

# Conformal Mapping and its Application in Electromagnetism

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## Abstract

Conformal mapping is a technique in complex analysis that preserves angles locally while transforming one region of complex plane into another through analytic functions. In electromagnetic, this property is invaluable because Laplace's equation, which governs static electric and magnetic fields in charge-free regions, is invariant under conformal transformations. This study presents geometric insights and investigates the application of conformal mapping techniques as an analytical method for simplifying and solving two-dimensional electromagnetic field problems. The method is based on the principle of transforming a complicated geometry in the complex  $z$ -plane into a simpler geometry in the complex  $w$ -plane using analytic functions that preserve Laplace's equation and the associated boundary conditions. Dirichlet boundary value problem configuration where the electric potential confined plates within a bounded conducting plates and electric potential on a lightning arrester were determined and also Mobius transformation a classical form of conformal map was employed to map cylinder into upper half-plane then electrical potential and electric field inside cylinder with two separate electrodes were determined in the mapped domain and inversely transformed to the original domain.

**KEYWORDS:** Conformal Map, Laplace Equation, Harmonic Function, Electrostatics, Boundary Value Problem

## Introduction

Complex analysis offers elegant theorems and provides geometric insights into functions, which can be more engaging than some aspects of real analysis. (Anagha , 2025)

The driving force behind many of the applications of complex analysis is the remarkable connection between complex analytic functions and harmonic functions. In two dimensions, the real and imaginary parts of any analytic function  $f(z)$  are harmonic this is why conformal mapping preserves harmonic functions. (Evans 2010). The composition of complex functions is also a complex functions, this manipulation can be observed as a complex change of variables that provide a conformal map of the form

$$w = f(z) \quad \dots 1$$

Conformal mapping is a central instrument in complex analysis because an analytic map with a non vanishing derivative on the infinitesimal level acts locally as rotation multiplied by a complex number and a dilation in the complex plane. This enables harmonic functions and potential fields to be mapped from one domain to another without destroying the form of Laplace-type equations (Alfors, 1979; Driscall & Trefethen, 2002; Naqos et al, 2024, Haider & Marwan, 2026).

The fundamental importance of conformal mappings stems from their dual nature - they are deeply rooted in complex function theory while offering powerful techniques for solving practical problems. Sajahan (2023)

Microstructural evolution is typically beyond the reach of mathematical analysis, but in two dimensions certain problems become tractable by complex analysis. Via the analogy between the geometry of the plane and the algebra of complex numbers, moving free boundary problems may be elegantly formulated in terms of conformal maps (Martin and Darren 2005)

Conformal mapping has many consequences in a wide range of physical problems, including fluid mechanics, aerodynamics, thermo mechanics, electrostatics, and elasticity (Peter 2019).

Conformal map preserves angles locally while transforming one region of complex plane into another through analytic functions. In electromagnetic,

this property is invaluable because Laplace's equation, which governs static electric and magnetic fields in charge-free regions, is invariant under conformal transformations. This means a difficult field problem in one geometry can be mapped to a simpler geometry where the solution is known then transformed back.

This study explores the efficacy of conformal mapping techniques in solving some electrostatics boundary value problems.

## Methodology

### Conformal Mapping

Conformal mapping technique could be used classically, in electrostatics, where the governing partial differential equation is Laplacian. This application rely on the conformal invariance property of the harmonic solution of a Dirichlet problem as well as the Carathéodory's boundary extension theorem. (Tri Quach, 2015).

Consider

$$z = x + iy \quad \dots 2$$

and

$$w = u + iv \quad \dots 3$$

a complex analytic function

$$w = f(z) \text{ or } u + iv = u(xy) + iv(xy) \quad \dots 4$$

can be re-interpreted as a mapping that takes a point  $z = x + iy$  belonging to a prescribed domain  $\Omega \in \mathbb{C}$  to a point  $w = u + iv$  belonging to the image domain  $D \in \mathbb{C}$ . In order to unambiguously relate functions on  $\Omega$  to functions on  $D$ , we require that the analytic mapping (4) be one-to-one, so that each point  $w \in D$  comes from a unique point  $z \in \Omega$ . As a result, the inverse function  $z = g^{-1}(w)$  is a well-defined map from  $D$  back to  $\Omega$ , which we assume is also analytic on all of  $D$ .

However, the central problem in the theory of conformal mapping is to determine a function  $f$  which maps a given region  $D \subset \mathbb{C}$  conformally onto another region  $G \subset \mathbb{C}$ . For instance . The Riemann Mapping Theorem, a cornerstone of this field, guarantees the existence of conformal maps between simply connected domains, providing a theoretical basis for numerous applications. ( Prem, 2019).

Existence of conformal mappings of simply connected domains onto the upper-half plane or the unit disk follows from the Riemann mapping theorem, a well-known result in complex analysis, and there are generalizations of this result for doubly and multiply connected domains. (Harri, 2019).

**Theorem1.** If  $\Omega \subseteq \mathbb{C}$  is any simply connected open subset, not equal to the entire complex plane, then there exists a one-to-one complex analytic map  $w = f(z)$ , satisfying the conformality condition  $f'(z) \neq 0$  for all  $z \in \Omega$ , that maps  $\Omega$  to the unit disk  $D = \{|w| < 1\}$ .

For instance if,

$$w = f(z) = z^2 \quad \dots 5$$

Therefore,

$$w = z^2 = u + iv$$

From,

$$z = re^{i\theta} = x + iy$$

So,

$$w = r^2 e^{i2\theta}$$

Therefore,

$$u = x^2 - y^2 = r^2 \cos 2\theta \quad \dots 6$$

and

$$v = 2xy = r^2 \sin 2\theta \quad \dots 7$$

Equations (6) and (7) shows that angles get doubled as  $\arg(w) = 2 \arg(z)$

Therefore (5) guarantees isomorphism between first quadrant and upper half plane.

So also if,

$$w = f(z) = \ln z \quad \dots 8$$

maps angles to strip width

$$w = \ln z = u + iv$$

from,

$$z = re^{i\theta} = x + iy$$

Then,

$$w = \ln z = \ln r + i\theta$$

$$u = \ln r = \ln|z|, v = \theta = \arg(z)$$

So that,  $w = \ln z$  splits polar coordinates into Cartesian coordinates. A classical conformal map that takes a half-plane and turns it into a strip is

$$w = \frac{1}{\pi} \log z \quad \dots$$

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$$= \frac{1}{\pi} \ln |z| + \frac{1}{\pi} i \arg z$$

Let,

$$w = u + iv = re^{i\theta}$$

Then,

$$u = \frac{1}{\pi} \ln r, \quad v = \frac{1}{\pi} \theta$$

$\log$  in (9) gives a strip and  $\frac{1}{\pi}$  makes the strip width

### Mobius Transformation

A map of the extended complex plane (which is conformally equivalent to a sphere) onto itself is conformal if and only if it is a Mobius transformation of the form

$$w = f(z) = \frac{az + b}{cz + d} \quad \dots 10$$

satisfying  $ad - bc \neq 0$  where  $a, b, c, d \in \mathbb{C}$ . Again, for the conjugate, angles are preserved, but orientation is reversed. (Yum-Tong Siu Math 115 (2006-2007)). The restriction on the determinant  $ad - bc$  is for the sake of the derivative (slope of the tangent line), which is

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \quad \dots$$

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If the determinant  $ad - bc = 0$ , then the original transformation,  $w$ , would be a constant function.

$$w = \text{Constant}$$

**Theorem 2;** Mobius transformation of form of (10) maps a circle in  $z$ -plane to a circle or straight line in  $w$ -plane.

For instance, if,

$$w = f(z) = i \frac{1 - z}{1 + z} \quad \dots 12$$

maps circle to line

### Laplace's equation and Potential function

Real and imaginary part of a complex analytic function

$$f(z) = \varphi(x, y) + i\psi(x, y) \quad \dots 13$$

are solutions to two dimensional Laplace's equation:

$$\nabla^2 \varphi = \nabla^2 \psi = 0 \quad \dots 14$$

If  $\varphi(x, y)$  represents the potential function, ( $\varphi = \text{constant}$ ) represents equipotential lines and ( $\psi = \text{constant}$ ) represents lines of force. Since  $\varphi = \varphi_0$  (vertical line) is always perpendicular to  $\psi = \psi_0$  (horizontal line) in the  $\varphi\psi$ -plane,

For a given real potential  $\varphi(x, y)$ , we can uniquely (except for an additive constant) determine a conjugate  $\psi(x, y)$  by Cauchy-Reimann equations, such that complex potential  $f(z = x + iy) = \varphi(x, y) + i\psi(x, y)$  is analytic. As a result,  $f(z)$  maps curves in the  $xy$ -plane onto curves in the  $\varphi\psi$ -plane "conformally" or vice versa, i.e. included angle is preserved during mapping

### Application in electrostatics

Electromagnetic field problems are described by Maxwell's equations. For two dimensions, static and quasi-static cases, these reduce to Laplace equations. This reduction occurs because in electrostatics  $\nabla \cdot D = 0$  and  $\nabla \times E = 0$ , so the electric potential  $V$  satisfies  $\nabla^2 V = 0$  in charge-free regions. (Sadiku, 2018)

In electrostatic potential  $\Phi$  between two parallel conducting plates is governed by equation (14) and boundary conditions  $\varphi\left(\frac{d}{2}\right) = \Phi_1$  and  $\varphi\left(-\frac{d}{2}\right) = \Phi_2$

The solution is a linear function

$$\varphi(x) = ax + b \quad \dots 15$$

where,

$$a = \frac{\Phi_1 - \Phi_2}{d} \text{ and } b = \frac{\Phi_1 + \Phi_2}{2}$$

The equipotential line  $\varphi = \varphi_0$  vertical line  $x = x_0$  parallel to the plates. (Kreyszig, 2011)

The electrical field.

$$= -\nabla \varphi \quad \dots 16$$

implies that,

$$= a$$

which is perpendicular to the plates.

For coaxial cylinder, if  $\varphi$  is independent of  $\theta$ ,

$$\begin{aligned}\nabla^2 \varphi &= r^2 \varphi_{rr} + r \varphi_r = 0, \\ \Rightarrow \varphi(r) &= a \ln r + b\end{aligned}\quad \dots 17$$

where  $a$ ,  $b$  are determined by boundary conditions;

$$\varphi(r_1) = \Phi_1 \text{ and } \varphi(r_2) = \Phi_2$$

The equipotential line is a circle

$$r = r_0.$$

The electric field is

$$\begin{aligned}E &= -\nabla \varphi, \\ E &= \frac{a}{r}\end{aligned}\quad \dots 18$$

which is in radial direction, perpendicular to the equipotential lines

For a region confined by two plates in the radial directions and with an included angle  $\alpha$ .  
Using complex analytic functions:

$$f(z) = \ln z = \ln |z| + i \arg(z) \quad \dots 19$$

Satisfied with the two boundary conditions:

$$u\left(\theta = -\frac{\alpha}{2}\right) = \Phi_1, \quad u\left(\theta = \frac{\alpha}{2}\right) = \Phi_2$$

and

$$u(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

$u(x, y)$  also satisfied with two dimensional Laplace's equation for it is the imaginary part of equation (19)

$$\varphi = a + b\theta \quad \dots 20$$

where,

$$a = \frac{\Phi_1 - \Phi_2}{2} \text{ and } b = \frac{\Phi_2 - \Phi_1}{\alpha}$$

the equipotential line  $\varphi = \varphi_0$  is a ray  $\theta = \theta_0$ .

The Electrical field

$$= -\nabla \varphi.$$

$$\Rightarrow$$

$$= -\frac{b}{r} \quad \dots 21$$

## Result

Conformal maps are clutch because Laplace's equation remains Laplace's equation after mapping. So we solve it in a computational geometry, then map back to the domain of interest

Illustration 1,

Electric potential and electric field intensity analysis in a domain  $\Omega$  bounded radially by two conducting plates with angle  $\beta$ , satisfied with  $u\left(\theta = -\frac{\beta}{2}\right) = V_1$ ,  $u\left(\theta = \frac{\beta}{2}\right) = V_2$ .

$u(x, y) = \tan^{-1}\left(\frac{y}{x}\right) (= \theta)$  as boundary conditions

Solution

Conformal map,

$$f(z) = \ln z \quad \dots 22$$

$$= \ln|z| + i \arg(z)$$

$u(x, y)$  is satisfied with two dimensional Laplace's equation for it is the imaginary part (22)

Therefore, the electric potential is given by

$$V = a + b\theta$$

$$\text{where } a = \frac{V_1 + V_2}{2} \quad \text{and} \quad b = \frac{V_2 - V_1}{\beta},$$

The equipotential line  $V = V_0$  is a ray  $\theta = \theta_0$ . (figure 1)

The Electrical field

is given by

$$= -\nabla \varphi.$$

$$\Rightarrow$$

$$= -\frac{b}{r}$$

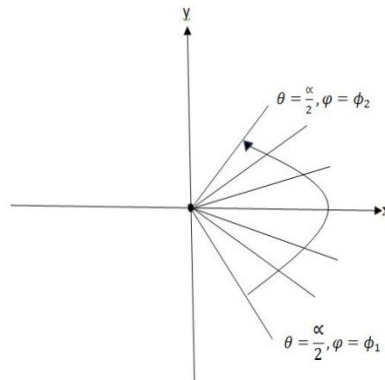


Figure 1

Illustration 2,

Determine the electric field intensity and electric potential  $V$  for a lightning arrester of height  $h$  which the ground represent by  $Imz=0$  on a given domain  $\Omega$  equivalent to the upper half-plane  $Imz>0$ . Consider electric potential  $u$  on the domain  $\Omega$  whose value at  $Imz=0$  and a lightning arrester is 0 and whose value has limit  $\infty$  as  $Imz \rightarrow \infty$ .

Solution.

Let lightning arrester represent by a line segment  $(0, hi)$  shown in figure 2

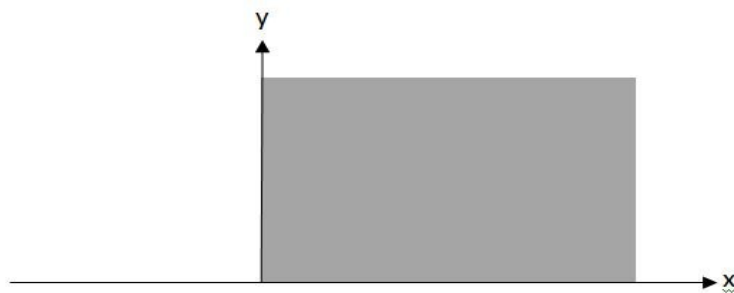


Figure 2

Conformal map that maps the domain  $\Omega$  to  $\mathbb{C}$  minus  $(\Im z=0, \Re \geq -h^2)$  is

$$z_1 = z^2 = (x + iy)^2 \quad \dots 23$$

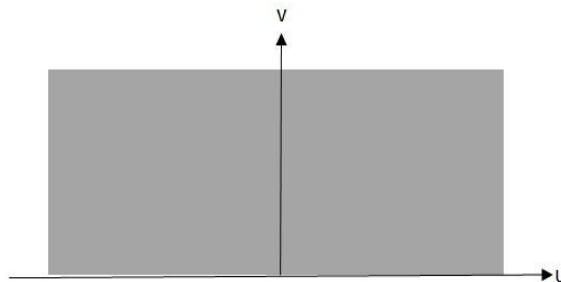


Figure 3

Therefore,

$$z_2 = z^2 + h^2$$

maps the domain  $\Omega$  to  $\mathbb{C}$  minus  $(\Im z = 0, \Re z \geq 0)$  as seen in figure 3.

Let  $w$  be the branch of  $\sqrt{z^2}$  defined by  $(0, 2\pi)$

Therefore, the electric potential  $V$  can be set as real part of

$$w = \sqrt{z^2 + h^2} \quad \dots 24$$

and the electric field is given by

$$\begin{aligned} &= -\nabla u \\ &\dots 25 \\ &= - (u_x, u_y) = - (u_x, -v_x) \\ &= - \frac{\partial(u - iv)}{\partial x} = \frac{\square}{dz} \\ &\dots 26 \end{aligned}$$

$$= - \frac{\square}{\sqrt{z^2 + h^2}}$$

### Illustration 3

Analysis of potential distribution and electric field inside cylindrical geometry containing two separate electrodes in which  $V=0$  on electrode at top boundary and  $V=1$  on electrode at bottom boundary

### Solution

Potential field inside cylinder with two electrodes can be represented as a potential distribution inside a unit disk as shown in figure 4

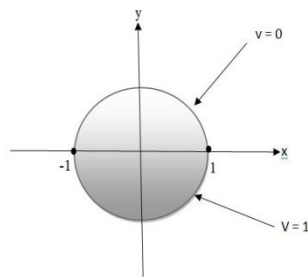


Figure 4

Since the boundary values specified, solving Laplace's equation on a disk is dirichlet problem

Given that,

$$V(x, y) = 0 \text{ if } x^2 + y^2 = 1 \text{ and } y > 0$$

and

$$V(x, y) = 1 \text{ if } x^2 + y^2 = 1 \text{ and } y < 0$$

Therefore, the voltage inside disk is

$$V_{xx} + V_{yy} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \dots$$

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$$\text{for } x^2 + y^2 < 1$$

Conformal mapping that take the points  $z=1$  and  $z=-1$  from original domain send it to points 0 and  $\infty$  respectively on transformed domain shown in figure 5a and 5b is

$$w = f(z) = i \frac{1-z}{1+z} \quad \dots$$

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which is clearly a mobius transformation

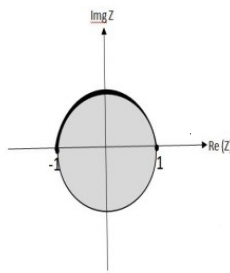


Figure (5a)

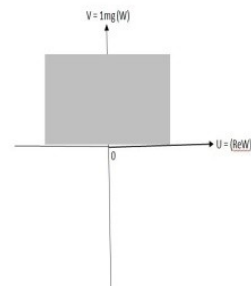


Figure (2b)

Therefore, Solve Laplace's equation on the upper half plane  $w$ -plane

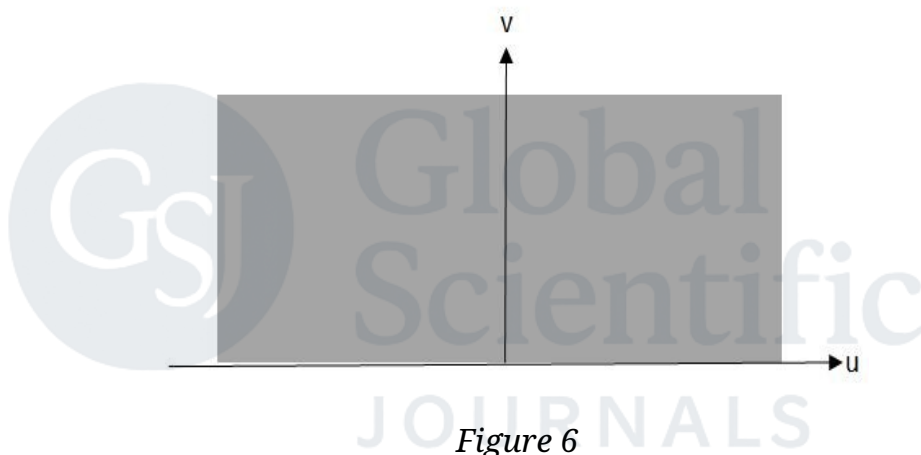


Figure 6

$$V(u, v) = 0 \quad \text{for } v = 0, \quad u > 0$$

$$V(u, v) = 1 \quad \text{for } v = 0, \quad u < 0$$

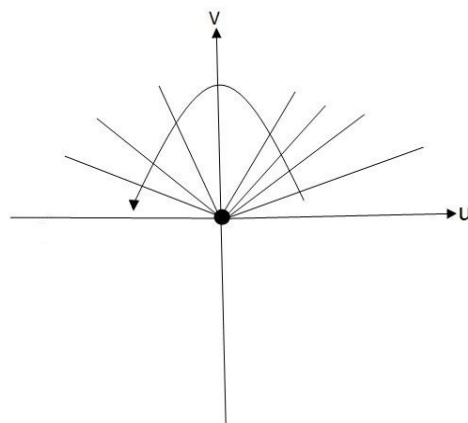


Figure 7

Note that, if

$$W(w) = \frac{1}{\pi} \log w \dots 29$$

$$= \frac{1}{\pi} \ln |w| + \frac{1}{\pi} i \arg w \quad -r < \arg w < r$$

In upper half plane  $0 < \arg < \pi$

$W$  is analytic so imaginary part is harmonic also imaginary part satisfies boundary conditions. so (existence and uniqueness of solutions to Laplace's equation)

$$V(u, v) = \frac{1}{\pi} \arg(u + iv) = \frac{1}{\pi} \arctan\left(\frac{v}{u}\right) \dots$$

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is required solution

Now transform  $V(u, v)$  to  $V(x, y)$  on the disk using

$$w = f(z) = i \frac{1-z}{1+z}$$

$$i \frac{1-z}{1+z} = i \frac{1-x-iy}{1+x+iy}$$

$$i \frac{1-x-iy}{1+x+iy} = \frac{i(1-2iy-x^2-y^2)}{1+2x+x^2+y^2}$$

$$w = u + iv = \frac{2y}{1+2x+x^2+y^2} + i \frac{1-x^2-y^2}{1+2x+x^2+y^2}$$

$$V(u, v) = \frac{1}{\pi} \arctan\left(\frac{v}{u}\right)$$

$$V(u, v) = \frac{1}{\pi} \arctan\left(\frac{1-x^2-y^2}{2y}\right)$$

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## Conclusion

This paper has presented the application of conformal mapping as an effective analytical tool for solving complex electromagnetic boundary value problems.

The study has shown that by employing appropriate conformal transformations, intricate

and irregular geometries in the physical  $z$ -plane can be mapped into simpler canonical domains in the  $w$ -plane, such as the upper half-plane, unit circle, or parallel-plate configuration, where Laplace's equation can be readily solved. The solutions obtained in the mapped domain were then inversely transformed to determine the electric potential and electric field in the original domain.

Results obtained for confined plates and lightning arrester configurations demonstrate that conformal mapping preserves both Laplace's equation and the boundary conditions, thereby providing exact closed-form solutions.

While the method is limited to two-dimensional electrostatic problems, its ability to provide accurate analytical insight, eliminate mesh generation errors, and simplify field computations makes it highly valuable in the design of high-voltage insulators, microstrip lines, MEMS devices, and shielded transmission lines.

Future work will extend the application to quasi-conformal mapping for handling inhomogeneous media and explore its integration with numerical methods for solving three-dimensional electromagnetic problems.

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### References

- Ahlfors, L. V. (1979). Complex analysis: An introduction to the theory of analytic functions of one complex variable (3rd ed.). McGraw-Hill.  
<https://archive.org/details/complexanalysis00ahlf>
- Anagha A. Bade (2025), Real and Complex analysis, *International Journal of Advanced Research in Computer and Communication Engineering*, & Vol. 14, Issue 4, ISSN (P) 2319-5940 DOI: 10.17148/IJARCCE.2025.144101

- Driscoll, T. A., & Trefethen, L. N. (2002). Schwarz-Christoffel mapping. Cambridge University Press. <https://doi.org/10.1017/CBO9780511546808>
- Evans L . C (2010) Partial Differential Equations (2<sup>nd</sup> ed). springer
- Haider A . M and Marwan A. L (2026). Applications of Matching Conformal Mappings in Analytic Functions, *Journal of Pure and Applied Sciences* 26, (4) (2026), 65 – 93  
<https://atbuscienceforum.com.ng/index.php/jpashttps://doi.org/10.70882/emig5e08>
- Kreyszig E. (2011) Advanced engineering Mathematics (10<sup>th</sup> ed.). John Willey & Sons
- Martin Z. B & Darren C., (2005). Conformal Mapping Methods for Interfacial Dynamics, Handbook of Materials Modeling. Volume I: Methods and Models, 1–35. Springer. Printed in the Netherlands
- Naqos, S., Murid, A. H. M., Nasser, M. M. S., & Yeak, S. H. (2024). Computing the Dirichlet to-Neumann map via an integral equation with the adjoint generalized Neumann kernel. *Partial Differential Equations in Applied Mathematics*, 12, Article 100967. <https://doi.org/10.1016/j.padiff.2024.100967>
- Tri Quach, (2015). Computational Method in Conformal and Harmonic Mappings, Doctoral Dissertations, school of Science, Aalto University Helsinki, Finland.
- Peter J. O., (2019). Complex Analysis and Conformal Mapping, Phd Thesis, University of Minnesota.
- Sadiqu, M. N. O (2018). Elements of electromagnetics (7<sup>th</sup> ed.) Oxford university Press.
- Sajahan Seikh (2023). Conformal Mappings in Complex Analysis, *International Journal of Innovative Research in Technology*, Volume 10 Issue 1 ISSN: 2349-6002 IJIRT 181705
- Yum-Tong Siu Math 115 (2006-2007). Conformal Mappings and Application to Electrostatics, Lecture notes Harvard University. <https://legacy-www.math.harvard.edu>

