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A STOCHASTIC ALGORITHM IN SOLVING VARIATIONAL INEQUALITY SCHEME FOR AN ECONOMIC EQUILIBRIUM PROBLEM

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Abstract:

The existence of equilibrium in Walrasian economic model of pure exchange economic problem is considered. A stochastic iteration method in solving variational Inequality scheme of an economic equilibrium problem is examined. Let Ψ_+^l be a price vector, and Let P be a real price vector space. Let $p \to P$ be a continuous linear monotone operator and K is a nonempty closed convex subset of P. From an arbitrary initial point $x_0 \in K$, a stochastic iteration scheme $\{x_n\}$ is defined as follows: $x_{n+1} = x_n - aF^*(x_n),\ n \geq 0,\ \ where\ \ F^*(x_n), n \geq 0$ is a strong approximation of $Tx_n - b\ ,\ for\, b\ (possible\ zero) \in P$ and $a_n \in (0,1]$. Under suitable condition on a_n we show that $\{x_n\}$ converges strongly to the solution of variational inequality formulation of Walrasian equilibrium.

Key words: Linear monotone operator, Hilbert space, Stochastic approximation.
MSC Classification: 62L20, 65J15, 74G25

1. INTRODUCTION:

Mathematical models of equilibrium in economics is design to capture the effect of competing interests among different 'agents' in the face of limited availability of goods and other resources. The constant interaction among these agents around the existence of prices for goods under which the optimization carried out by these agents, individually, leads to a balance between supply and demand.

The fundamental ideas dated back to Walras and others, (see [1]) works of Arrow and Debreu, initiated the mathematical form of the subject.

In almost all real applications of these techniques, the data defining the model is "noisy" in the sense of having a fairly wide distribution of values (see [2])

The computation of economic game equilibria has been a great interest as highlighted in the works of Lemke and Howson (see [3]) and the work by Scarf H.E. and Hansen T. ([4]). The work of Dafermos.S.[5] is the extension of Smith's work [6] as a variational inequality problem

Another interest work was by Lars Mathiesen, who attempted to solve the Walrasian or general equilibrium model of economic activities for the nonlinear complementarity problem. In the economic reviewed literatures, fixed-point theory proposed by Scarf.H. [4] provided the environment for establishing the existence of equilibrium. The problem of how agents might arrive at equilibrium through Walras type procedures of tatonnement has been of interest by which the market clearing condition is eventually met. The question of whether equilibrium exists can be very subtle, even in a pure economic framework.

In the work of P.T. Harker and J-S. Pang [7], they highlighted the stochastic variational inequality approach necessary to capture the noises associated with the processes of attaining equilibrium.

This work proposed a stochastic algorithm in solving variational inequality associated to an equilibrium economic problem which converges strongly to the price and solves the problem posed by P.T. Harker and J-S. Pang [7].

2. PROBLEM FORMULATION:

The interest and importance of constructing the solution proposed in this paper is given any price vector

$$p(t) \in P_+^n \ ; \ P_+^n = \{x \in p^n : x \ge 0\}$$

$$\frac{dp(t)}{dt} = z(p) \ , \ p(0) = p_0 \qquad (1)$$

$z(p)$ is the excess demand mapping and represented by

$z(p) = z\big(D(p) - S(p)\big),\ \ where\ D\ and\ S$ are the usual demand and supply mapping respectively.

Then, the problem of finding an equilibrium price vector is as follows:

$$Find\ \ p^* \in \mathbb{P}_+^n\ \ such\ that\ \ z(p^*) = 0 \qquad (2)$$

Such equation (1) stem mainly from diverse areas of application, where problems could be reduced to finding solutions to equilibrium problems , such as in the initial value problems of mathematical physics of the type of (1), and in projected dynamical systems.

Other important examples are in physics, optimization, economics, and traffic analysis of the type.

$$\frac{dp(t)}{dt} - z(p) = b \ , \ p(o) = p_0 \qquad (3)$$

$where\ b = 0$, it has strict connection with fixed-point problems of

$x^* = Q_k(x^* - sTx^*)$, where Q_k is the nearest point projection map (4)

Let $<,>$ denote the usual inner product of two vectors and $\|x\|^2 = < x, x >$. It is well known that if the excess demand $z(p)$ of the market satisfies the following:

(a) $< z(p), p >= 0$ that is Walrasian law is satisfied

(b) $z(\alpha p) = z(p), \alpha > 0$, that is homogeneity of degree zero, (see [8]) (5)

The following theorem establishes that Walrasian price vector can be characterized as a solution of a variational inequality.

Theorem 1. (Variational Inequality Formulation of Walrasian Equilibrium):

A price vector $p^* \in \mathbb{P}$ is a Walrasian equilibrium if and only if it satisfies the variational inequality:
$< z(p^*), p - p^* > \leq 0 \; \forall p \in \mathbb{P}$, where $\mathbb{P} = \{p: p > 0, p \in \mathbb{P}\}$ (6)

These equilibrium problems can be recast as co-variational inequality problem involving a continuous linear monotone map.

Find a unique point $p^* \in \mathbb{P}$ $such\ that\ CVI(T, p): < Tp^* - b, p - p^* > \forall p \in \mathbb{P}$ (7)

The fundamental connection between (5) and (7) is in theorem 2.

Theorem 2: Let $T: p \to p$ be a linear continuous monotone operator then the following statements are equivalent.

(a) p^* satisfies the CVI $< Tp^* - b, p - p^* >, \forall\, p \in \mathbb{P}$

(b) $Tp^* = b$ (8)

It is known that if T is continuous and strongly monotone then

$CVI\ (T, p): < Tp^* - b, p - p^* > \; \forall\, p \in \mathbb{P}$

has a unique solution (see [10],[11]). Most of the research efforts on construction of iterative methods for approximating equilibrium using the famous Mann, Isikawa ([13]) and other iterative processes have been devoted to the case where T possesses strong monotonicity properties. That is the mapping $T: p \to \mathbb{P}$ is said to strongly monotone over $T\ if\ \exists\ an\ \alpha > 0\ \ such\ that$

$< F(p) - F(p^*) >< p - p^* > \geq \alpha \|p - p^*\|^2 \; \forall\, p, p^* \in \mathbb{P}$ (9)

For practical consideration, one wants to generate an approximation of p^* from experimental observations, a deliberate introduction of uncertainty often provides a more accurate method of investigation, which suggest a stochastic approximation method.

Our aim in this paper is to present a stochastic iteration process which converges strongly to the unique solution of (1) and consequently to that of (7).

3. FORMULATION OF STOCHASTIC ITERATION SCHEME

We associated with each random vector defined on a probability space (Ω, f, P) and a fixed $a \in H$ the expectation operator E such that Eu is defined as

$$E < a, u > = < a, Eu > \; \forall\, a \in H\ if\ E\|u\| < \infty \quad (10)$$

where $\|u\|, < u, v >, < a, u >$ are random variables in the usual sense. It is know that there exists a continuous , convex and everywhere differentiable scalar function

$$f: H \to R\ \ such\ that\ \ \Phi(x) = \frac{1}{2} < Tp, p > - < b, p > \quad (11)$$

The gradient mapping $\nabla\Phi$ is determined for a given $T\ and\ b$ coincides with $Tp - b \; \forall\, p$

Using the property of convex function [9] determining p^* such that $Tp^* = b$ and

$< Tp^* - b, p - p^* > \; \geq 0 \; \forall\, p \in \mathbb{P}$ reducing to finding the unique zero

$p^*\ of\ \nabla\Phi(\mathrm{p}^*)$ such that $\nabla\Phi(\mathrm{p}^*) = 0$ (12)

Given that (12) is a strict monotone operator then, it suffice that approximating iteration scheme $p^{k+1} = p^k - a_k \nabla\Phi(\mathrm{p}^k)$ (13)

converges strongly to a zero of $\nabla\Phi$ using the sequence $\{a_k\}$.

We consider the usual form of stochastic approximation recursion for determining the unique price $p^* \in \mathbb{P}$ *given as* $p^{k+1} = p^k - a_k d^k$ (14)

where $\nabla\Phi(\mathrm{p}^k) = \int \mathrm{d}(\theta)\mathbb{P}_{\mathrm{p}_k}(d\theta)$ (15)

and $\mathbb{P}_{p_k}$ is the probability law of $d(p^k) = d^k$ (16)

d^k is an approximation, $\{a_k\}$ is a sequence of scalars that converges to zero.

The introduction of randomness into the iteration process speeds convergence and makes the algorithm less sensitive to modeling errors.

We consider a sequence of random vectors generated by d^k , computed from different prices $p \in \mathbb{P}$ for each k *and* $j = 1, \dots, N$ in such a way that d^k strongly approximates $\nabla\Phi(\mathrm{p}^k)$ and consistent in mean square such that the distance between true and estimated gradient vectors are minimized as follows:

$$E\left\|d^k - \nabla\Phi(\mathrm{p}^k)\right\| = 0 \;\; for\ each\ k \qquad (17)$$

and consistent with $\nabla\Phi(\mathrm{p}^k)$, in mean square if $E\left\|d^k - \nabla\Phi(\mathrm{p}^k\right\|^2 \to 0$ *as* $N \to \infty$ (18)

4. CONVERGENCE THEOREM FOR THE ITERATIVE ALGORITHM

Theorem 3: Let the sequence $\{x_k\}, k \geq 0$ of positive numbers satisfies the following conditions:

(i) $a_0 = 1, 0 < a_k < 1 \;, \forall\, k > 1$,

(ii) $\sum_{k=0}^{\infty} a_k = \infty$, and

(iii) $\sum_{k=0}^{\infty} (a_k)^2 < \infty$

Given that $\{d^k\}$ is a sequence of random vector satisfying (17) and (18), then the stochastic sequence $\{p^k\}_{k\geq 0}$ *in* $\mathbb{P}$ defined iteratively from $p_0 \in D(\Phi)$ by (14) converges strongly to the unique price $\{p^* : \nabla\Phi(\mathrm{p}^*) = 0\} \in \mathbb{P}$ $a.s.$

Proof: Let $\Lambda_k = a_k\left\|d^k - \nabla\Phi(\mathrm{p}^k)\right\|$ and

Let $\left\|d^2 - \nabla\Phi(\mathrm{p}^k)\right\| = \sigma^2/N, \sigma^2 \in (0, \infty)$, then $\{\Lambda_k\}$ is a sequence of independent random variables. From (17) , $E\Lambda_k = 0$ *for each* k , thus

$S_k = \sum_{j=1}^{k} \Lambda_\mathrm{k}$ is a Martingale

But $ES_k^2 = \sum_{j=1}^{k} K\Lambda_k^2 = \sum_{j=1}^{k} a_j^2 \; E\left\|d^2 - \nabla\Phi(\mathrm{p}^j)\right\|^2$

$$= \frac{\sigma^2}{N}\sum_{j=1}^{k} a_j^2$$

$$\Longrightarrow \sum_{j=1}^{\infty} E\Lambda_j^2 < \infty$$

Thus , by Martingale convergence theorem [12], we have

$$\sum_{k=1}^{\infty} \Lambda_k < \infty$$

$$\Rightarrow \lim_{k\to\infty} a^k\left\|d^k - \nabla\Phi(\mathrm{p}^k)\right\| = 0$$

$$\Rightarrow \;\; d^k \longrightarrow \nabla\Phi(\mathrm{p}^k).$$

4

The stochastic iterative scheme yields the following algorithms generated by the sequence $\{d^k\}$ is stated below.

Let p^k be an estimate of p^*

(a) compute $d^k \approx \nabla\Phi(\mathrm{p}^{\mathrm{k}})$

(b) compute a_k

(c) compute $p^{k+1} = p^k - a_k d^k$

(d) check for convergence

If $\|p^{k+1} - p^k\| < \delta\,, \delta > 0$

Is Yes: set $p^{k+1} = p^k$

Otherwise: set $k = k + 1$ and return to (a).

6. CONCLUSION:

For any $p^* \in \mathbb{P}$ *such that* $z(p^*) = 0$ there exist a Walrasian equilibrium price vectors p^* such that the price adjustment process governed by the stochastic iterative scheme converges to $p^* \in W$, W is the set containing all market equilibrium price vector.

In an economy with Walrasian equilibrium, there are no price vector other than Walrasian price vector that have the property of global stability under the stochastic price adjustment process.

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