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USING TRIANGULAR NUMBER AS AN APPROXIMATION FORMULAE FOR SUM OF ARITHMETIC PROGRESSION

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Abstract

Comparing links between triangular numbers (t_n) to the partial summing series of positive integers $(\mathbb{Z}^+)$ in relation to sum of arithmetic progression (S_n) were investigated with simple routine algebraic process leading to a new innovation to be called an approximation formulae which will serve as an alternative formulae for sum of arithmetic progression (S_n) in series and sequences.

Key words. Triangular numbers, Sum of arithmetic Progression, Positive Integers, Series and Sequences, Figurative numbers, Partial Sum and Number Theory

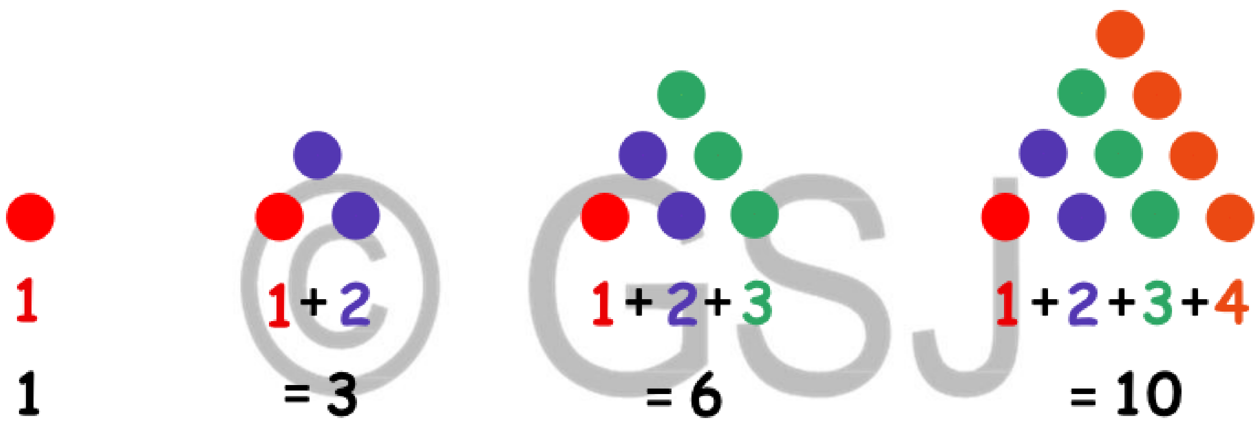
1 Introduction

Castillo (2016) regarded mathematics as the queen of sciences which was attributed to Gauss' attestation on the fundamentals of number theory to be the queen of mathematics, a thought-provoking statement. Mathematics with a contesting problem in series and sequences made the study of number figurative sequences a source of attraction to many mathematicians from ancient times especially in the theories of numbers (Panda & Ray, 2011).

Little wonder, mathematicians have been focusing their attention on the study of number theory paving ways for mathematical discoveries and fascinating properties of figurative numbers (e.g. triangular numbers, square triangular numbers, perfect square numbers, pronic numbers, metasquare numbers, diamond numbers, octagon numbers, pentagonal numbers, tetrahedral numbers, etc.) for a more detailed explorations in science, engineering and technology (especially in real estate, accounting, economics, medicine, manufacturing, aviation, banking and ecology) through an exactness of its conceptions and purity of its truth extolling the pattern for other sciences as an inexhaustible source of all mathematical knowledge and prodigal of incitements to investigation in other departments of mathematics. Ono, Robins & Wahl (1995) considered the problem of determining formulas for the number of representations of a natural number n by a sum of k triangular numbers and derived many applications, including the one connecting these numbers with the number of representations of n as a sum of k odd square integers, they also obtained an application to the number of lattice points in the k-dimensional sphere.

Beldon and Gardiner (2002) compared triangular numbers with perfect squares and established that the sum of any two consecutive triangular numbers is always a perfect square. This fact was known to the ancient Greeks and attributed the results to Theon of Smyrna (Heath, 1981) with the name triangular numbers stemming from the Greek interest in figurative numbers whereas interest in square triangular number by Burn (1991) metamorphosed into continued fractions.

Kleiner (2012) introduced cardinal numbers as an example of figurative numbers in the theories of number that can of course be studied as a mathematical topic without reference to history but when viewed in an historical setting as the resolution of centuries of groupings for the meaning of the infinite in mathematics, they acquire special significance as revealed in Tattersall (1999) that Pythagoreans study on polygonal numbers by depicted triangular numbers as been represented by triangular array of dots and sum of positive integers



This research work intends to investigate the relation between the partial sums of positive integers ($\mathbb{Z}^+$) to triangular numbers (t_n) and the sum of arithmetic progression (S_n) as to enacting a new approximating formulae as an alternative for sum of arithmetic progression to aid better understanding and improve knowledge in areas of series and sequences.

2 Partial Sum of Positive Integers and Triangular Numbers

Let sequence of positive integers ($\mathbb{Z}^+$) be define as

$$1, 2, 3, \dots, n \tag{1}$$

with series

$$t_n = 1 + 2 + 3 + \cdots + n \tag{2}$$

The partial sum presented by (S_n^*) for the sequence of positive integers ($\mathbb{Z}^+$) is the sum of n^{th} terms of the sequence in equation (1).

Suppose

$$S_1^* = 1 \Rightarrow T_1$$

$$S_2^* = 1 + 2 = 3 \Rightarrow T_1 + T_2$$

$$S_3^* = 1 + 2 + 3 = 6 \Rightarrow T_1 + T_2 + T_3$$

then

$$S_n^* = 1 + 2 + 3 + \cdots + n \Rightarrow T_1 + T_2 + T_3 + \cdots + T_n$$

Adding together series of positive integers ($\mathbb{Z}^+$) in ascending order of 1 to n and descending order of n to 1 gives,

$$t_n = 1 + 2 + 3 + \cdots + n$$

$$t_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$$

$$t_n + t_n = 2t_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

in n factors,

$$2t_n = n(n+1)$$

$$t_n = \frac{n(n+1)}{2}; n \in \mathbb{Z}^+ \quad (3)$$

or

$$t_n = \frac{n}{2}(n+1); n \in \mathbb{Z}^+$$

Breiteig (2015) defined the n^{th} term of a triangular number as

$$t_n = \frac{1}{2}n(n+1); n \in \mathbb{Z}^+ \quad (4)$$

compared to other study by Okagbue, Adamu, Iyase and Opanuga (2015) that generated sequence of integers for the summing digits to form an integer patterns or progressions which best described by algebraic formulae, recurrences and identities as

$$t_n = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{1}{2}n(n+1); n \in \mathbb{Z}^+ \quad (5)$$

3 Sum of Arithmetic Progression for Positive Integers ($\mathbb{Z}^+$)

The sum of arithmetic progression denoted by (S_n) is given as

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(T_1 + T_n) = \frac{n}{2}(a + T_n) \quad (6)$$

where, $T_n = a + (n-1)d$ and $T_1 = a$ indicating the last term of the series and the first term respectively. Applying partial sum of arithmetic progression (S_n) on series of positive integers in equation (2) gives

$$S_1 = 1 = T_1$$

$$S_2 = 1 + 2 = 3 = T_1 + T_2$$

$$S_3 = 1 + 2 + 3 = 6 = T_1 + T_2 + T_3$$

$$\vdots$$

$$S_n = 1 + 2 + 3 + \cdots + n = T_1 + T_2 + T_3 + \cdots + T_n$$

The partial sum (S_n^*) on sequence of positive integers in equation (1) and its sum of arithmetic progression (S_n) on the series of positive integers in equation (2) as seen in S_n^* and S_n respectively may be solved with equation (6)

$$S_n = \frac{n(n+1)}{2} \quad (7)$$

which is an indication that

$$S_n = S_n^* = \frac{n(n+1)}{2} = t_n \quad (8)$$

Equation (8) leads to investigating the relationship with view to reforming the formulae for sum of arithmetic progression (S_n) with an alternative to be deduced from triangular numbers which is the core interest of this research work.

4 Modification to Approximation Formulae

Representing the modification of equation (8) by A_n, then

$$A_n = \frac{l(l+1)}{2} = \frac{l(1+l)}{2(1)} = \frac{l(a+l)}{2d} \quad (9)$$

where, a is the first term, d is the common difference and l is the last term.

By routine algebra,

$$S_n = \frac{n(a+l)}{2} = \frac{n}{2}[2a + (n-1)d] = \frac{2na + n^2d - nd}{2} \quad (10)$$

Adjustment of equation (9) gives

$$A_n = \frac{[a+(n-1)d][a+a+(n-1)d]}{2d} \quad (11)$$

if $a = d$,

$$A_n = \left\{\frac{[d+(n-1)d]}{d}\right\}\left\{\frac{[2a+(n-1)d]}{2}\right\} = \frac{n}{2}[2a + (n-1)d] \quad (12)$$

indicating equation (12) as an exact formulae for equation (10)

if $a \neq d$, equation (9) is readjusted by replacing d for a

$$A_n^* = \frac{l(l+d)}{2d} \quad (13)$$

By routine algebra for further clarification,

$$S_n - A_n^* = \frac{a(d-a)}{2d}$$

indicating equation (13) as an approximation formulae for equation (10)

$$1 + 3 + 5 + \cdots + 101$$

$$S_n = 2601$$

$$A_n = 2575.5 \approx 2576 \simeq 2601$$

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$$A_n^* = 2600.75 \approx 2601$$

A_n^* is a better approximation formulae for S_n compared to A_n when $a \neq d$.

Comparing equation (12) for A_n when $a = d$ which is equivalent in equation (13) to equation (10) for S_n,

$$A_n^* = \frac{[a+(n-1)d][a+(n-1)d+d]}{2d} = \frac{(a+nd-d)(a+nd)}{2d} = \frac{a^2+2nad+n^2d^2-ad-nd}{2d} \tag{14}$$

$$S_n = \frac{n(a+l)}{2} = \frac{n}{2}[2a+(n-1)d] = \frac{2na+n^2d-nd}{2} = na + \frac{n^2d}{2} - \frac{nd}{2}$$

$$S_n - A_n^* = \frac{2na+n^2d-nd}{2} - \frac{a^2+2nad+n^2d^2-ad-nd}{2d} = \frac{a(d-a)}{2d} \tag{15}$$

Equation (15) is an indication that $S_n > A_n^*$ with the factor $\frac{a(d-a)}{2d}$ to be represented by D. In other words,

$$S_n - (A_n^* + D) = 0$$

it implies,

$$S_n - A_n^* - D$$

$$\frac{n}{2}[2a+(n-1)d] - \frac{l(l+d)}{2d} - \frac{a(d-a)}{2d} = \frac{nd[2a+(n-1)d] - l(l+d) - a(d-a)}{2d} = \frac{0}{2d} = 0$$

since $S_n - (A_n^* + D) = 0$, then an approximation formulae for S_n is $A_n^* + D$ which will now make an exact formulae for the well known sum of arithmetic progression (S_n).

$S_n^* = A_n^* + D$, that is, S_n^* is to be called an exact formulae for sum of arithmetic progression,

$$S_n^* = \frac{l(l+d)}{2d} + \frac{a(d-a)}{2d} \tag{16}$$

$$S_n^* = \frac{l(l+d)}{2d} + \frac{a(d-a)}{2d} = \frac{l^2+dl+ad-a^2}{2d}$$
$$= \frac{[a+(n-1)d]^2 + d[a+(n-1)d] + ad - a^2}{2d}$$

$$\frac{2nad - ad + a + n^2d^2 - 2nd^2 + nd + d^2 - d}{2d}$$

Solving the following problems with S_n and the proposed S_n^* in equation (16),

$$2 + 4 + 6 + \cdots + 19$$

$$S_n = 90$$

$$S_n^* = \frac{18(18+2)}{2(2)} + \frac{2(2-2)}{2(2)} = 90$$

$$6 + 12 + 18 + \cdots + 6n$$

$$S_n = 3n(n+1)$$

$$S_n^* = \frac{6n(6n+6)}{2(6)} + \frac{6(6-6)}{2(6)} = 3n(n+1)$$

$$1 + 4 + 7 + \cdots + (3n-2)$$

$$S_n = \frac{n(3n-1)}{2}$$

$$S_n^* = \frac{(3n-2)[(3n-2)+3]}{2(3)} + \frac{1(3-1)}{2(3)} = \frac{(3n-2)(3n-2+3)}{6} + \frac{2}{6} = \frac{n(3n-1)}{2}$$

6 Conclusion

In cognizance of previous research works on triangular numbers, figurative of triangular numbers and sums of triangular (t_n) in line with the summing series of positive integers $(\mathbb{Z}^+)$ called partial sum compared to sum of arithmetic progression on series of positive integers $(\mathbb{Z}^+)$ was the focus of this research work thereby paving ways for enacting a new approximation formulae to serve as an alternative for sum of arithmetic progression (S_n) (see equations 12, 13 & 16) in solving problems of series and sequences.

The newly enacted formulae (see equation 13) and its further adjustment (see equation 16) for sum of arithmetic progression (S_n) shall add values to number theory and pave ways for further research in recreational mathematics like figurative numbers and expand theories of numbers. This will encourage other users of series and sequences especially students of mathematics and other related disciplines in understanding the endowments embedded in theories of numbers by performing routine algebraic operations on existing formulae to creating a new innovations and inventions.

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