

A simple proof of Euler's formula and DeMoivre's theorem

The objective of this paper is to prove Euler's formula and DeMoivre's theorem without the use of infinite series. An elementary justification for Euler's formula is found in [1].

Our starting point is the following identity

$$\cos^2 n\theta + \sin^2 n\theta = 1. \quad (1)$$

Equation (1) can be factorized and rewritten as following

$$(\cos n\theta + i \sin n\theta)(\cos n\theta - i \sin n\theta) = 1, \quad (2)$$

where $i = \sqrt{-1}$.

Let

$$f(\theta) = \cos n\theta + i \sin n\theta \quad \& \quad g(\theta) = \cos n\theta - i \sin n\theta \quad (3)$$

It is clear that

$$f(0) = 1 \quad \& \quad g(0) = 1 \quad (4)$$

We note that

$$f'(\theta) = -n \sin n\theta + ni \cos n\theta = ni^2 \sin n\theta + ni \cos n\theta = ni(\cos n\theta + i \sin n\theta) = nif(\theta) \quad (5)$$

Similarly

$$g'(\theta) = -nig(\theta). \quad (6)$$

Therefore (2) can be written as

$$fg = 1. \quad (7)$$

Differentiate (7), we get

$$fg' + gf' = 0. \quad (8)$$

Dividing bothsides of (8) by fg , we get

$$\frac{g'}{g} + \frac{f'}{f} = 0. \quad (9)$$

Now, from (3), (5) and (9), we get

$$\frac{g'}{g} + ni = 0. \quad (10)$$

Integrating (10) yields

$$g(\theta) = c_1 e^{-i\theta}. \quad (11)$$

Since $g(0) = 1$, we get $c_1 = 1$, so

$$g(\theta) = e^{-ni\theta} = \cos n\theta - i \sin n\theta. \quad (12)$$

Similarly from (3), (6), and (9), we get

$$\frac{f'}{f} - in = 0. \quad (13)$$

Integrating (13), yields

$$f(\theta) = c_2 e^{ni\theta}. \quad (14)$$

Since $f(0) = 1$, we get $c_2 = 1$, so

$$f(\theta) = e^{ni\theta} = \cos n\theta + i \sin n\theta. \quad (15)$$

Euler's formula can be obtained if we put $n = 1$ in (15)

We have $\cos n\theta + i \sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos \theta + i \sin \theta)^n$ which proves DeMoivre's theorem.

References

[1] C. B. Allendoerfer, Editorial-The proof of Euler's Equation, American Mathematical Monthly Volume 55(1948) pp. 94-95.

Or (p. 125 of Selected papers on Calculus from MAA).

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