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Travelling Salesman Problem – ARM heuristic

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Summary: The paper discusses the travelling salesman heuristic using Hungarian Assignment Method, Nearest Neighbourhood, Minimum Spanning Tree and Branch and Bound Method and develops an improved heuristic or a modification of Hungarian Assignment Method called ARM heuristic. The criteria used for ARM Method is proved mathematically in this paper.

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Keywords: Travelling salesman problem, mathematics, operations research, heuristic, approximation, minimise travel distance, ARM heuristic, Hungarian Assignment Method

Section 1

Assignment Problem:

In the travelling salesman problem, a salesman has to travel to n locations and visit each location only once and return to the starting point. The aim is to visit all locations and minimise the travel distance. As all locations need to be visited, the solution to the travelling salesman problem is independent of starting location.

The assignment problem can be formalised as given by (Swarup Kanti et al 2015)

If X_{ijk} is the delta function

$X_{ijk} = 1$, if k^{th} movement is from node i to node j or $X_{ijk} = 0$ otherwise.

Where i, j, k are integers

Constraints:

Only 1 movement can be assigned to a specific k

$\sum_j \sum_k X_{ijk} = 1$ where $i \neq j$

Only 1 other node can be reached from specific node i

$\sum_j \sum_k X_{ijk} = 1$

Given k^{th} movement ends at j, $(k+1)^{th}$ next movement must start at j

$\sum_{i \neq j} X_{ijk} = \sum_{r \neq j} X_{jr(k+1)}$ for all j and k

Min $\sum_i \sum_j \sum_k d_{ij} X_{ijk}$ where dij is the distance between node i and node j

Section 2

ARM heuristic:

Now, consider a network with movements $\{x_i x_j\}$ where $i \neq j$ and i, j go from 1 to n

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If $x_k x_m + (\sum_n x_i x_j)/n\Psi > x_k x_p + x_p x_m$ for all n nodes (The LHS can be approximately equal to RHS at a minimum value for $\Psi = 1$, when number of nodes approach infinity). The $\Psi = \{a: a \rightarrow R \text{ and } 1 \leq a \leq 2\}$

$x_k x_p + x_p x_m$ becomes a segment of trial network.

Segments x_1 to x_k and x_m to x_1 which is network ($x_k - x_1 - x_m$) can be later optimized using Hungarian Assignment Method.

Similarly, for a segment combination of 2 movements, $x_k x_l$ and $x_l x_m$,

If $x_k x_l + x_l x_m + (\sum_n x_i x_j)/n\Psi > x_k x_p + x_p x_q + x_q x_m$

where there is some possibility that the replacement movement may work. This equation should be used for a small number of nodes. Also important is that the number of nodes on LHS of the equation is at least 1 less than number of nodes on RHS. If the difference is equal to or higher than 2 the trial should be checked before any other trial and the second term (average term) in LHS should be multiplied by 2.

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At least n(k, l, m) +1 = n(k, p, q, m)

Where l can be included in the replacement trial segment

However, if the number of nodes in LHS is equal to RHS the second term on LHS (average/ Ψ) should be omitted.

n(k, l, m) = n(k, p, m)

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If $x_kx_l + x_lx_m > x_kx_p + x_px_m$ the replacement must be checked. This equation should be used if the number of nodes is extremely large.

segment $x_k\, x_p\, x_m$ can be locked for trial. The remaining segment $x_kx_1\, x_m$ can be optimized by Hungarian Assignment Method as mentioned before and the optimum sequence can be added to $x_k\, x_p\, x_m$ if above inequality is true.

In general,

$\mathbf{x_ix_j + x_jx_k + \ldots + x_{ni} + (\sum_n x_ix_j)/n\Psi > x_ix_p + \ldots\ldots + x_zx_i}$ **----- equation 1**

$\mathbf{x_ix_j + x_jx_k + \ldots\ldots\ldots + x_{ni} > x_ix_p + \ldots\ldots\ldots\ldots + x_zx_i}$ **------ equation 2**

The above equations are necessary conditions of the ARM heuristic.

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Proof:

Expected value $E(x_ix_j)$ is $(\sum_n x_ix_j)/n$ as the number of nodes in the trial increase to infinity. If n is the number of nodes on LHS of above equation then LHS becomes,

$x_ix_j + x_jx_k + \ldots + x_{ni} + (\sum_n x_ix_j)/n = (n+1)\, E(x_ix_j)$

If number of nodes on RHS of the equation is 1 more than number of nodes on LHS,

$x_ix_p + \ldots\ldots\ldots\ldots + x_zx_i = (n+1)\, E(x_ix_j)$

The number of nodes in a first trial movement segment (T) must be greater than the absolute value of half of the total nodes. After first trial the number of nodes should be increased and decreased by

1 node at a time. Though a shorter segment may give a mileage smaller than the mileage given by Hungarian Assignment Method, it will be tedious and repetitious to try every possible segment.

T $\geq$ abs (n/2) ------ equation 3

Equation 3 is not a necessary condition of the heuristic but is used to speed the process of finding a better solution than found by regular Hungarian assignment method

Further trials should be performed immediately below or above T.

Section 3

Example:

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A distance matrix of 6 nodes is shown below in miles. The current Hungarian Assignment Method is used first followed by proposed ARM Method which is an extension of Hungarian Assignment Method.

	A	B	C	D	E	F
A	-	11	29	22	12	9
B	11	-	6	19	26	26
C	29	6	-	10	13	9
D	22	19	10	-	3	22
E	12	26	13	3	-	3
F	9	26	9	22	3	-

	A	B	C	D	E	F
A	-	2	20	13	3	0
B	5	-	0	13	20	20
C	23	0	-	4	7	3
D	19	16	7	-	0	19
E	9	23	10	0	-	0

F	6	23	6	19	0	-

The above table is updated by subtracting lowest distance in a row from all cells in a row.

	A	B	C	D	E	F
A	-	2	20	13	3	0
B	0	-	0	13	20	20
C	18	0	-	4	7	3
D	14	16	7	-	0	19
E	4	23	10	0	-	0
F	1	23	6	19	0	-

The same subtracting procedure is used for columns. The cells coloured yellow are selected for optimization and the repeat zeros in same row/ column are excluded and coloured light green. As number of yellows is less than the order of the matrix which is 6, the matrix is updated as below.

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	A	B	C	D	E	F
A	-	2	20	13	3	0
B	0	-	0	13	20	20
C	18	0	-	4	7	3
D	14	16	7	-	0	19
E	4	23	10	0	-	0
F	1	23	6	19	0	-

The rows without assigned zero are not crossed by horizontal lines while columns with zeros in unassigned rows are crossed by vertical lines. The remaining matrix is updated by subtracting the lowest value from all cells. A new assigned zero is obtained in the unassigned row.

	A	B	C	D	E	F
A	-	2	20	13	3	0
B	0	-	0	13	20	20
C	18	0	-	4	7	3

D	13	15	6	-	0	18
E	4	23	10	0	-	0
F	0	22	5	18	0	-

	A	B	C	D	E	F
A	-	2	20	13	3	0
B	0	-	0	13	20	20
C	18	0	-	4	7	3
D	13	15	6	-	0	18
E	4	23	10	0	-	0
F	0	22	5	18	0	-

A-F, B-C, D-E segments are found and row B and row C are further optimised by subtracting the lowest movement from the entire row.

	A	B	C	D	E	F
A	-	2	20	13	3	0
B	0	-	0	0	7	7
C	18	0	-	4	7	3
D	13	15	6	-	0	18
E	0	19	6	0	-	0
F	0	22	5	18	0	-

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C-B-D-E-A-F-C = 58 miles is the solution of the regular Hungarian Assignment Method.

	A	B	C	D	E	F
A	-	11	29	22	12	9
B	11	-	6	19	26	26
C	29	6	-	10	13	9
D	22	19	10	-	3	22
E	12	26	13	3	-	3
F	9	26	9	22	3	-

ARM Heuristic

C-B-D = 25 is selected first as the distance of the segment is highest for any 3 node segment.

C-F-E-D = 15

25 + average of 15 = 40 >15

40-15 = 25

B-C-F-E-D-A-B = 54 gives a first optimal solution better than regular Hungarian Assignment Method.

B-D-E = 22 is the next largest segment in the optimal solution found by Hungarian Assignment Method.

B-C-F-E = 18

22 + average of 15= 37

The difference between 2 segments is 37-18 = 19

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B-C-F-E-D-A-B = 21+22+11 = 54

E-A-F = 21 is the third largest segment

21 + average of 15 = 36

No easy alternatives found mathematically

A-F-C= 18 is the fourth largest segment

A-F-E-D-C = 25

18+15 x 2= 48>25

The average is multiplied by 2 because the difference between total nodes in original and trial segment is 2. Because the difference is large there is a strong possibility of a good solution.

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The difference between 2 segments is 48-25 = 23

A-F-E-D-C-B-A = 42 miles the final optimal solution to ARM heuristic.

It is possible to look for multiple replacements in 1 trial however the number of trials will become computationally difficult.

The solution obtained with Hungarian Assignment Method is C-B-D-E-A-F-C with the total distance of 58 miles. *A map should be drawn after completing original Hungarian Assignment Method for visualization of alternate segments*.

Using ARM Method, now consider the combined movement A-F-C with a total mileage of 18. A-F-E-D-C was selected because it has the highest total mileage for a combined movement of 5 nodes and the optimal solution received above and the number of nodes in it is 2 more than nodes in A-F-C. The original combined movement is replaced by A-F-E-D-C with a total combined movement of 25 miles because using the equation 1 presented above, 25(movement C-B-D) = 18 + 15= 43 > 25 (movement A-F-E-D-C). Ψ value of 1 was used for this example. Higher Ψ values will restrict finding of sub-optimal solutions. The trial segment also meets the equation 2 inequality. Omitting this combined movement the remaining nodes are optimised by Hungarian Assignment Method discussed above. We get, A-F-E-D-C-B-A with a total distance of 42 miles! An improvement over regular Hungarian Assignment Method mileage of 58 miles obtained as discussed above. Larger combined movements should be checked for the main equation inequality before smaller combined movements. Many combined movements can be simultaneously replaced however the heuristic will lose its simplicity.

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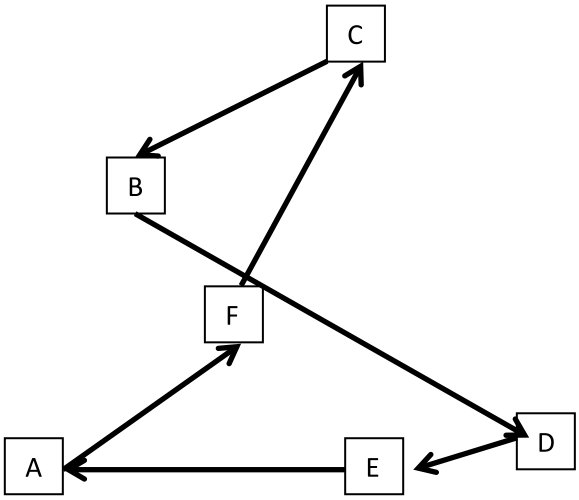

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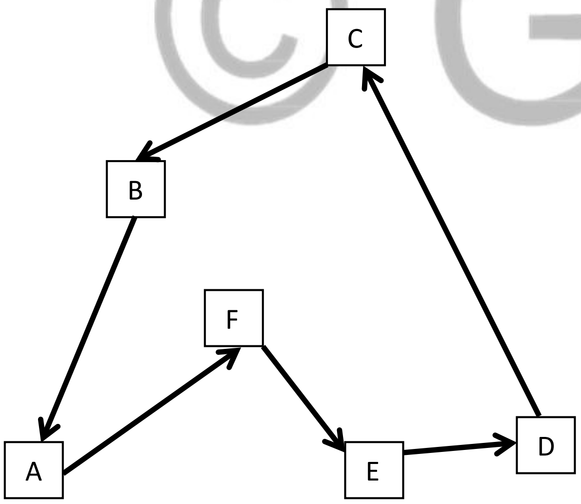

Fig 1. The alternate movement segment using ARM heuristic is shown in comparison with movement segment found by current Hungarian Assignment Method in black.

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Section 4

Alternative heuristics:

Nearest Neighbourhood

	A	B	C	D	E	F
A	-	11	29	22	12	9
B	11	-	6	19	26	26
C	29	6	-	10	13	9
D	22	19	10	-	3	22
E	12	26	13	3	-	3
F	9	26	9	22	3	-

Nearest Neighbourhood heuristic starts with a random city and moves to the closest city from that random city. If 2 or more cities are equally close a random city is chosen. 2 Nearest Neighbourhood solutions are given below for the same example discussed before.

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A-F-E-D-C-B-A = 9+3+3+10+6+11 = 42

The above solutions give a movement combination with lower mileage than ARM Method or any other method discussed in this paper, which was 42 miles. But, as it is already known, the Hungarian Assignment Method gives a better solution than Nearest Neighbourhood Method though is worked on this occasion.

Minimum Spanning Tre

	A	B	C	D	E	F
A	-	11	29	22	12	9
B	11	-	6	19	26	26
C	29	6	-	10	13	9
D	22	19	10	-	3	22
E	12	26	13	3	-	3
F	9	26	9	22	3	-

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If E-D movement is selected, we have {E, D} and {A, B, C, F}

Finding the minimum distances from E and F nodes to each of the A, B, C, F nodes and selecting the minimum, we get

The minimum spanning tree is {D, A, F, E, C, B}

So, the movement is B-D-A-F-E-C-B = 72 which is higher than the distance 54 miles found by ARM Method.

Branch and Bound Method:

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	A	B	C	D	E	F
A	-	11	29	22	12	9
B	11	-	6	19	26	26
C	29	6	-	10	13	9
D	22	19	10	-	3	22
E	12	26	13	3	-	3
F	9	26	9	22	3	-

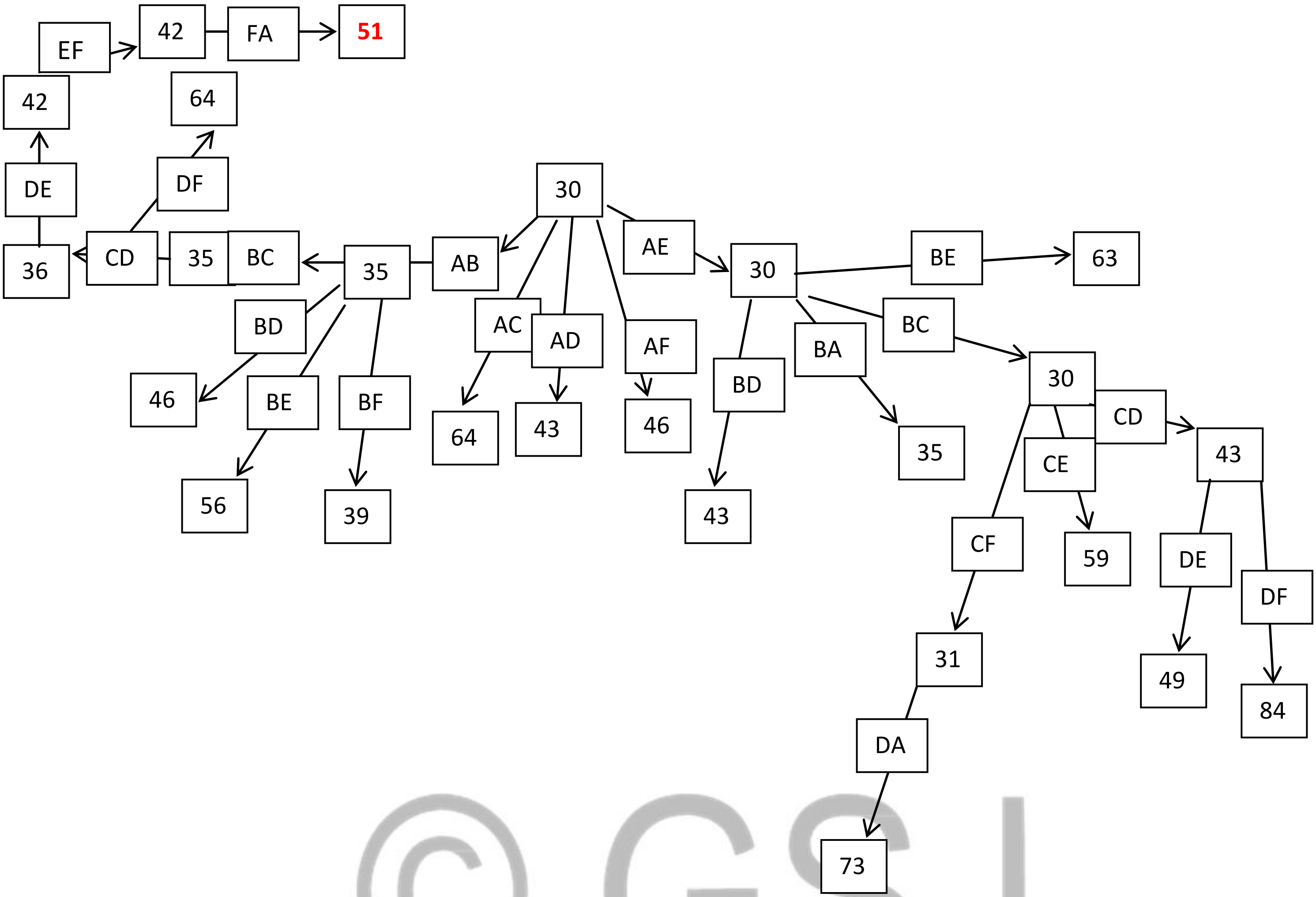

Fig 2. The Branch and Bound Method

The minimum movement in every row is summed to get a total initial minimum of 30 miles. Starting with location A, the shortest distance is found for all possible movements followed by location B etc. A-B-C-D-E-F-A segment found by Branch and Bound method gives distances of 42+9 = 51 miles which is better than the solution found by ARM Method. However the Branch and Bound Method becomes laborious with the increase in number of nodes while ARM heuristic is not laborious for large number of nodes. ARM Method is recommended to be used with the node map if available for quick choice of alternative segments.

Simpler Branch and Bound heuristics are available but not discussed in this paper.

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Section 5

Conclusion:

A heuristic is a compromise between accuracy and simplicity. The exact solution can be obtained by simulating each and every possible combination of movements. The ARM heuristic is quicker and more accurate than Branch and Bound Method discussed in this paper. *More, it is a mathematically supported and visually assisted heuristic.*

In equation 1 the value of Ψ can be increased for finding movements close to original Hungarian Assignment Method and reduce calculation time. If the number of nodes in the replacement segment, in ARM heuristic, is more than original segment by 2, the second term in LHS should be multiplied by 2. The number of nodes in the trial replacement segment should be about half of the total nodes in the problem. It is advantageous to have the picture/ map of nodes and movements to assist ARM heuristic for faster optimal solution.

Many trials were performed with ARM Method and alternate heuristics on a map of different number of nodes and ARM heuristic was found to be less computationally intensive.

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