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Efficiency of a Generalized Diversity Index: *a New Way to Look at Bio-diversity*

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ABSTRACT

In the present article, we have explored the concept of bio-diversity, which is a crucial aspect of environmental services. Bio-diversity encompasses all living things, including microbes, plants, animals, and humans. Bio-diversities are difficult to define and measure. Central to the issues is the concept of species, which has proved most elusive to generations of biologists. Most diversity indices do not consider differences between species when summarizing data on the relative abundance of the community. Therefore, it is important to have realistic diversity measures that factor in differences between species and their contributions to the ecosystem. For this, we introduce here a new generalized index of diversity to consider any difference between the abundances of species based on the relative abundances of the different species in the community.

1. Introduction

Bio-diversity is a concept that refers to the variety of life. It can be studied on many levels. At the highest level, one can look at all the different species in the community by transferring relative abundance or introducing a new species. One can study bio-diversity within a pond ecosystem or a neighbourhood park on a much smaller scale. The ecosystem is a biological community plus all abiotic factors influencing the community Bio-diversity, short for biological diversity, is now a common word and refers to the variety of life on the planet. For ecologists, bio-diversity is also the diversity of durable interactions among species. Of course, there are several different definitions of bio-diversity. It has been defined in the Convention on Biological Diversity, United Nations, adopted in 1992 at the Rio de Janeiro Earth Summit as "the variability among living organisms from all sources including inter alias, terrestrial, marine and other aquatic ecosystems and the ecological complexes of which they are part; this includes diversity within species, between species and of ecosystems". According to the definition above, biodiversity can be divided into species diversity, genetic diversity, and ecosystem diversity. Bio-diversity is the number of species in a given ecosystem (community) at species levels. A diversity index is a quantitative measure that reflects how many different types (such as species) there are in a dataset (a community). These indices are statistical representations of bio-diversity in various aspects. Diversity usually implies a measure of species number and the equitability of the abundance present. By measuring diversity, organizations, and communities can better understand the makeup of their ecosystems and work towards creating more inclusive and equitable environments.

[12] suggested that to understand the discussions of the subject, it is first necessary to understand the terminology and to grasp the distinctions between the principle concepts involved. Again, he suggested that the oldest and most fundamental component of diversity is species number (species richness). It is a measure of the number of different species present in an ecosystem, while the second component termed equitability (species evenness) measures the "relative abundance" of the different populations present in an ecosystem. Also, he refers to three implicit assumptions correlated with all species diversity

indices:

- All species are equally important.
- All individuals assigned to a species are equal.
- Species diversity has been measured in appropriate units.

Again, Peet suggested an index of heterogeneity based on the sum of squares of deviation from an even distribution. Other heterogeneity indices are those based on information theory given by the Shannon formulation. For more details see [9].

Traditional ecological diversity indices are based on an abundance of species present in the community. [21] Shannon's index is the most popular heterogeneity index "Heterogeneity is the term given to the use of richness and evenness to describe diversity" be most sensitive to changes in the importance of rare species in a sample. [22] Simpson's index measures the probability that two individuals selected randomly from a sample will belong to the same species and be most sensitive to changes in the importance of common or dominant species in a sample.
[14] gave clear derivations of the Shannon formula. Again [13], optimal these formulas equitability indices (J). Later [15] proved that Hill's index, N_α, [5] tends to the Shannon formula as $"\alpha"$ tends to one. In [16], she proposed two useful characteristics which an index of diversity should possess, namely,

P1: for fixed s, the index increases as the π_i become more equal,
P2: for equal π_i, the index is an increasing function of "s".

[24] proposed a measure that takes account of the species set and distances between species but ignores species abundances.
[23] defines functional diversity (*FD*) as "those components of biodiversity that influence how an ecosystem operates or functions". It has been suggested that ecosystems with a greater *FD* will operate more efficiently in terms of productivity.
[17,18] shows it is possible to connect traditional ecological diversity and biodiversity indices based on Shannon's entropy.

[1,2] proposed further generalized $H(\alpha,\beta)$ index due to Good by allowing (α,β) to take values in the real plane. For more details see [20].

Section 2: Shows theoretical considerations of applying properties due to [10,11] to improve the $H(\alpha,\beta)$ index due to [4] on the change of diversity, either by a shift in the species abundance or by introducing a new species. It provides a helpful methodology for statistical analysis of molecular data in bio-diversity studies. In section 3: We consider two different species models and examine the behaviour of the optimal index, $H(\alpha,\beta,\gamma)$, for each model. By comparing both models we can deduce an acceptable for optimal choices of (α,β,γ). Present results extending the parameters (α,β,γ) of the index $H(\alpha,\beta,\gamma)$ to include any real values under two desirable properties (**P1, P2)** due to [16]. These two main properties give a combined acceptable region of (α,β,γ). In section 4: We deal with obtaining a combined acceptable region of the newly optimal generalized index, $H(\alpha,\beta,\gamma)$. Section 5: Discusses how the index $H(\alpha,\beta,\gamma)$ behaves as a diversity measure concerning species evenness and richness components.

2. Theoretical Considerations

Understanding the mechanisms that shape biodiversity is one of the main goals of ecology. Diversity in ecology as a concept was first introduced by [3]. In the ecological field, the most useful way to compare heterogeneity between communities is by the concept of intrinsic diversity ordering [11].

A short review of recent developments in diversity indices attempts to summarize both the total number of species in a community and the degree of evenness of the relative abundances. For a given community which includes s-species with the relative abundance data $\boldsymbol{\pi}=(\pi_1,\pi_2,\ldots,\pi_s)$, where the total number of individuals within a community is N, and N_i is the number of individuals belonging to a species-i, such that $0\le\pi_i\le1$, and $\Sigma\pi_i=1$. The value of a traditional heterogeneity index with equal probabilities can be considered a richness measure. This is because these indices are monotone-increasing functions of the number of species (s).
[20] suggested a further generalization of Good's index that included as special cases both Shannon's and Simpson's indices for integer values of (α,β), Generalized Diversity Index (GDI); $H(\alpha,\beta)$, by discussing its optimal bounds of $(\alpha,\beta)\epsilon\Re$ within a closed region in the quadrant $0<\alpha\le2$ and $-2.5\le\beta\le2.5$. For ecological applications, it is given by

$$H(\alpha,\beta)=\sum_{i=1}^{s}\pi_i{}^{\alpha}\,(-\ln(\pi_i))^{\beta};\ (\alpha,\beta)\in\Re. \tag{1}$$

Where π_i is the relative abundance of taxon-i (species-i) and (α,β) define structures of the algorithm for real values $\Re$, for more details see [1,2,20].

In this work, we briefly review diversity as a concept introduced by [10,11]. Suppose a certain quantity is distributed among a countable set of categories, labelled by (i=1,2,...,s) with π_i as the proportionate share received by category i and $\sum\pi_i$=1. For concreteness of further discussion on the concept and measurement of diversity, they consider a community of biological organisms grouped into species and call $\boldsymbol{\pi}$=($\pi_1,\pi_2,\ldots$) *the species* (*relative*) *abundance vector*. Assume that s is the finite number of species that are physically present in the community, where $\pi_1\geq\pi_2\geq\ldots\geq\pi_s$ are obtained by arranging the components $\boldsymbol{\pi}$ in a decreasing order. Given a community $C(s,\boldsymbol{\pi})$, let $R(i;\boldsymbol{\pi})$ simply as $R(\boldsymbol{\pi})$ denote a numerical measure of rarity to be associated with species *i.* Then a diversity measure of the community is an average rarity of its species, and *the diversity index* associated with the measure of rarity is defined by $\Delta(\boldsymbol{\pi})=\sum\pi_i R(i;\pi_i)$ as an average rarity index of diversity, where $R(i;\pi_i)$ is decreasing in $\boldsymbol{\pi}$=($\pi_1,\pi_2,\ldots$) so that $R(i;\pi_i)\leq R(j,\pi_j)$; $\pi_i\geq\pi_j$. Shannon index is an example of measuring of rarity as average community rarity and the Simpson index may also be interpreted as well. In the form of $Eq(1)$ the function of rarity $R(i;\pi_i)=(\pi_i)^{(\alpha-1)}\{-ln(\pi_i)\}^{\beta}$ is a decreasing function of π_i for all values $\alpha>0$ and $\beta\geq 0$ within the region; say $\boldsymbol{R}_1$. Then introducing a new species increases the value of the diversity in this region. For two simple properties which a good diversity index should possess, we determined the range of valid α and β values for this generalized index, see [2].

Since rarer species correspond to smaller values of $\boldsymbol{\pi}$, $R(\boldsymbol{\pi})$ should be a decreasing function of $\boldsymbol{\pi}$. [11] suggested two criteria for optimal diversity measures. The first refers to the increase in diversity if the rarity of the ith species in the community, $R(i;\pi_i)$, is a decreasing function in π_i. The rarity of the ith species depends only on its relative abundance π_i. Suppose the rarity of a species depends only on its (decreasing) rank and not explicitly on the numerical values of the components of $\boldsymbol{\pi}$; as the second criterion. This second criterion refers to increasing the diversity index of a community either by transferring relative abundance or by introducing a new species, which is precisely the probability that a given encounter is interspecific. This concept of interspecific versus intraspecific encounters is explored further in what follows. Note that the function $R(i;\pi_i)$ measuring rarity is defined on the interval (0,1].

In this manner, let (Y+1) be equal to the number of differences in species up to and including the first interspecific between species as a geometric experiment. Clearly,

$$E[1/Y+1\backslash\pi_i]=\pi_i\left\{\frac{\{-log(\pi_i)\}}{(1-\pi_i)}\right\}. \tag{2}$$

Since large values of Y are associated with the rarity of the species-i. This implies that, for

$$R(i;\pi_i)=\left\{\frac{\{-log(\pi_i)\}}{(1-\pi_i)}\right\}. \tag{3}$$

Here, let Y be the number of interspecific. This average ratio rises to a different index. So for our work, it can be written that,

$$R(i;\pi_i)=\pi_i{}^{(\alpha-1)}\left\{\frac{\{-ln(\pi_i)\}^{\beta}}{(1-\pi_i)^{\gamma}}\right\}. \tag{4}$$

For more details, see [10,11].

Traditional ecological diversity measures can be improved to take into account *any difference between the abundances of species*. Therefore, we can introduce a further generalization to the index, $H(\alpha,\beta)$, that includes more parameters, to more general values of (α,β,γ). Thus, following the idea of Good [4], a new generalized form for such expression is presented in such equation:

$$H(\alpha,\beta,\gamma)=\sum_{i}^{s}{\pi_i}^{\alpha}\left\{\frac{\{-ln(\pi_i)\}^{\beta}}{(1-\pi_i)^{\gamma}}\right\};\ \ (\alpha,\beta,\gamma)=0,1,2,\ldots\ . \tag{5}$$

Where π_i refers to the relative abundance of species in the community.

The objective of this study is to present a further generalization of the ecological diversity index already generalized as the average rarity within a community "through possible interrelationships between individuals", for more details see [11]. Such an index used frequently in the ecological literature thus has a meaningful interpretation within the average community rarity formulation, in which we introduce the additional parameter *"γ"*. [1,2] and [20] have suggested that choosing different values for *"α"* and "β" entails particularly noticeable variations in diversity. Such variation can occur in the richness or the relative abundance of species. In this work, we will investigate structures of the algorithm for *"γ"* in real values.

Diversity indices play a central role in ecological theory. This work aims to provide a reliable theoretical and applicable framework for performing statistical analysis on ecological diversity. Patil and Taillie introduced the concept of intrinsic diversity ordering which can be determined by using the diversity profile. It may be noted that the diversity profile is a non-negative and convex curve that consists of a sequence of measurements as a function of a given *"γ"* parameter. To overcome this problem, a transformation is proposed which can be constrained to be non-negative real values $\Re$ and convex.

The optimal generalized diversity index, $H(\alpha,\beta,\gamma)$, is a Schur-concave function. If

$$h(\pi_i)={\pi_i}^{\alpha}\left\{\frac{\{-ln(\pi_i)\}^{\beta}}{(1-\pi_i)^{\gamma}}\right\}. \tag{6}$$

Is a Schur-concave function too.

Notice that $h(\pi_i)$ is required to be bounded as π_i tends to zero. As π_i tends to one $h(\pi_i)$ does not be finite as this corresponds to the case s=1 and π_i=1 for some "i".

To check whether $H(\alpha,\beta,\gamma)$ of its "inverse" satisfies the two key properties (**P1**, **P2**) it is only necessary to determine whether

$$f(x)=\frac{x^{\alpha}.[-ln(x)]^{\beta}}{(1-x)^{\gamma}};\quad x\epsilon[0,1], \tag{7}$$

is a Schur-concave function of x in the interval [0,1] so that $f(x)'<0$.

Now

$$f'(x)=\frac{x^{(\alpha-1)}.[-ln(x)]^{(\beta-1)}}{(1-x)^{\gamma}}\left\{\frac{1}{(x-1)}\left[[a-(\alpha-\gamma)x][-ln(x)]+\beta(1-x)\right]\right\}$$

and

$$f''(x)=\frac{x^{(\propto-2)}.[-ln(x)]^{(\beta-2)}}{(1-x)^{\gamma}}\times\left\{\frac{1}{(x-1)^2}\left[\begin{matrix}[[(\gamma-\alpha)(\gamma-\alpha+1)]x^2+[2\alpha(\gamma-\alpha+1)]x+\alpha(\alpha-1)]\\.[-ln(x)]^2-\left[\begin{matrix}[(2\alpha-1)\beta-2\beta\gamma]x^2+\\[2\beta\gamma-2(2\alpha-1)\beta]x+(2\alpha-1)\beta\end{matrix}\right].[-ln(x)]\\+\beta(\beta-1)[x^2-2x+1]\end{matrix}\right]\right\} \tag{8}$$

Since $\left\{x^{(\propto-2)}.[-ln(x)]^{(\beta-2)}/(1-x)^{\gamma}\right\}\geq 0$ for all $x\epsilon[0,1]$. Making the substitution z = [-ln(x)] to have g(z), it follows that the sign of $f(x)''$ is determined by that of the function:

$$\mathrm{g}(z)=\left\{\frac{1}{(x-1)^2}\left[\begin{matrix}[[\gamma-\alpha)(\gamma-\alpha+1)]x^2+[2\alpha(\gamma-\alpha+1)]x+\alpha(\alpha-1)].z^2\\-[[(2\alpha-1)\beta-2\beta\gamma]x^2+[2\beta\gamma-2(2\alpha-1)\beta]x+(2\alpha-1)\beta\ ].z\\+\beta(\beta-1)[x^2-2x+1]\end{matrix}\right]\right\} \tag{9}$$

For γ=0, it can be written that:

$$g(z) = \left\{ \frac{1}{(x-1)^2} \begin{bmatrix} [\alpha(\alpha-1)]x^2 - [2\alpha(\alpha-1)]x + \alpha(\alpha-1)].z^2 \\ -[[(2\alpha-1)\beta]x^2 - [2(2\alpha-1)\beta]x + (2\alpha-1)\beta\ ].z \\ +\beta(\beta-1)[x^2-2x+1] \end{bmatrix} \right\}.$$

Or

$$g(z) = \alpha(\alpha-1)z^2 - (2\alpha-1)\beta z + \beta(\beta-1). \quad (10)$$

Hence $f(x)$ is a concave function of x in the interval [0,1] if $f(x)''< 0$ which is true when $g(z)<0$, $\forall z\epsilon[0,\infty)$. The sign of $g(z)$ for $z \geq 0$ depends upon the location of any real roots of the $g(z)= 0$.

If β $[4\alpha(1-\alpha)+\beta]<0$ then both roots of $Eq(10)$ are complex. So if $\alpha(\alpha-1)<0$, then $g(z)$ will be wholly negative $\forall z\epsilon[0,\infty)$ either if the roots of $g(z)=0$ are both complex or if they are both negative.

The former case corresponds with the region $0<\beta<4\alpha(1-\alpha)$ for $0<\alpha<1$. The latter case corresponds with the region $4\alpha(1-\alpha)<\beta<1$ for $\alpha\geq½$. These give the region within which $g(z)$ is concave as $0<\beta<4\alpha(1-\alpha)$ for $0<\alpha<½$, and $0<\beta<1$ for $½\leq\alpha<1$.

If $\alpha=1$, so there is one value of $z=(\beta-1)$, then $g(z)=\{-\beta z+\beta(\beta-1)\}$ and clearly $g(z)<0$ $\forall z\geq 0$ if $0<\beta<1$. Similarly, if $\beta=1$ in $g(z)$, so $z=(2\alpha-1)/\alpha(\alpha-1)$, then $g(z)<0$ $\forall z\geq 0$ if $0<\alpha<1$. This gives the region within which $g(z)$ is concave if $½\leq\alpha<1$. For $0<\alpha<1$, $f(x)=0$ is a quadratic equation with a maximum at $z=[(2\alpha-1)\beta]/[2\alpha(\alpha-1)]$.

Thus $f(x)$ is a concave function and bounded $\forall x\epsilon[0,1]$ if,

i. $0<\alpha<½$ and $0\leq\beta\leq4\alpha(1-\alpha)$, or
ii. $½\leq\alpha<1$ and $0\leq\beta\leq1$, or
iii. $\alpha=1$ and $0<\beta\leq1$.

For giving (α,γ) values, it is of interest to know for which values β of does $g(z)=0$ have a root $z=ln(2)$, this case corresponding to $f(x)$ having a point of inflection at $x=½$. The equation $Eq(9)$ can be re-written as

$$\beta^2 - ((((2\alpha-1)-2\gamma)+4\gamma)z+1)\beta + \Big((\gamma-\alpha)(\gamma-\alpha+1)+4\big(\alpha(\gamma-\alpha+1)+\alpha(\alpha-1)\big)\Big)z^2 = 0.$$

For $\alpha\ \epsilon(0,1]$ and $\gamma\epsilon[0,1]$ only the positive value β for which there is a root $z=ln(2)$ is required. As example:

1) α= 0.25 and γ=0.00; give a root if β =0.77,
2) α= 0.25 and γ=0.50; give a root if β =0.92,
3) α= 0.50 and γ=0.50; give a root if β =1.33,
4) α= 1.25 and γ=0.00; give a root if β =1.69,
5) α= 0.75 and γ=1.75; give a root if β =2.15,

Consequently, $H(\alpha,\beta,\gamma)$ is Schur-concave function for $(\alpha,\beta,\gamma=0)$ in the region $\boldsymbol{R_1}$, $\alpha\epsilon(0,½)$, $\beta\epsilon[0,4\alpha(1-\alpha)]$; $\alpha\epsilon[½,1)$, $\beta\epsilon[0,1]$; $\alpha=1$, $\beta\epsilon(0,1]$. This result is comforted with [1,2] and [20].

For positive values of "γ" within the same regions $(\alpha,\beta,\gamma>0)$, the optimal generalized diversity index is a Schur-concave function too. To do so numerically for each value of (α,γ), the range of values of β that yield negative values of $f''(x)$ is given in Table 1.

Table 1: The values of β yielding negative values of $f''(x)$ for $x\epsilon[0,½]$.

α	β	γ	α	β	γ	α	β	γ
0.05	≤ 0.395	0.20	0.45	≤ 1.42	1.08	0.75	≤ 2.14	2.00
0.10	≤ 0.551	0.28	0.45	≤ 1.43	1.11	0.80	≤ 2.30	2.00
0.15	≤ 0.683	0.37	0.50	≤ 1.49	0.95	0.85	≤ 2.40	2.00
0.20	≤ 0.808	0.47	0.50	≤ 1.51	0.99	0.90	≤ 2.49	1.99
0.25	≤ 0.933	0.59	0.55	≤ 1.67	1.35	0.95	≤ 2.57	1.99
0.30	≤ 1.057	0.73	0.55	≤ 1.67	1.36	1.00	≤ 2.26	1.99
0.30	≤ 1.013	0.85	0.60	≤ 1.65	0.93	1.05	≤ 2.73	2.00

0.35	≤ 1.182	0.86	0.60	≤ 1.66	0.97	1.10	≤ 2.80	2.00
0.35	≤ 1.163	0.95	0.65	≤ 1.92	1.61	1.15	≤ 2.87	2.00
0.40	≤ 1.305	0.96	0.65	≤ 1.92	1.63	1.20	≤ 2.92	1.99
0.40	≤ 1.306	1.00	0.70	≤ 2.05	1.79	1.25	≤ 3.00	2.00

From the information in the above table, it can be seen that β and γ are related to each other when the parameter α fixed, increasing the parameters β and γ within the combined acceptable region $\boldsymbol{R_1}$; if we relaxed the restriction that the inequality holds for all $s \geq 1$ where $(0< \alpha \leq 1.00, 0\leq \beta \leq 2.265, 0\leq \gamma < 2.00)$ and it could be extended, for example, to $s \leq 200$ if $(\alpha >1.00, \beta \leq 3.00, \gamma \leq 2.00)$. So the limitation of β yields the upper bound of the $\boldsymbol{R_1}$-region.

3. Species Abundance Models

The distribution of species abundance can be classified into statistical or ecological models. Statistical models describe observed patterns, while ecological models attempt to explain them. There are several statistical models widely applied such as log series and log-normal models, on the other hand, the random fraction and broken-stick models are two clear examples of ecological models.

Two species abundance models are used as examples in this paper; these are the equiprobable model and the broken-stick model due to [6] which is an example of a typical uneven abundance distribution ecological data.

a. The Equiprobable Model

The generalized diversity index, $H_{Eq}(\alpha,\beta,\gamma)$, for the equiprobable case, is defined by

$$H_{Eq}(\alpha,\beta,\gamma) = \frac{S^{1-\alpha+\gamma}.[ln(S)]^{\beta}}{(1-S)^{\gamma}}, \tag{11}$$

To examine the behaviour of $H(\alpha,\beta,\gamma)$ with s species, we suppose it to be a continuous function of s for $s \geq 1$. Then we have

$$\frac{H_{Eq}(\alpha,\beta,\gamma)}{ds} = (S-1)^{1-\gamma}S^{-\alpha+\gamma}.[ln(S)]^{\beta-1}.\{[(\alpha-\gamma)-1-(\alpha-1)S].[ln(S)]+(S-1)\beta\}.$$

For $\beta \neq 0$ and for $\beta=0$ we have

$$\frac{H_{Eq}(\alpha,0,\gamma)}{ds} = (S-1)^{1-\gamma}S^{-\alpha+\gamma}.\{(\alpha-\gamma)-1-(\alpha-1)S\}. \tag{12}$$

The sign of $H_{Eq}(\alpha,\beta,\gamma)$ depends upon the sign of $[((\alpha-\gamma)-1-(\alpha-1)s)(\ln(s))+(s-1)\beta]$ for real positive of β. We now consider the behaviour of GDI using real values of (α,β,γ). It means that $H_{Eq}(\alpha,\beta,\gamma)$ satisfies property (**P2**), being a monotonic increasing function of s for all $s \geq 1$, provided $0<\alpha \leq 1$, $\beta>0$, and $0\leq\gamma\leq 0.5$, but excluding the cases where (α,β,γ) equals (0,0,0) and (1,0,0).

For the case $\alpha \leq 1$, $\beta=0$, and $\gamma \leq 1$, $H_{Eq}(\alpha,\beta,\gamma)$ is an increasing function of s for all $s \geq 1$, excluding the case $(\alpha,\beta,\gamma)=(1,1,0)$. In addition, $H_{Eq}(\alpha,0,0)$ is a strictly monotonic decreasing function of s.

For non-negative values of $(\alpha-\gamma) =1$ and $\beta<0$, the differentiation of $H_{Eq}(\alpha,\beta,\gamma)$ concerning s can be written that

$$\frac{H_{Eq}(\alpha,\beta,\gamma)}{ds} = \frac{(S-1)^{1-\gamma}}{S}.[ln(S)]^{\beta-1}.\{(S-1)\beta-[(\alpha-1)S].[ln(S)]\}. \tag{13}$$

If $\alpha \geq 1$, $\beta \leq 0$, and $\gamma \geq 0$, but excluding the case $(\alpha,\beta,\gamma)=(1,0,1)$, then $H_{Eq}(\alpha,\beta,\gamma)$ is a monotonic decreasing function of s for all $s \geq 1$, so that a suitable "inverse" of $H_{Eq}(\alpha,\beta,\gamma)$ would satisfy property (**P2**). For small values of (α,γ) for less than or equal to one and small negative values β (close to zeros), $H_{Eq}(\alpha,\beta,\gamma)$ has a turning point for some s at $s \approx e^{\beta/(\alpha-\gamma-1)}$.

When $\alpha > 1$, it can be seen that the differentiation of $H_{Eq}(\alpha,\beta,\gamma)$ concerning s for all $\beta< 0$. Thus $H_{Eq}(\alpha,\beta,\gamma)$ decreases as s increases for all $s \geq 2$. Similarly, with $\beta<0$, $H_{Eq}(\alpha,\beta,\gamma)$ is a strictly monotonic decreasing function of s in the case $\alpha=1$. In addition, $H_{Eq}(\alpha,0,0)$

is a strictly monotonic decreasing function of s for $\alpha \geq 1$ and $\beta=0$, but excluding the cases where (α,β,γ) equals (1,0,0).

In practice, most values of "s " encountered would lie within two and, typically, two-handed. A behaviour holding for $s \geq 2$ would hold for most s cases encountered in practice.

In summary, $H_{Eq}(\alpha,\beta,\gamma)$ satisfies the main property (**P2**), being a monotonic increasing function of s for all $s \geq 1$, provided $0<\alpha \leq 1$, $\beta \geq 0$, and $\gamma \geq 0$. If $\alpha \geq 1$, $\beta \leq 0$, and $\gamma \geq 1$, then $H_{Eq}(\alpha,\beta,\gamma)$ a monotonic decreasing function of s for all $s \geq 1$, so that a suitable inverting transformation of $H_{Eq}(\alpha,\beta,\gamma)$ would be required to satisfy **P2**.

b. MacArthur's broken-stick model

[6] proposed the broken-stick model and likened it to an assemblage of a stick randomly into s pieces. It is a very uniform distribution perhaps the most uniform found in natural communities [8]. Such a model is used to predict the average species abundance distribution.

In practice, when the abundance data are not available, we apply the broken-stick model to estimate the non-equally relative abundances (p_i), which is the probability of the ith rarest species, it is given by

$$p_i = \frac{1}{S}\sum_{r=1}^{i}\frac{1}{S-r+1} \; for \; i = 1,2,\dots,s. \tag{14}$$

To examine the behaviour of an improved diversity index $H(\alpha,\beta,\gamma)$ as a function of s, we numerically evaluate the index for different values of (α,β,γ). Denote this index by $H_{Bs}(\alpha,\beta,\gamma)$.

4. Combined Acceptable Region of $H(\alpha,\beta,\gamma)$:

Not all values of (α,β,γ) lead to suitable measures of diversity; for example, $H(1,0,0) \equiv 1$ and so is not a useful index; $H(0,0,0) \equiv s$ measures species richness but gives no information about the species abundances π_i's.

In this paper, we examine whether the general index $H(\alpha,\beta,\gamma)$ satisfies the main two key properties requirements for an optimal diversity index for arbitrary real α, β, and γ. We refer to the regions of (α,β,γ) which satisfy these two properties by $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ respectively, and thus for which $H(\alpha,\beta,\gamma)$ is a valid diversity index, as the "acceptable" regions.

To satisfy property (**P1**) we require that $H_{Eq}(\alpha,\beta,\gamma) \geq H_{Bs}(\alpha,\beta,\gamma)$ in regions for which $H(\alpha,\beta,\gamma)$ is an increasing function of s, or $H_{Eq}(\alpha,\beta,\gamma) \leq H_{Bs}(\alpha,\beta,\gamma)$ in regions for with $H(\alpha,\beta,\gamma)$ a decreasing function of s so that a suitable "inverse" of $H(\alpha,\beta,\gamma)$ would satisfy property (**P2**). We refer to these regions of (α,β,γ) as $\boldsymbol{R}_1$- region and $\boldsymbol{R}_2$-region respectively.

In practice, most values of s encountered would lie between two and typically, two hundred. Relaxing the monotonicity requirement for $s>1$ to this range of s values extends the two regions given above. Relaxing the monotonicity from holding for all $s \geq 1$ extends the region say $\boldsymbol{R_1}$ for $\alpha \leq 1$, $\beta>0$, and $\gamma \geq 0$. Again, relaxing the monotonicity of form holding for all $s \geq 1$ extends the region above say $\boldsymbol{R_2}$ for $\alpha \geq 1$, $\beta \leq 0$, and $\gamma \geq 0$.

The behaviour of $H(\alpha,\beta,\gamma)$ using the equiprobable and broken-stick models is shown in the following figures for different values of (α,β,γ) in the combined acceptable regions: $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ for s =50 species as an example.

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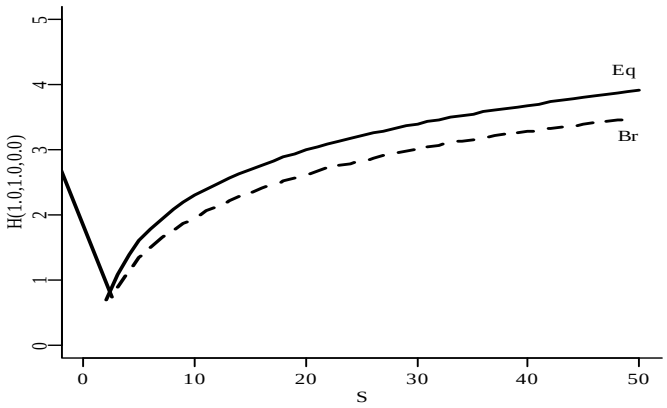

Figure 1: The behaviour of $H(1.0,1.0,0.0)$ for Equiprobable (*Eq*) and Broken-stick (*Bs*) models for 50 species.

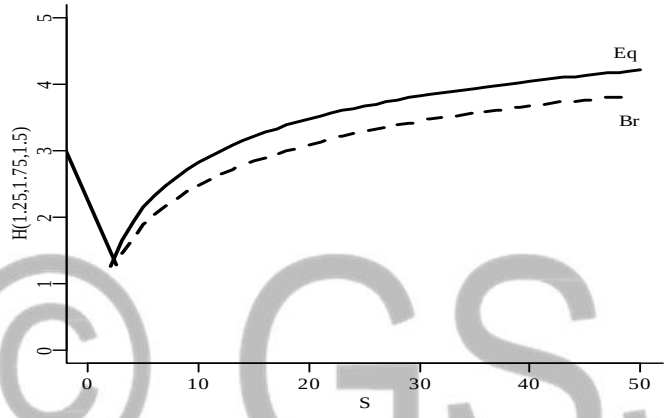

Figure 2: The behaviour of $H(1.25,1.75,1.5)$ for Equiprobable (*Eq*) and Broken-stick (*Bs*) models for 50 species.

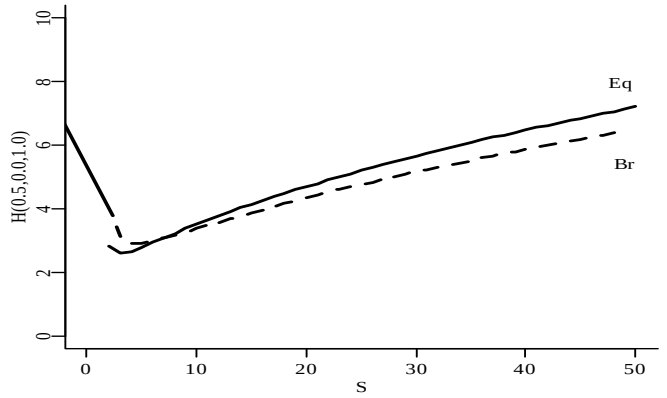

Figure 3: The behaviour of $H(0.5,0.0,1.0)$ for Equiprobable (*Eq*) and Broken-stick (*Bs*) models for 50 species.

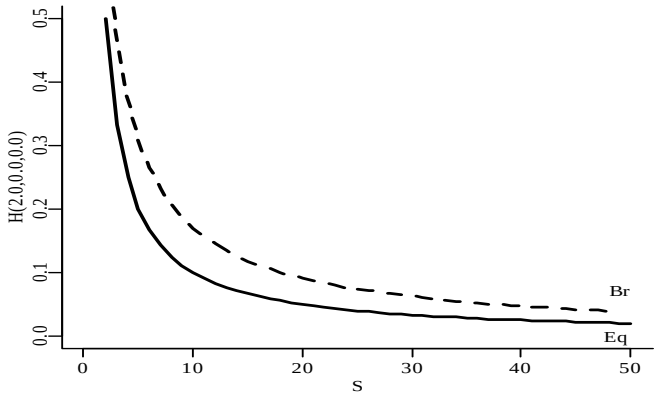

Figure 4: The behaviour of $H(2.0,0.0,0.0)$ for Equiprobable (*Eq*) and Broken-stick (*Bs*) models for 50 species.

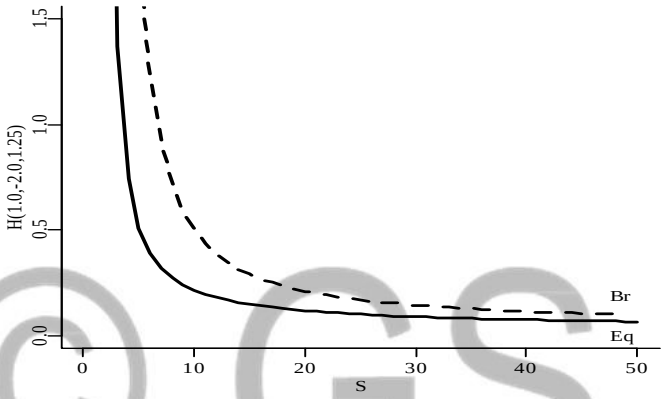

Figure 5: The behaviour of $H(1.0,-2.0,1.5)$ for Equiprobable (*Eq*) and Broken-stick (*Br*) models for 50 species.

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It is clear that $H(\alpha,\beta,\gamma)$ within $\boldsymbol{R_1}$-region is a monotonic increasing function of s as an example for the Shannon index $H(1,1,0)$ for $s\geq1$ in **Figure 1** and $H(1.25,1.75,1.5)$ for $s\geq2$ in **Figure 2**. When $\alpha\leq1$, $\beta\leq0$, and $\gamma\leq1$, $H(\alpha,\beta,\gamma)$ has a turning point for small values of s as an example for $H(0.5,0,1)$ has a turning point (say, $s\approx3$); see **Figure 3**. The behaviour of $H(\alpha,\beta,\gamma)$ within the $\boldsymbol{R_2}$-region is a monotonic decreasing function of s as example for the Simpson index H(2,0,0) for $s\geq1$ in **Figure 4** and in **Figure 5** with same pattern.

5. Interpretation of $H(\alpha,\beta,\gamma)$:

To examine the sensitivity of $H(\alpha,\beta,\gamma)$ to the changes in the abundance of species we compare the change in the index value. Such an index could be sensitive to evenness and richness components. Thus diversity indices may be categorised as those most sensitive to changes, in the rare species in $\boldsymbol{R_1}$, an example being Shannon's index, $H(1,1.0)$, and those indices which are most sensitive to changes in the common species in $\boldsymbol{R_2}$, an example being the inverse of Simpson's index, $H(2,0.0)$. To assess the extent to which $H(\alpha,\beta,\gamma)$ is sensitive to evenness define the ratio,

$$J(\alpha,\beta,\gamma)=\frac{H(\alpha,\beta,\gamma;\pi_i)}{H(\alpha,\beta,\gamma;1/S)}\,, \tag{15}$$

where $J(\alpha,\beta,\gamma)$ is an equitability measure. We observe that $J(\alpha,\beta,\gamma)\leq1$ for different values of (α,β,γ) in the acceptable regions. Sensitivity to evenness is indicated by the ratio $J(\alpha,\beta,\gamma)$. If $J(\alpha,\beta,\gamma)\approx1$ for some (α,β,γ) and given s, then $H(\alpha,\beta,\gamma)$ is not a good discriminator between the equiprobable model and the broken-stick model. Accordingly small values of $J(\alpha,\beta,\gamma)$ in acceptable regions indicate optimal choices of (α,β,γ). In practice, of course, the estimated value of $H(\alpha,\beta,\gamma)$ would be evaluated from the data which would introduce additional variability into the discrimination problem. Consequently to find the optimal choices of

(α,β,γ) we search numerically for small values of J-ratio in the acceptable region $\boldsymbol{R_1}$ or large values in the region $\boldsymbol{R_2}$.

Tables 1 and 2 show J-ratios for different choices of (α,β,γ) in the acceptable regions $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ for the cases s=10, s =50, and s =100 respectively.

Table 1: The values of J-ratios in the region R1 for different numbers of species.

α	β	γ	J-ratio; s=10	J-ratio; s=50	J-ratio; s=100
0.25	**0.00**	**1.00**	0.960	0.917	0.912
0.25	**0.50**	**0.75**	0.957	0.932	0.927
0.50	**0.25**	**1.50**	0.964	0.898	0.890
0.50	**0.75**	**1.25**	0.924	0.889	0.885
0.75	**1.00**	**0.50**	0.873	0.872	0.875
0.75	**1.50**	**1.75**	0.898	0.863	0.869
1.00	**1.00**	**0.00**	0.856	0.898	0.911
1.00	**1.75**	**0.50**	0.837	0.859	0.871
1.25	**1.50**	**0.75**	0.867	0.927	0.948

Table 2: The values of J-ratios in the region R_2 for different numbers of species.

α	β	γ	J-ratio; s=10	J-ratio; s=50	J-ratio; s=100
1.00	**-0.50**	**1.00**	1.264	1.098	1.073
1.25	**0.00**	**1.50**	1.311	1.167	1.172
1.50	**-0.25**	**0.50**	1.451	1.399	1.387
1.50	**0.00**	**0.25**	1.289	1.318	1.185
1.75	**-0.75**	**1.25**	2.375	1.965	1.894
1.75	**0.00**	**1.50**	1.795	1.639	1.050
2.00	**-0.50**	**0.25**	2.188	2.203	2.193
2.00	**0.00**	**0.00**	1.706	1.908	1.948

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It can be seen from **Table 1** that J-ratios for some ($\alpha,\beta\geq1,\gamma$) and given s, the optimal regions are found to be $\alpha\approx1$, $\beta\geq1$, and $\gamma\geq0$ in the region $\boldsymbol{R_1}$ as species number s decreases. For Shannon's index, the J-ratio increases as s increases, reflecting the known fact that it emphasizes species abundance more than species richness. In contrast, from **Table 2** the optimal region $\boldsymbol{R_2}$ for large positive $\alpha\geq1.75$, $\beta<0$, and $\gamma\geq0$; J-ratios increase as species number s increase. Unfortunately, this result is not true throughout the two regions for all cases.

6. Summary and Conclusion:

This paper has focused on numerical studies of the optimal generalized diversity index, $H(\alpha,\beta,\gamma)$, by comparing Equiprobable (Eq) and Broken-stick (Bs) models for 50 species for two acceptable regions $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ of (α,β,γ). The $\boldsymbol{R_1}$-region is a convex null in the upper right quadrant for which $H(\alpha\leq1,\beta>0,\gamma\geq0)$ is an increasing function of s, and the $\boldsymbol{R_2}$-region is the left quadrant $\alpha\geq1$, $\beta\leq0$, and $\gamma\geq0$, for which $H(\alpha,\beta,\gamma)$ is a decreasing function of s. Within these two regions, we have examined the optimal choice of (α,β,γ) by using the discriminatory ability of $H(\alpha,\beta,\gamma)$ in comparing the Br-model with the Eq-model. Generally, when the parameter α is fixed, increasing the parameters β and γ leads to an increase in the diversity index, $H(\alpha,\beta,\gamma)$. This index shows a directly proportional relationship with the broken-stick model when these parameters are applied. It can be demonstrated that two regions ($\boldsymbol{R_1}$, $\boldsymbol{R_2}$) are maintained when other models are used.

Other authors have proposed criteria additional to (**P1**, **P2**) properties desirable in a diversity index due to [16]. [19] has suggested that any index, $N(\alpha,\beta,\gamma)$, measuring the diversity of large communities should be a function of the relative abundances (p_i) satisfying the property (**P3**), that $N(\alpha,\beta,\gamma)\equiv s$ in the Equiprobable Eq-model and $N(\alpha,\beta,\gamma)<s$ otherwise. Suppose that the relative abundances of species in the community of the s species are $\pi_1, \pi_2,\ldots, \pi_s$. Then the $N(\alpha,\beta,\gamma)$, which gives more weight to

rare than to common for ($0<\alpha\leq1$, $\beta\geq0$) in $\boldsymbol{R_1}$-region otherwise it gives more weight to common than to the rare for ($\alpha\geq1$, $\beta\leq0$) in $\boldsymbol{R_2}$-region, is given by

$$N(\alpha,\beta,\gamma)=\left\{\frac{H(\alpha,\beta,\gamma)}{(ln(s))^{\beta}/(S-1)^{\gamma}}\right\}^{\frac{1}{1-a+\gamma}},(\alpha,\beta,\gamma)\in\Re;(\alpha-\gamma)\neq1.$$

and

$$N(\alpha,\beta,\gamma)=Exp\left\{\frac{H(\alpha,\beta,\gamma)}{(S-1)^{-\gamma}}\right\}^{1/\beta},(\alpha,\beta,\gamma)\in\Re;(\alpha-\gamma)=1\ and\ \beta\neq0. \quad (16)$$

The transformation provides the class of diversity index, which is $N_{\alpha}=\left[\sum\pi_i^{\alpha}\right]^{1/(1-\alpha)}$ for some $\alpha>0$ and $\beta=\gamma=0$ as [5] index. [7] denoted $N_1=e^{H'}$ for $\alpha=1=\beta$ and $\gamma=0$; where $H(1,1,0)$ is equivalent to the diversity index due to [21]. In this study, We used $H(\alpha,\beta,\gamma)$ base (e) instead of $log(10)$, expressed mathematically.

$$N(1,1,0)=Exp\{H(1,1,0)\}=Exp\left\{\sum_{i=1}^{S}\pi_i\ [-ln(\pi_i)]\right\},$$

and

$$N(2,0,0)=\{H(2,0,0)\}^{-1}=\left\{\sum_{i=1}^{S}\pi_i^2\right\}^{-1}=\frac{1}{\sum_{i=1}^{S}\pi_i^2}.$$

Which is equivalent to the Simpson diversity index satisfying (**P1**, **P2**).

From the study findings, one can conclude that the behaviour of a Generalized Diversity Index (GDI) using real values of ($\alpha\leq1,\beta>0,\gamma\geq0$) within $\boldsymbol{R_1}$-region is an increasing function of s, and for $\alpha\geq1$, $\beta\leq0$, and $\gamma\geq0$ within $\boldsymbol{R_2}$-region, $H(\alpha,\beta,\gamma)$ is a decreasing function of s. Within these two regions we have examined the optimal chose of (α,β,γ) by discriminatory ability of $H(\alpha,\beta,\gamma)$ in comparing the broken-stick (Bs) and Equibprobable (Eq) models. In case ($\alpha\leq1,\beta\leq0,\gamma\geq0$), we recommend the use of $\Lambda(\alpha,\beta,\gamma)$ as a inverse transformation of *GDI* than the reciprocal $N(\alpha,\beta,\gamma)$.

This eassy was devoted to studying the efficiency of *GDI* by introduceing additional parameter "γ" to consider any difference between the abundances of species.

In conclusion, we recommend using the $H(\alpha,\beta,\gamma)$ index for two or more communities, which performs the efficiency of a Generalized Diversity Index, a new way to look at the construction of biodiversity with useful statistical properties regardless of the level of heterogeneity in the underlying population. In other words, to study the properties of the optimal diversity index, which will behave more smoothly as one moves between different underlying levels of diversity. This index will provide a helpful methodology in biodiversity studies.

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