 2: The circumference quadratic relationship



I enunciated the Theorem of Infinite Right Triangles as a property of the circumference, and phrased it as follows:

> *Every circumference circumscribes a group of an infinite number of right triangles, all of them with a common hypotenuse, which is the diameter of the circumference. The legs of the infinite right triangles of a same group may freely vary, provided the sum of the squares of the numbers that represent the legs remains constant, and equal to the square of the number that represents the diameter of the circumference. Clearly, the circumference diameter also is the hypotenuse of every right triangle belonging to a same group of right triangles.*

This Theorem implies the following corollary:

> *The existence of one right triangle implies the existence of a whole group of an infinite number of right triangles with a same hypotenuse ("h") and variable legs ("x" and "y"), their unique hypotenuse being the diameter of the circumference that circumscribes all of said right triangles, whatever its value. All right triangles belonging to that group follow the quadratic relationship "$x^2 + y^2 = h^2 = Constant$".*

A group of right triangles in a particular circumference may or may not encompass a subgroup of Pythagorean triples, as well as a subgroup of powers that satisfy Beal's Conjecture.

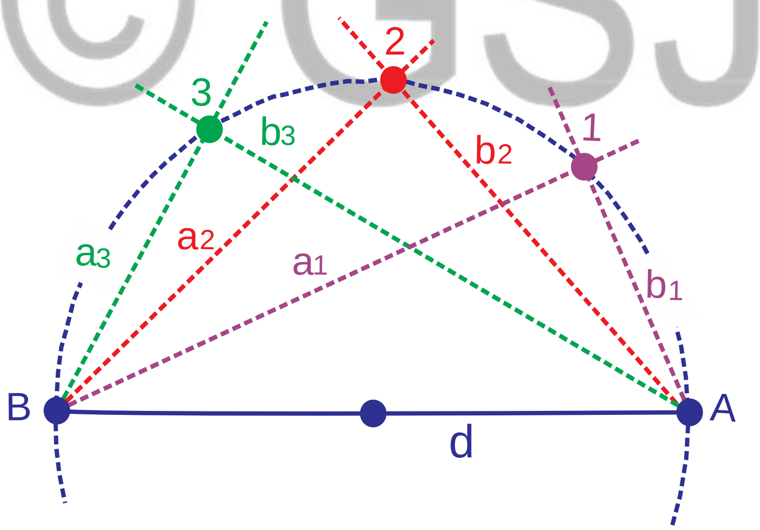
Algebraically speaking, as we see in Figure 3:

$$a_1^2 + b_1^2 = a_2^2 + b_2^2 = a_3^2 + b_3^2 \ldots = d^2 = \text{Constant}$$

Said quadratic relationship remains whatever the position of a point in the circumference contour line (1, 2, 3 …). When that point coincides with one of the ends of the diameter used as the hypotenuse, as in the case of point "B", when "$a = 0$" and "$b = c$", no right triangle occurs, but the relationship remains ("$0^2 + b^2 = c^2$"). We may mathematically generalize the circumference quadratic relationship[5] as follows:

$$x^2 + y^2 = \text{Constant}$$

Figure 3: A group of an infinite number of right triangles with a same hypotenuse

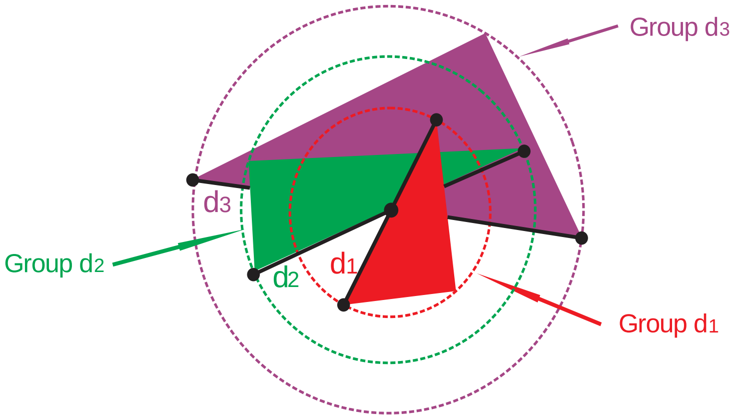


For each new value of a hypotenuse, we will have a new circumference whose diameter is equal to the new hypotenuse, and a new group of an infinite number of right triangles into that circumference, as in Figure 4. If we put together all these groups of right triangles, we have a universal family of all possible right triangles, each individual group ruled by the circumference quadratic relationship, "$x^2 + y^2 = \text{Constant}$", where

[5] There are other reasons to accept the statement, as listed in the referred book published in 2019.

said "Constant" is the square of the diameter of the elected circumference ("d_1", "d_2", "d_3" …).

Figure 4: Universal family of right triangles



I am aware of a well-known geometric property, which says that the geometric place generated when we move the right angle formed by two straight-line segments tied to the ends of a third straight-line segment is a semi-circle. However, I did not find a proper and comprehensive understanding of the broader meaning and applications of that circumference quadratic relationship, ruling an infinite number of right triangles with a same hypotenuse inscribed into a circumference beyond the restricted application to the right triangles, as I did in previous works and as I do in this paper. This extended concept reaches Pythagoras' Theorem, Fermat's Conjecture, Beal's Conjecture and inscribed polygonal lines in general[6].

In previous works, as well as in this paper, I decided to think "outside the box" and question some of the so far untouchable math fundamentals and premises, with the purpose to identify the causes of some strange results and open questions we face under current math[7]. Next, adopting a reasoning free from any pre-established concept or understanding, I questioned myself about what to do to overcome these oddities and improve the use of math as a scientific language, by putting together apparently independent areas of knowledge to deal with these questionable math matters. I used the interdependence of geometry and number theory, algebra and trigonometry to handle and explain the subjects of this paper.

Circumference chords

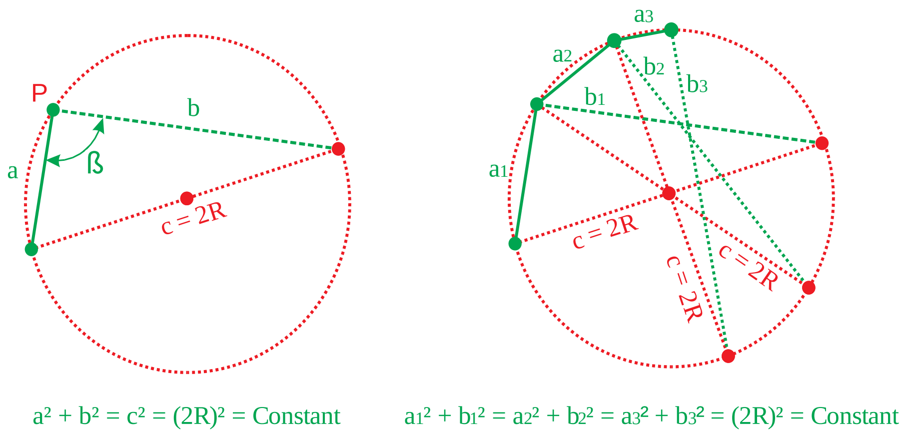
As we see in Figure 5, two circumference chords, "a" and "b", from a point "P" in a circumference contour line, tied to the ends of any diameter "c", whatever their lengths, encompass a 180-degree arch. It means they form a right angle "ß" of a right triangle with legs "a" and "b", and a hypotenuse "c" equal to the circumference diameter "2R". If we stay within a same circumference, all possible pair of chords, "a_1, b_1", "a_2, b_2", …, will comply with that circumference quadratic relationship "$a_1^2 + b_1^2 = a_2^2 + b_2^2 = … =$

[6] Including irregular polygons, whenever their vertices remain in the contour line of a same circumference.

[7] AMUI. Sandoval, You may not enjoy mathematics (but you do not have to hate it), AYA Editora, 2022.

$(2R)^2$ = Constant", whatever their lengths. My Theorem of Infinite Right Triangles expresses this circumference property.

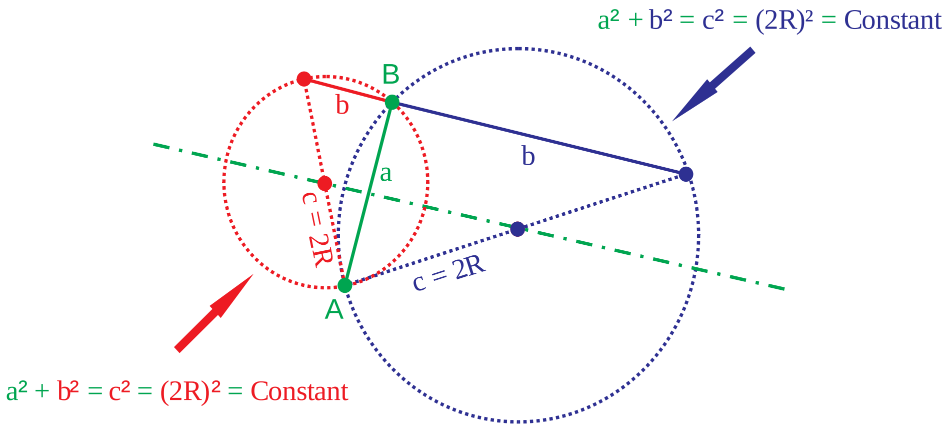
Figure 5: Circumference chords and polygon sides



In this paper I will move further and state that the referred quadratic relationship, "$x^2 + y^2 = (2R)^2$ = Constant", as well as my Theorem of Infinite Right Triangles have a much broader scope than I previously said. This quadratic relationship (explicitly or not) rules every regular and irregular polygon inscribed in a circumference, as well as three famous subjects in number theory: Pythagoras' Theorem, Fermat's Conjecture (Fermat's Last Theorem) and Beal's Conjecture.

Each side of a polygon inscribed in a circumference is a chord in the circumference. Given a straight-line segment "AB", whatever its length, it may be a chord in an infinite number of circumferences, as we see in Figure 6. In each one of these Infinite circumferences, that chord "a" will comply with the quadratic relationship determined by the diameter "2R" of the chosen circumference, "$a^2 + b^2 = c^2 = (2R)^2$ = Constant". Within that chosen circumference, any chord "x" randomly defined will follow the same quadratic relationship, "$x^2 + y^2 = (2R)^2$ = Constant", and remains covered in the Theorem of Infinite Right Triangles. Given a circumference, its diameter "2R" rules the quadratic relationship, while any randomly chosen chord "x" in that circumference simply complies with it.

Figure 6: Circumference chords and the quadratic relationship



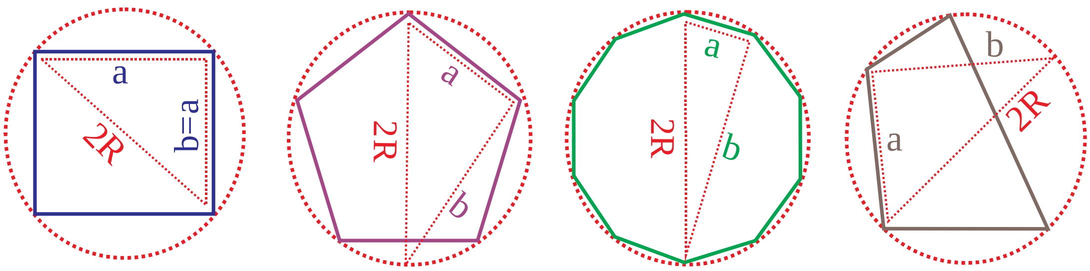
As we saw in Figure 5, any polygonal line with “n” straight-line segments (open or closed) whose vertices remain in the contour line of a same circumference will follow the quadratic relationship:

$$(a_1^2 + a_2^2 + \ldots + a_n^2) + (b_1^2 + b_2^2 + \ldots + b_n^2) = n(2R)^2$$

In the particular case of a regular polygon with “n” equal sides “a”, as some polygons in Figure 7:

$$na^2 + nb^2 = n(2R)^2 \qquad \text{or} \qquad a^2 + b^2 = (2R)^2$$

Figure 7: Polygons and the circumference



Consequently, the right triangle, as well as other polygons inscribed in a circumference are geometric figures, which in some way follow that circumference property. Because of that, we may express their side lengths, perimeters, areas and other properties in function of the circumference radius. Either explicitly or not, all these polygons comply with the circumference quadratic relationship, “$x^2 + y^2 = (2R)^2$ = Constant”, and the Theorem of Infinite Right Triangles.

I will go even further by saying that Pythagoras’ Theorem, Fermat’s Conjecture (Fermat’s Last Theorem) and Beal’s Conjecture are all instances of that circumference property (and its quadratic relationship, “$x^2 + y^2 = (2R)^2$ = Constant”), and that all three subjects are particular views of the Theorem of Infinite Right Triangles. Nuances of a same subject in number theory and geometry. I will handle the three subjects under the statement that the circumference quadratic relationship (explicitly or not) encompasses the previously referred universal family of all possible right triangles (Pythagorean and non-Pythagorean, as known in math), whatever the math expression, which in some way, refers to right triangles (as Pythagoras’, Fermat’s and Beal’s expressions).

PYTHAGORAS’ THEOREM

Pythagoras’s Theorem is perhaps the oldest and certainly the most famous math theorem. Although named after the Greek mathematician Pythagoras, there are references to it in some much older sources, as in archaeologic remains from ancient Mesopotamian civilizations.

According to the prevailing understanding, this theorem is a property of the right triangle under the well-known wording that “In any right triangle, the sum of the squares of the numbers that represent the legs (“a” and “b”) is equal to the square of the number that represents the hypotenuse (“c”)”: “$a^2 + b^2 = c^2$”. That statement is correct, but as I say, it is only a particular case of the circumference quadratic relationship, and of my Theorem of Infinite Right Triangles[8].

Pythagoras' Theorem as a particular case of the Theorem of Infinite Right Triangles

Traditionally, Pythagoras' Theorem refers to a single right triangle, with legs "a" and "b", and a hypotenuse "c", in a manner that "$a^2 + b^2 = c^2$". The Theorem of Infinite Right Triangles refers to a universal family of an infinite number of right triangles, formed by individual groups, each group with an infinite number of right triangles with a same hypotenuse and variable legs "x" and "y", whatever the value of said hypotenuse, while it remains constant: "$x^2 + y^2 =$ Constant".

From my previous comments, we see that the existence of a single right triangle implies the existence of a whole group of an infinite number of different right triangles, all of them inscribed in a same circumference, and with a common hypotenuse. That is why I stated that:

> *Pythagoras' Theorem express a property of the circumference, and is a particular view of the Theorem of Infinite Right Triangles[9].*

FERMAT'S CONJECTURE

Possibly, no other math question challenged so many mathematicians and non-mathematicians worldwide and for so many years as Fermat's Conjecture did. Fermat enunciated his conjecture in the Latin language, and here below, I offer a free translation of it into English.

> *With whole numbers[10], it is impossible to express a third power as the sum of two third powers or a fourth power as the sum of two fourth powers or, in general, any number to a power greater than the second power as the sum of two powers with the same exponent.*

Fermat said he had a truly marvelous demonstration for that statement, but he did not disclose it. Fermat's comment suggests that his proof, if indeed he had one, should be a rather concise approach. Fermat's Conjecture lasted unproven for almost four centuries, when the math community accepted a proof presented by the British mathematician Andrew Wiles. After that, people refer to said Conjecture as "Fermat's Last Theorem". As a side comment, Wiles proof, even if it is correct[11], cannot be the proof Fermat said he had, because in his complex proof with hundreds of pages and equations, Wiles used concepts not available at Fermat's time.

[8] I previously stated that Pythagoras Theorem also is a particular case of Ptolemy's Theorem.
[9] The circumference exists as an independent geometric figure, defined by a proper law. The right triangle reflects a circumference property. There is no right triangle disconnected from a circumference and its quadratic relationship.
[10] I will show that the exponent may be greater than "2" when we deal with non-integer numbers. Even though, Fermat's equality implicitly complies with the referred quadratic relationship of the circumference.
[11] As far as I know, Wiles proof is not unanimously recognized.

Fermat's Conjecture as a corollary of the Theorem of Infinite Right Triangles

In December 2022, I published a paper[12] with my third proof of Fermat's Conjecture and stated that Fermat's Conjecture was a corollary of the Theorem of Infinite Right Triangles[13]. I rewrote Fermat's Conjecture as "$(x^{n/2})^2 + (y^{n/2})^2 = (z^{n/2})^2$", which to be a true equality requires the numbers "$x^{n/2}$","$y^{n/2}$" and "$z^{n/2}$" to represent straight-line segments that form a right triangle, and the legs "$x^{n/2}$" and "$y^{n/2}$" to freely vary while keeping "$z^{n/2}$" constant, as per the referred Theorem. It is necessary to keep in mind that, algebraically speaking, we do not alter the equality when we rewrite it as above.

> *In order to allow legs "$x^{n/2}$" and "$y^{n/2}$" to freely vary, when dealing with integers, only the exponent "n" may change, since we cannot allow "x", "y" and "z" to change, without altering the expression under analysis. Clearly, that is an impossibility, because if we alter "n", "$z^{n/2}$" will not remain constant (and contradicts the referred Theorem), what allows us to state that the exponent "n' cannot be different from "1" or "2". With whole numbers, Fermat's math expression is not a true equality when "n" is greater than "2".*

As a remark, Fermat restricted his conjecture to integer numbers. Nevertheless, when dealing with non-integer numbers, I say that the quadratic relationship will be implicitly present, whenever the expression "$(x^{n/2})^2 + (y^{n/2})^2 = (z^{n/2})^2$" is an acceptable equality, even if the exponent "n" is different from "2". In that case, the numbers "$x^{n/2}$", "$y^{n/2}$" and "$z^{n/2}$" form a right triangle, and necessarily "$x^{n/2} = a$", "$y^{n/2} = b$" and "$z^{n/2} = c$", in a manner that "$a^2 + b^2 = c^2$". We are still within a circumference of a diameter "$c = z^{n/2}$", with a group of right triangles with variable legs, all of them with a hypotenuse "$c = z^{n/2}$".

To illustrate the point, consider the (approximate) equality of non-integer numbers, "$(3.000)^4 + (3.464)^4 = (3.873)^4$. Apparently, this expression contradicts the circumference quadratic relationship (and my Theorem). However, if we rewrite that expression as "$(3.000^{4/2})^2 + (3.464^{4/2})^2 = (3.873^{4/2})^2$" we will see that it is (roughly) equivalent to "$9.000^2 + 11.999^2 = 15.000^2$", what means the quadratic relationship implicitly rules the given equality[14].

When dealing with integers, it is possible to say that to be a true equality, the exponent "n" in Fermat's algebraic expression, "$x^n + y^n = z^n$", cannot be greater than "2" (unless we accept approximate results, called "near-miss solutions"). If we accept non-integer values for "x", "y" and "z", it is possible to find an equality with "$n \neq 2$", in which case the quadratic relationship would be implicitly present. The values of "$x^{n/2}$", "$y^{n/2}$" and "$z^{n/2}$" would be numbers "a", "b" and "c", which represent straight-line segments that form a right triangle (mathematically speaking, non-Pythagorean triples). Fermat's Conjecture denies the existence of any right triangle under the conditions imposed on the numbers, which form Fermat's math expression.

In fact, as we see in Figure 8, the symbiosis between geometry and math allows us to state that Fermat's Conjecture is a self-evident statement. Each straight-line segment of

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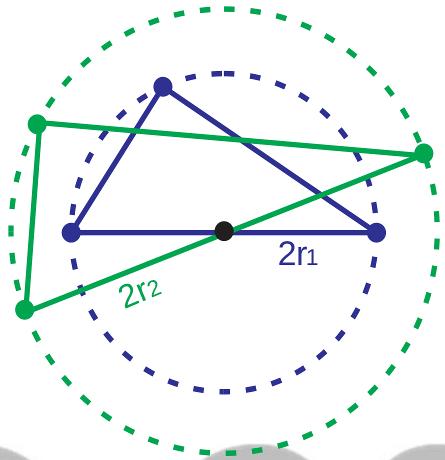
[12] AMUI, Sandoval, Two famous conjectures (Pierre de Fermat and Andrew Beal), AYA Editora, Brazil, 2022.

[13] A symbiosis between geometry and mathematics is the basis of my proof, since I believe it is not feasible to prove Fermat's Conjecture otherwise.

[14] If we accept non-integer numbers, we will deal with approximate results, not with exact equalities.

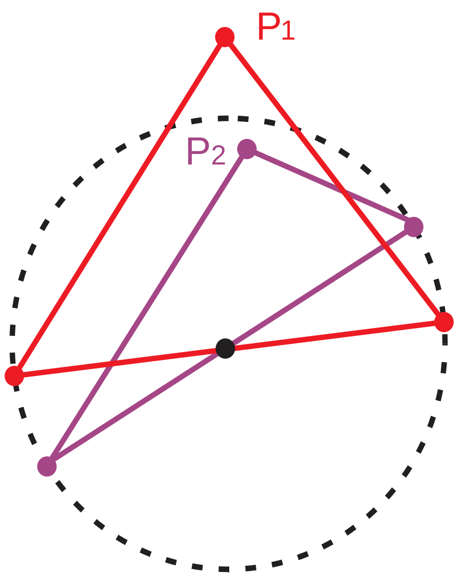
length "$2r_1$" taken as the diameter of a circumference encompasses a whole group of right triangles with a same hypotenuse "$2r_1$" (blue line)). If we change the straight-line segment to "$2r_2$", we will have a different circumference with another whole group of right triangles, again with a common hypotenuse "$2r_2$" (green line). The three vertices of the triangles pertaining to the same circumference remain in the contour line of said circumference, and the numbers that represent the sides of the right triangles (integers or non-integers) comply with the quadratic relationship, "$x^2 + y^2 = (2r)^2$", without exception.

Figure 8: Groups of right triangles



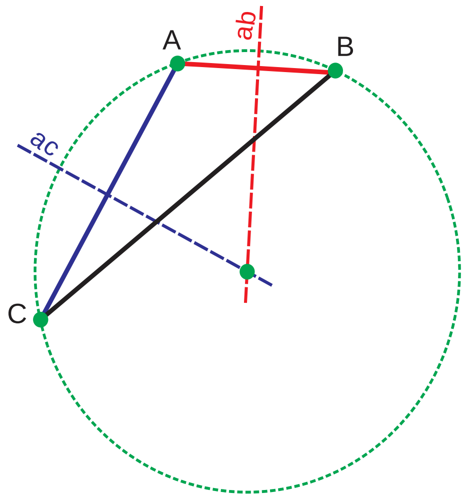
In case one of the three vertices of the resulting triangle (P_1, P_2, …) is not in the contour line of a circumference formed by the given straight-line segment ("$2r_1$", "$2r_2$", …), the other two vertices are, as in Figure 9, the resulting triangle will be an acute triangle (red line) or an obtuse triangle (purple line), but never a right triangle. The numbers that represent the sides of all these non-right triangles (integers or non-integers) do not comply with the quadratic relationship, "$x^2 + y^2 = (2r)^2$", because math cannot violate geometry requirements.

Figure 9: Acute or obtuse triangle



We know we can draw a circumference passing through any three points “A”, “B” and “C”, as in Figure 10. However, the triangle formed by the three points “ABC” will be a right triangle if, and only if, one of its three sides is the diameter of the resulting circumference. In case it occurs, the quadratic relationship will prevail.

Figure 10: A circumference passing through any three points



Then, explicitly or not, to represent a right triangle, the exponent “n” in Fermat’s equality must be equal to “2”.

In Summary:

> *Fermat’s Conjecture (Fermat’s Last Theorem) is a valid statement, and a corollary of the Theorem of Infinite Right Triangles, since its math expression “$x^n + y^n = z^n$” must comply with the circumference quadratic relationship, “$x^2 + y^2 =$ Constant”. That compliance is explicit when “$n = 2$” and implicit when “$n \neq 2$”.*

BEAL’S CONJECTURE

Beal’s Conjecture is also a famous challenge. The Beal Prize Committee offers a significant prize for either a proof or a counterexample of the Conjecture. Often referred to as a generalization of Fermat’s Conjecture, this Conjecture states that:

> *If “$A^x + B^y = C^z$”, and “A”, “B” and “C” are whole numbers greater than “1” and “x”, “y” and “z” are whole numbers greater than “2”, the base numbers of the three powers must have a common prime factor.*

In that paper of December 2022[15], I suggested a proof of Beal’s Conjecture based on elementary concepts of arithmetic, trigonometry and geometry (again, an approach in line with the symbiosis between geometry and mathematics). I offered elements that, in order to be a true equality, the base numbers of the powers of Beal’s expression must have a common prime factor. I also presented an explanation why two out of three of the base numbers must share a common exponent, either explicitly or not.

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[15] AMUI, Sandoval, Two famous conjectures (Pierre de Fermat and Andrew Beal), AYA Editora, 2022.

Besides the requirements that the three base numbers share a common prime factor and that two of them share a common exponent, Beal's Conjecture also is an instance of the previously referred circumference quadratic property, expressed as "$x^2 + y^2 = (2R)^2 =$ Constant", and a particular view of the Theorem of Infinite Right Triangles. Beal's Conjecture mathematically represents a specific subfamily of the universal family of right triangles.

Beal's Conjecture as a particular view of the Theorem of Infinite Right Triangles

This paper aims to show that, as Pythagoras' Theorem and Fermat's Conjecture, Beal's Conjecture also is an instance of the circumference quadratic property, and a particular view of the Theorem of Infinite Right Triangles. As a complement to the suggested proof offered in December 2022, I handled this second analysis of Beal's Conjecture by rewriting its math expression as "$(A^{x/2})^2 + (B^{y/2})^2 = (C^{z/2})^2$", in the same way I handled Fermat's Conjecture. Again, we have to stress that we do not alter the equality when we rewrite it as above.

As the algebraic expression that represents Pythagoras' Theorem is a particular case of Fermat's expression, the latter is a particular case of the algebraic expression that represents Beal's Conjecture. The math expression of Beal's Conjecture "$A^x + B^y = C^z$" is then a generalized math expression, which encompasses the other two. When "$x = y = z = n$", we see Fermat's Conjecture, "$x^n + y^n = z^n$"; when "$x = y = z = 2$", we see Pythagoras' Theorem, $a^2 + b^2 = c^2$".

According to Pythagoras' Theorem the sides' lengths that form a right triangle may be integer or non-integer numbers, as "a, "b" and "c". Except when the exponent "n" is equal to "2", right triangles under Fermat's math expression may only occur when dealing with non-integer numbers.

Beal's math expression "$(A^{x/2})^2 + (B^{y/2})^2 = (C^{z/2})^2$" implies that "$A^{x/2}$", "$B^{y/2}$" and "$C^{z/2}$" represent straight-line segments, which form right triangles. This means that in order to exist one right triangle, it is mandatory to exist a whole group with an infinite number of right triangles with a same hypotenuse.

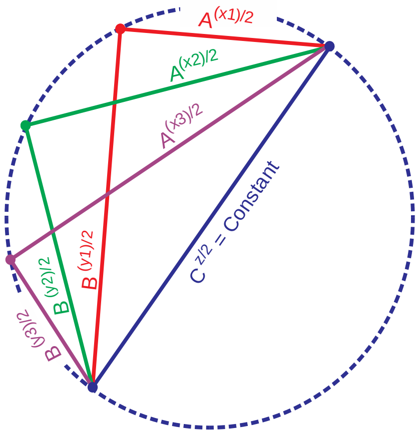
If we let the base numbers and the exponents in Beal's math expression to assume any values, said expression encompasses an infinite number of right triangles, the previously referred universal family of right triangles. In each particular group, the right triangles will have variable legs, "$A^{x/2}$" and "$B^{y/2}$", and a same hypotenuse equal to "$C^{z/2}$", whatever the values of "C" and "z", while "$C^{z/2}$" remains constant. The compliance of said math expression with the quadratic relationship makes it a particular view of the Theorem of Infinite Right Triangles.

However, not all of these right triangles, which simultaneously satisfy the circumference quadratic relationship and Beal's math expression, will also satisfy the requirements of Beal's Conjecture[16].

Figure 11 illustrates these statements.

[16] Beal's math expression comprises the universal family of right triangles. Beal's Conjecture comprises a subfamily of said universal family. Both groups with an infinite number of right triangles. Math curiosities.

Figure 11: Compliance of Beal's math expression with the quadratic relationship[17]



As an algebraic example of powers that satisfy Beal's Conjecture:

$2^3 + 2^3 = 2^4$

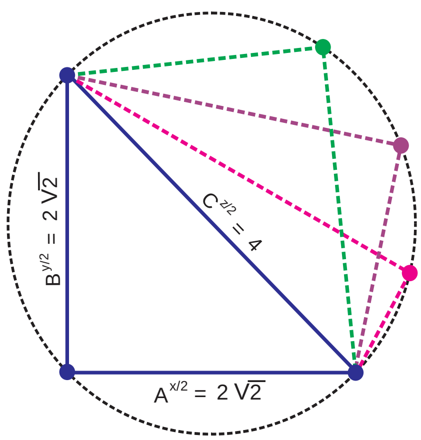
$(2^{3/2})^2 + (2^{3/2})^2 = (2^{4/2})^2$ or $(\sqrt{2^3})^2 + (\sqrt{2^3})^2 = (\sqrt{2^4})^2$

$(2\sqrt{2})^2 + (2\sqrt{2})^2 = (4)^2$

$8 + 8 = 16$

As we see in Figure 12, we can find other pairs of numbers, "$A^{x/2}$" and "$B^{y/2}$", to replace "$2^{3/2}$", which will maintain the equality when keeping "2^4" constant (not necessarily satisfying Beal's Conjecture). These other pairs of numbers will represent straight-line segments forming a group of right triangles with a common hypotenuse represented by "4", what would obey the circumference quadratic relationship. It means that while keeping "$(C^{z/2})^2$" constant, we can vary "x" and "y" with the purpose to allow "$A^{x/2}$" and "$B^{y/2}$" to assume different values, as required by the Theorem of Infinite Right Triangles.

Figure 12: Right triangle under the requirements of Beal's Conjecture



[17] Not necessarily satisfying the requirements of Beal's Conjecture.

However, in the example above ($2^3 + 2^3 = 2^4$), only one pair of numbers "$A^{x/2}$" and "$B^{y/2}$" ($2^{3/2}$ and $2^{3/2}$) satisfies Beal's Conjecture. In each group of right triangles that satisfy Beal's math expression, "$(C^{z/2})^2$ = constant", we may or may not have a particular subgroup of right triangles that meet Beal's Conjecture.

In order to be equalities, the math expressions of Pythagoras, Fermat and Beal, explicitly or not, have to comply with the circumference quadratic relationship, "$x^2 + y^2$ = Constant", which rules all possible right triangles ever conceived (the universal family of right triangles). There are groups of right triangles with a same hypotenuse behind each one of the three math expressions, as required by the Theorem of Infinite Right Triangles. Otherwise, it is not possible to find numbers and/or powers to satisfy the requirements of the corresponding theorems and/or conjectures.

In terms of algebra:

Pythagoras' Theorem:	$a^2 + b^2 = c^2$	All numbers are constants
Fermat's Conjecture:	$(x^{n/2})^2 + (y^{n/2})^2 = (z^{n/2})^2$ = Constant	
Beal's Conjecture:	$(A^{x/2})^2 + (B^{y/2})^2 = (C^{z/2})^2$ = Constant	

Contrarily to the requirements applicable to Fermat's Conjecture, in which the three powers have a common exponent, Beal's Conjecture accepts powers with different exponents, what implies that we can vary the exponents of "A^x" and "B^y", without altering the exponent of "C^z = Constant". Then, if we accept that the base numbers and their exponents can assume any real number, the math expression of Beal's Conjecture allows us to vary the legs of a right triangle while keeping its hypotenuse constant. As we see in Figures 11 and 12, it does not contradict the Theorem of Infinite Right Triangles, as Fermat's Conjecture does when dealing with whole numbers and "$n > 2$".

In other words:

> *Assuming that the square root of a power formed by an integer base number greater than "1" and an integer exponent greater than "2" represents the hypotenuse of a right triangle, we will find an infinite number of right triangles with that same hypotenuse[18]. However, only a limited number of these right triangles, if any, will have legs represented by the square roots of powers formed by integer base numbers greater than "1" and integer exponents greater than "2".*

Common prime factor of the base numbers and common exponent of two out of three of the base numbers

Beal's Conjecture imposes the condition "$A^x + B^y = C^z$". In this equality, "A", "B" and "C" are whole numbers greater than "1", while "x", "y" and "z" are whole numbers greater than "2". That Conjecture also requires "A", "B" and "C" to have a common prime factor. Although not a requirement of the Conjecture, but as a circumstantial

[18] Similarly, we know that every circumference encompasses a whole family of an infinite number of right triangles, which share an integer common hypotenuse, but not all of them with legs represented by whole numbers (Pythagorean triples).

requirement imposed by arithmetic properties, two out of three of the base numbers must share a common exponent.

As I stated in previous papers, we have two ways to analyze Beal's Conjecture: either as an addition of two powers, "$(A^{x/2})^2 + (B^{y/2})^2$", or as a subtraction of one power from another, "$(C^{z/2})^2 – (A^{x/2})^2$" or "$(C^{z/2})^2 – (B^{y/2})^2$". In both arithmetic operations, we need to find a third power, "$(C^{z/2})^2$", "$(B^{x/2})^2$ or "$(A^{y/2})^2$", which complies with the same restrictions applicable to the base numbers and exponents of the powers involved in the relevant arithmetic operation.

In addition to the compliance with the Theorem of Infinite Right Triangles, Beal's math expression must comply with arithmetic rules applicable to addition and subtraction of powers as required by Beal's Conjecture. That is why the base numbers of the three powers of Beal's math expression must share a common prime factor, and two out of three of the powers must share a common exponent[19]. Otherwise, it is not possible to find powers that make the math expression of Beal's Conjecture a true equality.

To clarify the statement, let us perform an addition or a subtraction of two powers, knowing that their base numbers have a common factor "m" and the same exponent "k".

Addition, as "$A^x + B^y = ?$", and "$x = y = k$":

$$(ma)^x + (mb)^y = (m)^x(a)^x + (m)^y(b)^y = (m)^k(a)^k + (m)^k(b)^k = (m)^k[(a)^k + (b)^k]$$

We know the result has to be a power, what implies its base number to have the same factor "m": $[(a)^k + (b)^k] = m^{k'}$, and

$$(m)^k[(a)^k + (b)^k] = (m)^k(m^{k'}) = (m)^{k+k'}$$

Numerically,

$$3^3 + 6^3 = (3.1)^3 + (3.2)^3 = 3^3.1^3 + 3^3.2^3 = 3^3(1^3 + 2^3) = 3^3(1 + 8) = 3^3(9) = 3^3.3^2 = 3^{3+2} = 3^5$$

Then,

$$3^3 + 6^3 = 3^5$$

Subtraction under the same assumptions, as "$C^z – A^x = ?$", and "$z = x = k$":

$$(mc)^z – (ma)^x = (m)^z(c)^z – (m)^x(a)^x = (m)^k(c)^k – (m)^k(a)^k = (m)^k[(c)^k – (a)^k] =$$

We know the result has to be a power, what implies its base number to have the same factor "m": $[(c)^k – (a)^k] = m^{k'}$, and

$$(m)^k[(c)^k – (a)^k] = (m)^k(m^{k'}) = (m)^{k+k'}$$

Numerically,

[19] AMUI, Sandoval, Two famous conjectures (Pierre de Fermat and Andrew Beal), AYA Editora, 2022.

$98^3 - 7^6 = (7.14)^3 - (7.1)^6 = (7^3.14^3) - (7^6.1^6) = (7^3.14^3) - (7^3.1^3)^2 = 7^3(14^3 - 7^3.1^3)$
$= 7^3(7^3.2^3 - 7^3.1^3) = 7^3.7^3 = 7^3.7^3(2^3 - 1^3) = 7^6(7^1) = 7^7$

Then,

$98^3 - 7^6 = 7^7$

Beal's Conjecture requires compliance with some arithmetic operating properties, it being the reason why:

Given two powers formed by integer base numbers greater than "1", raised to integer exponents greater than "2", in order to perform an addition or a subtraction operation with these two powers to obtain a third power, which obeys the same restrictions the two given powers do, their base numbers must share a common prime factor and a common exponent. Additionally, the base number of the resulting power will have to have the same common prime factor too.

As an example, the straight-line segments represented by the numbers "$34^{5/2}$", "51^2" and "85^2" form a right triangle, with legs "$34^{5/2}$" and "51^2", and a hypotenuse "85^2" (Figure 13).

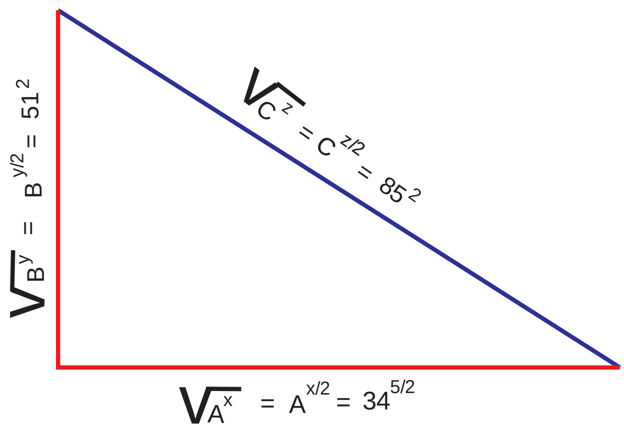
Additionally, their squares meet Beal's Conjecture.

$(34^{5/2})^2 + (51^2)^2 = (85^2)^2$

$34^5 + 51^4 = 85^4$

We can state that if we disregard the requirements of Beal's Conjecture in respect of the nature of the base numbers and exponents, it will be easy to meet Beal's math expression (the universal family of right triangles). However, not so easy, if we need to comply with the Conjecture requirements (a subfamily of the universal family or right triangles).

Figure 13: Square roots of powers forming a right triangle



In my view, we may enunciate Beal's Conjecture the other way around:

Assuming that "A", "B" and "C" are integers greater than "1", and "x", "y" and "z" are integers greater than "2", the math expression "$A^x + B^y = C^z$" is a true equality if, and only if, the three base numbers share a common prime factor, and two of them (explicitly or not) share a common exponent.

FINAL COMMENTS

Mathematics is scientific instrument developed by humankind over many centuries based upon certain axioms, postulates, premises and methods, here jointly referred to as "math fundamentals". Ever since, math has been growing under the questionable assumption that we have been using valid math fundamentals. In the development of my innovative and polemic ideas, I assumed the hypothesis that some mistaken fundamentals could be behind the poorly explained strange results and open questions we face under current math. With said hypothesis in mind, I granted myself the right to follow a completely free reasoning approach in order to question the prevailing math framework. Otherwise, it would not be possible to identify the causes of said undeniable math oddities and inconsistencies, and use that information with the purpose to improve the instrument.

As a scientific language, mathematics represents and describes the subjects of studies of other sciences, their pre-existing phenomena and/or properties, as in physics, astronomy and the like. That is why I stated that whenever dealing with a subject relating to any science, whatever the science (including geometry), mathematics must fully comply with the appropriate requirements ruling said subject.

The need to comply with the requirements of the served science is the reason why I say that algebra does not have the independence and widespread field of application presently accepted. As examples, when dealing with physics, if a math expression describes a natural phenomenon, physics must confirm it. Until said confirmation happens, any supposed phenomenon a math expression anticipates does not exist and is nothing else but a conjecture[20]. Similarly, in geometry, math may move from an existent geometric figure to the corresponding algebraic expression, as an ellipse. In the opposite direction, moving from a random algebraic expression to geometry, we may or may not find a meaningful geometric figure that corresponds to the given algebraic expression[21]. For instance, certain math expressions supposedly represent elliptic curves. Contrarily to real figures (conics, Cassini oval and others), which follow geometric laws, an elliptic curve is an abstraction, follows no geometric law, and nobody can tell how it looks like, since, as an abstraction, it does not exist in nature. The shapes we see in math books are mere assumptions, which results from the adoption of the Cartesian system and current math fundamentals. If we use different concepts, we will obtain different shapes[22].

Then, given a math expression, it is mandatory to verify if it represents any geometric figure, and if it does, what geometric figure it represents. In case it represents an existent geometric figure, that math expression must comply with the requirements of the represented figure[23]. That is what occurs with Pythagoras' Theorem, Fermat's Conjecture (Fermat's Last Theorem) and Beal's Conjecture, because their

[20] The supposed existence of multiverse is a good example, and a cause of disagreement between mathematicians and physicists.

[21] Differently from math, in our real world (geometry) there are three dimensions, only.

[22] See "Alternative Cartesian System", in AMUI, Sandoval, You may not enjoy mathematics (but you do not have to hate it), AYA Editora, 2022.

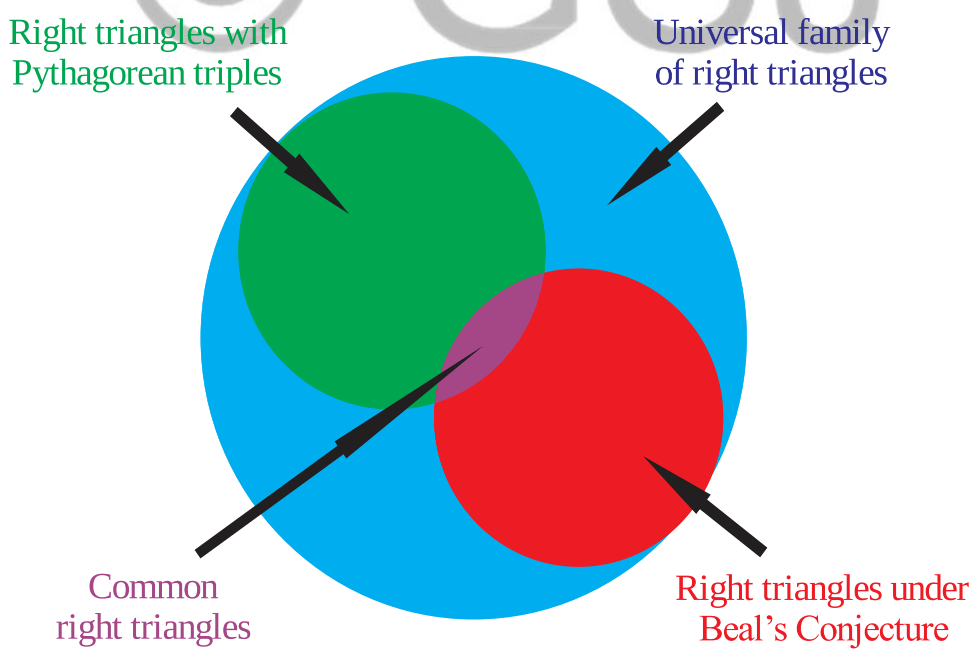
[23] This limitation does not apply in case a math expression represents an abstraction, an imaginary geometric figure, which does not exist in the real world (polynomials, elliptic curves and the like). An abstraction is beyond verification or rebuttal arguments. The limitation neither applies to scientific formulas (as in physics).

corresponding math expressions represent right triangles, and must comply with the circumference quadratic relationship. In brief, we cannot overlook the interdependence of geometry and some branches of math. If we do it, we may face inconsistencies or accept mistaken conclusions.

This understanding of the interdependence of geometry and some branches of mathematics is of the essence in the approach adopted in this paper, in which I stated that a circumference property, mathematically expressed by the quadratic relationship under the terms of the Theorem of Infinite Right Triangles, "$x^2 + y^2 = (2R)^2 =$ Constant", has a much broader scope than currently accepted. It rules open and closed polygonal lines inscribed in a circumference, Pythagoras' Theorem, Fermat's Conjecture (Fermat's Last Theorem), and Beal's Conjecture, since math cannot violate the geometric requirements pertinent to these subjects.

In Figure 14 we see the previously referred universal family of an infinite number of right triangles (Pythagorean and non-Pythagorean triples) represented by the Theorem of Infinite Right Triangles. Pythagoras' Theorem and Beal's equality also represent said universal family of right triangles if we do not impose any restriction on the numbers involved in their respective math expressions (blue area). Within said universal family of right triangles, we have a subfamily of an infinite number of right triangles with integer triples (green area). We also have another subfamily of an infinite number of right triangles, which meet Beal's Conjecture (red area)[24]. Fermat's Conjecture expresses the nonexistence of right triangles under the conditions it imposes on the numbers in Fermat's math expression (empty set).

Figure 14: Universal family of right triangles and its subfamilies



[24] Math curiosities: infinite sets comprised by another infinite set.

As a side comment, it seems that the numbers that represent the sides of the right triangles encompassed by Beal's Conjecture cannot be integer triples[25]. In other words, there is no right triangle, which belongs to the three groups, as the referred Figure 14 suggests (purple area). If this is true, we may state that, in the same way the conditions required by Fermat's Conjecture expresses the nonexistence of right triangles with integer triples, Beal's Conjecture also does that (empty sets).

The math expression of the Theorem of Infinite Right Triangles and its particular cases, Pythagoras' Theorem, Fermat's Conjecture and Beal's Conjecture, are simply mathematical representations of right triangles. As a result, they have to comply with the circumference quadratic relationship, "$x^2 + y^2 = \text{Constant}$".

All polygons inscribed in a circumference, as well as Pythagoras' Theorem, Fermat's Conjecture (Fermat's Last Theorem) and Beal's Conjecture are instances of that circumference property, and particular views of the Theorem of Infinite Right Triangles.

In searching for a unified theory of mathematics, besides logic and commonsense, it is mandatory to take into account the unbreakable connection between geometry and some branches of mathematics.

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[25] That is an assumption (the half of an exponent represented by an odd integer greater than "2" is a fractional number). I do not have that proof, and I do not know if it exists.