

A note on Bank-Laine sequences

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In this paper we show that the sequence $1, -3, 5, -7, 9, \dots$ forms a BL-sequence also we show that the product of two Bank-Laine functions need not be a Bank-Laine function.

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1. Introduction

Let A be an entire function, and let f_1, f_2 be linearly independent solutions of the differential equation

$$y'' + A(z)y = 0$$

normalized so that the Wronskian $W(f_1, f_2) = 1$. The Bank-Laine product $E = f_1 f_2$ satisfies $E'(z) = \pm 1$ at every zero of E . It was observed by Shen [3,4] that if (a_n) is a complex sequence tending to infinity without repetition, then there exists a Bank-Laine function F with zero-sequence (a_n) , the construction based on the Mittag-Leffler theorem. There are some sequences which form a BL-sequence and some do not form BL-sequence [5]. A natural question arising: for which sequence (a_n) , with finite exponent of convergence does there exist a Bank-Laine function E of finite order with zero sequence a_n . Langley [1,2] answered the question showing that the answer is negative for a large class of sequences.

Definition 1.1.[5] Let g be an entire function. We say that g has the BL property if $g'(z) = \pm 1$ at every zero of g .

2. Main results

Example 2.1. The sequence $1, -3, 5, -7, 9, -11, 13, -15, \dots$ forms a BL-sequence.

Proof: This sequence can be written as $\{1, 5, 9, 13, 17, \dots\} \cup \{-3, -7, -11, -15, \dots\}$. Now we know that [6]

$$f(z) = \cos \frac{\pi z}{4} - \sin \frac{\pi z}{4} = (1 - z)\left(1 + \frac{z}{3}\right)\left(1 - \frac{z}{5}\right)\left(1 + \frac{z}{7}\right)\left(1 - \frac{z}{9}\right) \dots$$

It is clear that the zeros of $f(z)$ is the sequence $1, -3, 5, -7, \dots$

$$\begin{aligned} g(z) &= \cos\left(\frac{\pi z}{4} + \frac{\pi}{4}\right) \\ &= \cos \frac{\pi z}{4} \cos \frac{\pi}{4} - \sin \frac{\pi z}{4} \sin \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}}\left(\cos \frac{\pi z}{4} - \sin \frac{\pi z}{4}\right) = \frac{1}{\sqrt{2}}f(z). \end{aligned}$$

Thus

$$\frac{4}{\pi}g(z) = \frac{4}{\pi\sqrt{2}}f(z)$$

is a Bank-Laine function of finite order.

Theorem 2.1. If f_1 and f_2 have the Bank-Laine property and no common zeros, then $f_1 f_2$ need not have a Bank-Laine property.

proof: Let $S_1 = \{a_1, a_2, a_3, \dots\}$ and $S_2 = \{b_1, b_2, b_3, \dots\}$ be the zeros of f_1 and f_2 respectively and $S_1 \cap S_2 = \emptyset$. Suppose that f_1 and f_2 have the Bank-Laine property and assume that $f_1 f_2$ on the contrary have the Bank-Laine property. Then there is an entire function E of finite order such that E has simple zeros in the set $S = S_1 \cup S_2$ and no other zeros and such that $E = 0$ implies $E' = \pm 1$. so

$$\begin{aligned} E(z) &= f_1 f_2 e^{\psi(z)} \\ E'(z) &= f_1 f_2' e^{\psi(z)} + f_2 f_1' e^{\psi(z)} + \psi'(z) f_1 f_2 e^{\psi(z)} \end{aligned}$$

If $z \in S_1$ then we have

$$E'(z) = \pm 1 = \pm f_2(z) e^{\psi(z)}$$

this implies

$$e^{\psi(z)} = \pm \frac{1}{f_2(z)} \text{ at each zero of } f_1$$

or

$$e^{re\psi(z)} = \frac{1}{|f_2|}$$

then

$$re(\psi(z)) = \sum_{k=0}^m re(b_k) z^k.$$

Thus $Re(\psi(z))$ is apolynomial in z , and so either it is constant or tends to infinity ,and so $e^{re\psi(z)}$. $e^{re\psi(z)}$ can go to infinity faster than $\frac{1}{|f_2|}$ and if $f_2(z)$ is bounded then $\frac{1}{|f_2|}$, and if $|f_2|$ is bounded then $\frac{1}{|f_2|}$ is also bounded but $e^{re\psi(z)}$ need not be bounded. Similarly if $z \in S_2$ then we have

$$e^{re\psi(z)} = \frac{1}{|f_1|}$$

in both cases we reach to contradiction.

Theorem 2.2 If α, β are complex numbers with α/β not a real rational number, $\alpha_1, \dots, \alpha_m$ are pair wise distinct non-zero complex numbers, $\beta_1, \dots, \beta_n$ are pairwise distinct non-zero complex numbers, then there is no Bank-Laine function of form

$$E(z) = \prod_{j=1}^m (e^{\alpha_j z} - \alpha_j) \prod_{k=1}^n (e^{\beta_k z} - \beta_k) e^{Q(z)}$$

with Q a polynomial.

Proof: Let z_{ri} be a solution of the equation $e^{\alpha_j z} - \alpha_j = 0$, $i = 1, \dots, m$, $z_{ri} = \frac{1}{\alpha_j} \log \alpha_j = \frac{1}{\alpha_j} (\log |\alpha_j| + i(\arg \alpha_j + 2r\pi))$, $r = 0, \pm 1, \pm 2, \pm 3, \dots$

$E(z_{ri}) = 0$, $\pm 1 = E'(z_{ri}) = \alpha_j \prod_{\substack{j=1 \\ i \neq j}}^m (\alpha_j - \alpha_i) \prod_{k=1}^n (\alpha_j^{\beta/\alpha} - \beta_k) e^{Q(z_{ri})}$ so, we have

$$1 = \left| \alpha_j \prod_{\substack{j=1 \\ i \neq j}}^m (\alpha_j - \alpha_i) \prod_{k=1}^n (\alpha_j^{\beta/\alpha} - \beta_k) e^{Q(z_{ri})} \right|$$

$$= |\alpha_j| \prod_{\substack{j=1 \\ i \neq j}}^m |\alpha_j - \alpha_i| \prod_{k=1}^n |\alpha_j^{\beta/\alpha} - \beta_k| e^{\operatorname{Re} Q(z_{ri})}$$

or

$$e^{-\operatorname{Re} Q(z_{ri})} = |\alpha_j| \prod_{\substack{j=1 \\ i \neq j}}^m |\alpha_j - \alpha_i| \prod_{k=1}^n |\alpha_j^{\beta/\alpha} - \beta_k|$$

Since α/β is not a real rational number (irrational or complex) then, we have $\beta_i^{\alpha/\beta}$ has infinitely many different values, so $|\beta_i^{\alpha/\beta} - \alpha_k|$ can not be estimated and the function $Q(z_{ri})$ can not be determined, so we are lead to a contradiction.

If z_{ri} is a solution of the equation $e^{\beta_k z} - \beta_k = 0$, $i = 1, 2, \dots, n$, $z_{ri} = \frac{1}{\beta_k} \log \beta_k = \frac{1}{\beta_k} (\log |\beta_k| + i(\arg \beta_k + 2r\pi))$, $r = 0, \pm 1, \pm 2, \dots$ $E(z_{ri}) = 0$, $\pm 1 = E'(z_{ri})$ or

$$\pm 1 = \beta_k \prod_{\substack{j=1 \\ i \neq j}}^m (\beta_k - \beta_j) \prod_{k=1}^n (\beta_k^{\alpha/\beta} - \alpha_k) e^{Q(z_{ri})}$$

we have

$$1 = \left| \beta_k \prod_{\substack{j=1 \\ i \neq j}}^m (\beta_k - \beta_j) \prod_{k=1}^n (\beta_k^{\alpha/\beta} - \alpha_k) e^{Q(z_{ri})} \right|$$

$$= |\beta_k| \prod_{\substack{j=1 \\ i \neq j}}^m |\beta_k - \beta_j| \prod_{k=1}^n |\beta_k^{\alpha/\beta} - \alpha_k| e^{\operatorname{Re} Q(z_{ri})}$$

or

$$e^{-reQ(z_{ri})} = |\beta\beta_i| \prod_{\substack{j=1 \\ i \neq j}}^m |\beta_i - \beta_j| \prod_{k=1}^n |\beta_i^{\alpha/\beta} - \alpha_k|$$

Since $\beta_i^{\alpha/\beta}$ has infinitely many different values, because α/β is irrational or complex number, so $|\beta_i^{\alpha/\beta} - \alpha_k|$ can not be estimated, so the function $Q(z_{ri})$ can not be determined, so we are lead to a contradiction.

example 2.1 The function $f_1 \cos z$ can be written in the form $\frac{1}{2}e^{-iz}(e^{2iz} + 1)$, and $f_2 \sin \pi z$ as $\frac{1}{2i}e^{-i\pi z}(e^{2\pi iz} - 1)$. It is clear that α/β is irrational, so $f_1 f_2$.

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