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Modeling the Optimal Control of the Transmission Dynamics of COVID – 19 Infection with Quarantine and Isolation

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Abstract

This present study discusses the spread of COVID-19 epidemic and its control. Regrettably,the virus is continuously spreading and its mortality rate is increasing daily. Here we used quarantine and treatment (isolation) strategies to control the spread of the disease. This work used mathematical modeling and optimal control approach to study the effect of the two control strategies as time-dependent intervention and also ascertain their contributions in the dynamic of the transmission of COVID-19. The model was well-posed as we proved that all its state variables are non negative for all time. The basic reproduction number was computed and was used it to carry out the sensitivity analysis that classify the serious parameter contributing to spread of COVID-19. The optimal control analysis was done using the Pontryagin's maximum principle to find out the optimal strategies needed to restrain the disease. The finding's of the optimal control analysis and numerical simulations showed that the combined implementation of the two interventions produced a fine result in reducing COVID-19 infection in the population. This implies that combined interventions need to be deployed early in order to reduce the virus infection to the barest minimum.

Keywords: COVID-19, optimal control, Treatment (Isolation), Quarantine, Basic Reproduction Number, Sensitivity analysis.

I. Introduction

COVID-19 is a rapidly spreading infectious disease with pandemic potential, caused by the novel virus SARS-COV-2 [1]. Coronaviruses are enveloped, single-stranded, positive-sense RNA viruses belonging to the family of coronaviridae [2]. The cause generally mild respiratory infections, eventhough they are occasionally latthal. Since their discovery and first characterization in1965 [3], three major, large-scale outbreaks have occurred, caused by emerging, highly pathogenic coronaviruses, namely, the "Severe Acute Respiratory Syndrome" (SARS) outbreak in 2003 in

mainland china [4], the “Middle East Respiratory Syndrome” (MERS) outbreak in 2012 in Saudi Arabia [5], [6] and the the MERS outbreak in South Korea [7], [8]. These outbreaks have resulted in more than 8000 and 2200 comfirmed SARS and MERS cases respectively [9]. Recently, a fourh coronavirus outbreak has occurred in wuhan, the capital of Hubei province and seventh largest city of peoples republic of China [10], [11], [12].

On 31 December 2019, the Chinese authority informed the World Health Organization (WHO) in China country of 27 cases of viral pneumoniain which a novel coronavirus was identified as the causative agent. WHO also released a wide range of interium guidance for all countries on how to get prepared to cope with the emergency control of further spread of the disease. The virus ius mainly spread from person to person, through respiratory droplets and the spread is more likely when people are close (within 6 feet) to each other (Center for Disease Control and Prevention (CDC)) [13]. This has prompted Healthauthoritiesto stress upon social distancing and wearing of face mask as a means of controlling the spread of the disease. Certainly, at this time in the absence of a vaccine and any confirmed anti-viral therapy, social distancing, quarantine and isolation with treatment are the only possible control strategies available [14]. Thevirus may also be transmitted by touching infected surfaces and then touching ones own mouth, nose or eyes hence disinfecting of surfaces and hands is also recommended as preventive measure. Symptoms of the disease which includes severe respiratory illness, fever, cough, chills, lost of taste or smell, and myalgia orfatique may appear 2 – 14 days after exposure [14], [15]. The elderlys who are 65 years and above and individuals with underlying medical conditions like chronic lung disease, serious heart condition, diabetes, chronic kidney disease and immunocompromised individuals are people with high risk of the infection [16].

II. Related Litrature

In the last decades, many mathematical models used to study the transmission dynamics and control of infectious disease have been proposed.these models are important such inpolicy making, emergency planning and risk assessment, definition of control programs, and improvement in various health -economic aspect [17].

Ali et al, [18] formulated a deterministic epidemic model for the spread of coronavirus disease (COVID-19)in which they include asymptomatic, quarantine and isolation compartments,since studies has stress on the importance of these population groups on the transmission of the disease. In their study, optimal quarantine and isolation strategies were deviced, noting that high levels need to be maintained during the early stages of the out break.

Tang et al, [19] deviced deterministic compartmental model based on the clinicalprogression of the disease, epidemiological status of the individuals, and intervention measures. Their sensitivity analysis show that intervensions such as intensive contact tracing followed by quarantine and isolation, can effectively reduce the control reproduction number and transmission risk, with the effect of travel restriction.

Libotte et al[20] stated in their work that the objective is to determine an optimal control strategy for vaccine administration in COVID-19 pandemic treatment considering real data from China. Two optimal control problem (mono and multi- objective) were proposed in which the first consists of minimizing the quantity of infected individuals during the treatment while the second considers minimizing together the quantity of infected individuals and the prescribed vaccine concentration during the treatment.

Yan and Zou [21] discussed the application of optimal and sub-optimal controls to SEQIJR SARS model via the Pontryagin's maximum principle. To this end, two control variables representing the quarantine and isolationstrategies are considered in the model. The simulation results demonstrated that the maximal applications of quarantine and isolation strategies in the early stage of the epidemic are of very critical impacts in both cases of optimal and sub-optimal control.

Ahmed et al [22] in their paper, review and introduce some models for the COVID-19 that can address important questions about the global health care and suggest important notes. The y suggested three well known numerical technique forsolving such equations, which are Euler's method, Runge-Kutta method of order two (RK2) and of order 4 (RK4). Results based on the suggested numerical techniques were provided, together with the approximate solutions which gives important key answers to the global issue.

Yadav, [23], discussed the spread of COVID-19 epidemic of India and its end by using SIR model and the discussion about the spread was greatly detailed using Euler's method. The result from the SIR model suggest that the Euler'smethod can be used to predict transmission and prevention of COVID-19 epidemic in India.

III. Model formulation

In this paper, we consider a deterministic compartmental model which divides the total human population size at time t denoted by $N(t)$, into susceptible individuals $S(t)$, exposed individuals $E(t)$, quarantined individuals $Q(t)$, infected individuals $I(t)$, treated (isolated) individuals $T(t)$ and recovered individuals $R(t)$. Based on the clinical progression of the individuals and intervention measures of the infection, epidemiological status of the individuals and intervention measures are presented in figure 1

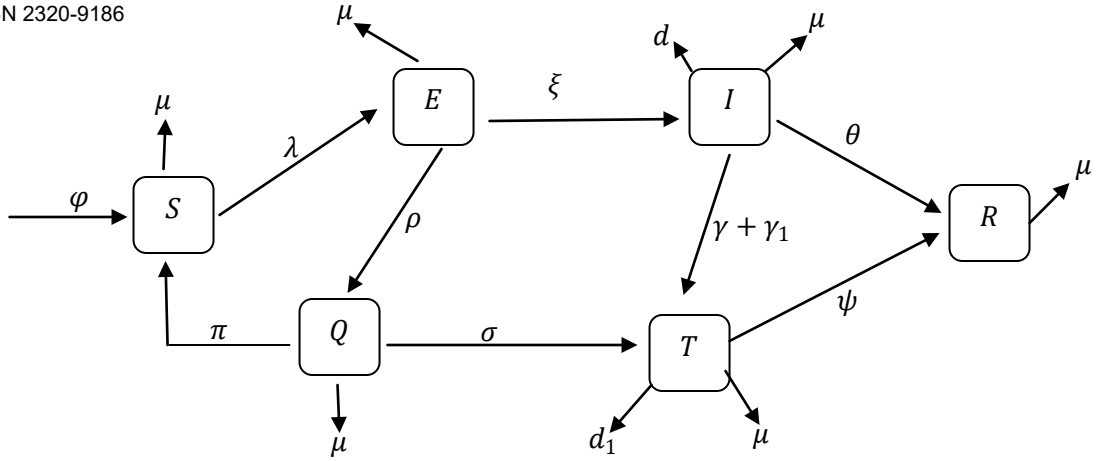

Figure 1: Schematic diagram of the model.

The model is based on the following assumptions. The mixing between individuals are homogeneous. The population infected with active COVID–19 is generated from exposed, infected, quarantined and treated individuals. The recovered individuals are assumed to develop immunity to COVID–19 and all the compartments exit through natural death at a rate μ. For an individual to become infectious, he/she must pass through the latent stage. The force of infection associated with COVID–19 infection denoted by λ is given by

$$\lambda = \frac{\beta(I + \eta E + \eta_1 Q + \eta_2 T)}{N} \quad (1)$$

Where β is the effective contact rate for COVID-19 infection and the parameters η, η_1, η_2 are the modification factors for the exposed, quarantined and treated (isolated) individuals. He parameters $\eta_1 \, and \, \eta_2$ are associated with hygiene consciousness of the quarantined and treated individuals. Susceptible individuals are recruited at the rate φ. The acquire COVID-19 infection through active contact at a rate λ. Exposed individuals are those that may have had contact with an infected person. Through contact tracing, the exposed individuals to the virus progresses to quarantine compartment for a period of fourteen (14) days (incubation period) at a rate ρ, and either move to treated (isolated) compartment at a rate σ, for treatment or back to susceptible compartment at a rate π, depending on whether they are effectively infected (i.e., if they develop the symptoms) or not. The other proportion of the exposed individuals that are not traced (missed quarantine) move to infected compartment at a rate ξ. Interpersonal contact tracing is carried out on all persons undergoing treatment in order to pull out more infected individuals in the population. The infected individuals can be traced at a rate γ and isolated for treatment, while others present themselves for treatment at a rate γ_1. Also the treated individuals may recover at a rate ψ, while individuals in the infected compartment and those isolated for treatment may die due to COVID-19 infection at a rate $d \, and \, d_1$ respectively. Proportion of the infected compartment who missed treatment may recover from the infection due to boost in immune system at a rate θ. It is assumed that individuals that died because of COVID-19 are buried immediately to prevent further transmission.

Table 1: Variable Description

Variables	Description
S	Susceptible Compartment
E	Exposed Compartment
Q	Quarantined Compartment
I	Infected Compartment
T	Treated (Isolated) Compartment
R	Recovered Compartment

Table 2: Parameter Description

Parameters	Description
φ	Recruitment rate into susceptible compartment.
μ	Natural death rate.
β	Transmission rate
θ	Recovery rate of infected individuals due to immune response.
ψ	Recovery rate of the treated individuals
d	Disease induced death rate of the infected individuals.
d_1	Disease induced death rate of the treated individuals.
π	Progression rate from quarantined to susceptible compartment.
ξ	Progression rate of exposed individuals to infection compartment
ρ	Progression rate of the exposed individuals to the quarantined compartment through contact tracing.
σ	The isolation rate of those that develop symptoms during quarantine period to treatment compartment.
λ	Force of infection.
γ	Progression rate of the infected individuals to the treatment compartment

	through contact tracing.
γ_1	Progression rate of the infected individuals to treatment compartment through self submission.

Putting the above formulations and assumptions together gives the following system of differential equations.

$$\frac{dS(t)}{dt} = \varphi + \pi Q(t) - \mu S(t) - \lambda S(t)$$

$$\frac{dE(t)}{dt} = \lambda S(t) - (\mu + \xi + \rho)E(t)$$

$$\frac{dQ(t)}{dt} = \rho E(t) - (\mu + \sigma + \pi)Q(t) \qquad (2)$$

$$\frac{dI(t)}{dt} = \xi E(t) - (d + \mu + \gamma + \gamma_1 + \theta)I(t)$$

$$\frac{dT(t)}{dt} = \sigma Q(t) + (\gamma + \gamma_1)I(t) - (\psi + \mu + d_1)T(t)$$

$$\frac{dR(t)}{dt} = \psi T(t) + \theta I(t) - \mu R(t)$$

where

$$S(0) = S_0, E(0) = E_0, Q(0) = Q_0, I(0) = I_0, T(0) = T_0, R(0) = R_0 \qquad (3)$$

are the initial conditions, assumed to be positive.

3.1 Positivity and Boundedness of Solutions

For the COVID-19 transmission model system (2) to be epidemiologically meaningful, it is important to prove that all its state variables are non negative for all time. In other words, solutions of the model system (2) with non negative initial data remain positive for all time $t > 0$.

The total human population can be determined by

$$N(t) = S(t) + E(t) + Q(t) + I(t) + T(t) + R(t)$$

By adding all the equations in system (2) gives the equation for $N(t)$, the total population.

$$N(t) \leq \varphi - \mu N - dI - d_1 Q \qquad (4)$$

Applying Barkhoff and Rota's [24] theorem having in mind that in the absence of COVID-19, $d = d_1 = 0$, then we obtain

$$0 \leq N \leq \frac{\varphi}{\mu} \ as \ t \longrightarrow \infty$$

Theorem 1: The solution set $\Gamma = \left\{(S, E, Q, I, T, R) \in \mathbb{R}_+^6 : N \leq \frac{\varphi}{\mu}\right\}$ of the epidemiological model system (2) with non negative initial data (3) remain non negative for all time $t > 0$.

Proof: Given that the initial data $S(0), E(0), Q(0), I(0), T(0), R(0)$ are non negative, it is clear from the first subsystem of the model system (2) that is

$$\frac{dS(t)}{dt} = \varphi + \pi Q(t) - \mu S(t) - \lambda S(t)$$

$$\geq -(\lambda + \mu)S(t) \qquad (5)$$

Solving (4) gives

$$S(t) \geq S(0)\exp\big(-(\lambda + \mu)\big)\, t \geq 0$$

This implies that $S(t) > 0 \ \forall \ t > 0$. Similarly, it can be proved that $E, Q, I, T \ and \ R$ are all non - negative, and this is done using the remaining sub-equations of system (2) and we have

$$E(t) \geq E(0)\exp\big(-(\mu + \xi + \rho)\big)\, t \geq 0$$

$$Q(t) \geq Q(0)\exp\big(-(\mu + \sigma + \pi)\big)\, t \geq 0$$

$$I(t) \geq I(0)\exp\big(-(d + \mu + \gamma + \gamma_1 + \theta)\big)\, t \geq 0$$

$$T(t) \geq T(0)\exp\big(-(\mu + \psi + d_1)\big)\, t \geq 0$$

$$R(t) \geq R(0)\exp(-\mu)\, t \geq 0$$

This completes the proof. It is crucial to note that model system (2) will be analyzed in a feasible region Γ given by

$$\Gamma = \left\{(S, E, Q, I, T, R) \in \mathbb{R}_+^6 : S + E + Q + I + T + R \leq \frac{\varphi}{\mu}\right\} \qquad (6)$$

which can easily be verified to be positively invariant with respect to model system (2). In what follows, model system (2) is epidemiologically and mathematically well posed in Γ [25].

Theorem 2: Assume that the initial condition of COVID-19 in model system (2) satisfies $N(0) \leq \frac{\varphi}{\mu}$. Then, whenever the solution exists on an interval P, it satisfies the following boundedness

$$N(t) \leq \frac{\varphi}{\mu}$$

Proof: Since $I(t) \geq 0$, we have from (4) that

$$\frac{dN(t)}{dt} < \varphi - \mu N$$

Using comparison theorem by Lakshmikantham et al, [26], Zhang [27], it can be shown that

$$N(t) \leq N(0)e^{-\mu t} + \frac{\varphi}{\mu}(1 - e^{-\mu t}) \quad (7)$$

Whenever$(0) \leq \frac{\varphi}{\mu}$, we have $N(t) \leq \frac{\varphi}{\mu}$. Consequently, $I(t) \leq \frac{\varphi}{\mu}$.

3.2 Asymptotic Stability of Disease Free Equilibrium (DFE)

At steady state, each of the equations in system (2) are equal to zero. This implies that the disease free equilibrium of the COVID-19 model is given by

$$H_o = [S, 0, 0, 0, 0, 0] = \left[\frac{\varphi}{\mu}, 0, 0, 0, 0, 0\right] \quad (8)$$

3.3 Basic Reproduction Number

The objective of a disease elimination programme is to reduce the basic reproduction ratio below one. Here the effective reproduction number $\mathcal{R}_e$ is calculated using the next generation approach [28], [29]. It follows that $F\ and\ V$ which stands for new infection and remaining transmission terms respectively are obtained. Then for F we have

$$E = \lambda S, Q = 0, I = 0, T = 0 \quad (9)$$

Then

$$\mathrm{F} := \frac{\mathrm{S}}{\mathrm{N}}\begin{pmatrix} \eta & \eta_1 & 1 & \eta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For V we have

$$E = (\mu + \xi + \rho)E = AE$$

$$Q = -\rho E + (\mu + \sigma + \pi)Q = -\rho E + BQ \quad (10)$$

$$I - \xi E + (d + \mu + \gamma + \gamma_1 + \theta)I = -\xi E + CI$$

$$T = -\sigma Q - (\gamma + \gamma_1)I + (\mu + \psi + d_1)T = -\sigma Q - DI + GT$$

Where, $A = (\mu + \xi + \rho), B = (\mu + \sigma + \pi), C = (d + \mu + \gamma + \gamma_1 + \theta), D = (\gamma + \gamma_1)\ and$

$$G = (\mu + \psi + d_1)$$

Then

$$V := \begin{pmatrix} A & 0 & 0 & 0 \\ -\rho & B & 0 & 0 \\ -\xi & 0 & C & 0 \\ 0 & -\sigma & -D & G \end{pmatrix}$$

And

$$V^{-1} \rightarrow \begin{pmatrix} \frac{1}{A} & 0 & 0 & 0 \\ \frac{\rho}{A \cdot B} & \frac{1}{B} & 0 & 0 \\ \frac{\xi}{A \cdot C} & 0 & \frac{1}{C} & 0 \\ \frac{B \cdot D \cdot \xi + C \cdot \rho \cdot \sigma}{A \cdot B \cdot C \cdot G} & \frac{\sigma}{B \cdot G} & \frac{D}{C \cdot G} & \frac{1}{G} \end{pmatrix}$$

$$F \cdot V^{-1} \rightarrow \begin{bmatrix} \frac{\eta \cdot S\beta}{A \cdot N} + \frac{\xi \cdot S\beta}{A \cdot C \cdot N} + \frac{\rho \cdot S\beta \cdot \eta_1}{A \cdot B \cdot N} + \frac{S\beta \cdot (B \cdot D \cdot \xi + C \cdot \rho \cdot \sigma) \cdot \eta_2}{A \cdot B \cdot C \cdot G \cdot N} & \frac{S\beta \cdot \eta_1}{B \cdot N} + \frac{\sigma \cdot S\beta \cdot \eta_2}{B \cdot G \cdot N} & \frac{S\beta}{C \cdot N} + \frac{D \cdot S\beta \cdot \eta_2}{C \cdot G \cdot N} & \frac{S\beta \cdot \eta_2}{G \cdot N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the basic reproduction ratio is the dominant eigenvalue of the next generation matrix of FV^{-1} and having that $\frac{S}{N} = 1$, we obtain

$$\mathcal{R}_e = \beta \left[\frac{\eta BCG + \xi BG + \rho \eta_1 CG + (\xi BD + \rho \sigma C)\eta_2}{ABCG} \right] \qquad (11)$$

The model was quantitatively analyzed for the stability of the disease free equilibrium using [28] theorem, which shows that the disease free equilibrium of COVID-19 model system (2) is locally asymptotically stable if $\mathcal{R}_e$ is less than unity and unstable if $\mathcal{R}_e$ is greater than unity.

3.4 Existence of COVID-19 Endemic Equilibrium State

Let $H_1^* = [S^*, E^*, Q^*, I^*, T^*, R^*]$ be any arbitrary equilibrium state of the model system (2) where the disease cannot be totally eradicated but remains in the population. Note that he force of infection is denoted by

$$\lambda = \frac{\beta(I + \eta E + \eta_1 Q + \eta_2 T)}{N}$$

If we solve system (2) simultaneously, by equating the sub-equations to zero we obtain

$$S^* = \frac{\varphi + \pi Q^*}{\mu + \lambda}, E^* = \frac{S^* \lambda}{A}, Q^* = \frac{\rho E^*}{B}, I^* = \frac{\xi E^*}{C}, T^* = \frac{\sigma Q^* + (\gamma + \gamma_1) I^*}{G}, R^* = \frac{\psi T^* + \theta I^*}{\mu}$$

Which we further solved to get

$$S^* = \frac{\varphi AB}{\lambda(AB - \rho\pi) + \mu AB}$$

$$E^* = \frac{\lambda\varphi B}{\lambda(AB - \rho\pi) + \mu AB}$$

$$Q^* = \frac{\lambda\varphi\rho}{\lambda(AB - \rho\pi) + \mu AB}$$

$$I^* = \frac{\lambda\xi\varphi B}{C[\lambda(AB - \rho\pi) + \mu AB]}$$

$$T^* = \frac{\lambda\varphi[\sigma\rho C + \xi BD]}{CG[\lambda(AB - \rho\pi) + \mu AB]}$$

$$R^* = \frac{\varphi\psi\lambda[\sigma\rho C + \xi BD] + \theta\xi\varphi\lambda BG}{\mu CG[\lambda(AB - \rho\pi) + \mu AB]}$$

From equation (1), we have

$$\lambda = \frac{\beta(I^* + \eta E^* + \eta_1 Q^* + \eta_2 T^*)}{N^*} \quad (12)$$

Substituting $S^*, E^*, Q^*, I^*, T^*, R^*$ in (12) where

$$N^* = S^* + E^* + Q^* + I^* + T^* + R^* \quad (13)$$

The, we obtain

$$N^* = \frac{\varphi AB + \lambda\varphi B + \lambda\varphi\rho}{\lambda(AB - \rho\pi) + \mu AB} + \frac{\lambda\xi\varphi BC + \lambda\varphi[\sigma\rho C + \xi BD]}{CG[\lambda(AB - \rho\pi) + \mu AB]} + \frac{\varphi\psi\lambda[\sigma\rho C + \xi BD] + \theta\xi\varphi\lambda BG}{\mu CG[\lambda(AB - \rho\pi) + \mu AB]}$$

$$N^* = \frac{\varphi\lambda[\mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG] + \mu\varphi ABCG}{\mu CG[\lambda(AB - \rho\pi) + \mu AB]} \quad (14)$$

and

$$\beta(I^* + \eta E^* + \eta_1 Q^* + \eta_2 T^*)$$

$$= \beta\left[\frac{\lambda\xi\varphi B}{C[\lambda(AB - \rho\pi) + \mu AB]} + \frac{\lambda\eta\varphi B + \lambda\eta_1\varphi\rho}{\lambda(AB - \rho\pi) + \mu AB} + \frac{\lambda\eta_2\varphi[\sigma\rho C + \xi BD]}{CG[\lambda(AB - \rho\pi) + \mu AB]}\right]$$

$$= \frac{\lambda\beta\varphi[\xi GB + \eta BCG + \eta_1\rho CG + \eta_2[\sigma\rho C + \xi BD]]}{CG[\lambda(AB - \rho\pi) + \mu AB]} \quad (15)$$

Substituting (14) and (15) into (12) gives

$$\lambda = \frac{\frac{\lambda\beta\varphi\,[\xi GB+\eta BCG+\eta_1\rho CG+\eta_2[\sigma\rho C+\xi BD]]}{CG[\lambda(AB-\rho\pi)+\mu AB]}}{\frac{\varphi\lambda[\mu BCG+\rho\mu CG+\mu\xi BG+[\sigma\rho C+\xi BD][\mu+\psi]+\theta\xi BG]+\mu\varphi ABCG}{\mu CG[\lambda(AB-\rho\pi)+\mu AB]}}$$

$$1 = \frac{\mu\beta\varphi\left[\xi GB + \eta BCG + \eta_1\rho CG + \eta_2[\sigma\rho C + \xi BD]\right]}{\lambda\varphi[\mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG] + \mu\varphi ABCG}$$

$$\lambda\varphi[\mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG] + \mu\varphi ABCG = \mu\varphi\beta\left[\xi GB + \eta BCG + \eta_1\rho CG + \eta_2[\sigma\rho C + \xi BD]\right]$$

$$\lambda\varphi[\mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG] + \mu\varphi\left[ABCG - \beta\left[\xi GB + \eta BCG + \eta_1\rho CG + \eta_2[\sigma\rho C + \xi BD]\right]\right] = 0$$

$$\lambda[\mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG] + \mu ABCG[1 - \mathcal{R}_e] = 0$$

Therefore, λ is a positive solution of the following equation

$$b_1\lambda + b_0 = 0 \qquad (16)$$

Where

$$b_1 = \mu BCG + \rho\mu CG + \mu\xi BG + [\sigma\rho C + \xi BD][\mu + \psi] + \theta\xi BG$$

$$b_0 = \mu ABCG[1 - \mathcal{R}_e]$$

It is worth noting that b_1 is positive and b_0 will be positive if and only if $\mathcal{R}_e$ is less than unity and negative if $\mathcal{R}_e$ is greater than unity. Therefore, if b_0 is positive, no endemic equilibrium state exists and if $b_0 > 0$, then a forward bifurcation occurs since $b_1 > 0$. This implies that an endemic equilibrium state only exists If and only if $b_0 < 0$. This is given by

$$\lambda = \frac{-b_0}{b_1} \Longrightarrow \lambda = \frac{\mu ABCG[\mathcal{R}_e - 1]}{b_1}$$

Then substituting for λ in $S^*, E^*, Q^*, I^*, T^*, R^*$ we get

$$S^* = \frac{\varphi AB}{\frac{\mu ABCG\,(\mathcal{R}_e-1)(AB-\rho\pi)}{b_1} + \mu AB} = \frac{\varphi b_1}{\mu[CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1]}$$

$$E^* = \frac{\frac{\varphi\mu A\,B^2 CG(\mathcal{R}_e-1)}{b_1}}{\frac{\mu ABCG\,(\mathcal{R}_e-1)(AB-\rho\pi)}{b_1} + \mu AB} = \frac{\varphi BCG(\mathcal{R}_e - 1)}{CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1}$$

$$Q^* = \frac{\frac{\varphi\rho\mu ABCG\ (\mathcal{R}_e-1)}{b_1}}{\frac{\mu ABCG\,(\mathcal{R}_e-1)(AB-\rho\pi)}{b_1} + \mu AB} = \frac{\varphi\rho CG(\mathcal{R}_e - 1)}{CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1}$$

$$I^* = \frac{\frac{\varphi\xi\mu A\, B^2 CG(\mathcal{R}_e - 1)}{b_1}}{C\left[\frac{\mu ABCG\,(\mathcal{R}_e - 1)(AB - \rho\pi)}{b_1} + \mu AB\right]} = \frac{\varphi\xi BG(\mathcal{R}_e - 1)}{CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1}$$

$$T^* = \frac{\frac{\varphi\mu ABCG\,(\mathcal{R}_e - 1)}{b_1}[\rho\sigma C + \xi BD]}{CG\left[\frac{\mu ABCG\,(\mathcal{R}_e - 1)(AB - \rho\pi)}{b_1} + \mu AB\right]} = \frac{\varphi(\mathcal{R}_e - 1)[\rho\sigma C + \xi BD]}{CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1}$$

$$R^* = \frac{\frac{\varphi\mu ABCG\,(\mathcal{R}_e - 1)}{b_1}[\psi[\rho\sigma C + \xi BD] + \theta\xi BG]}{\mu CG\left[\frac{\mu ABCG\,(\mathcal{R}_e - 1)(AB - \rho\pi)}{b_1} + \mu AB\right]} = \frac{\varphi(\mathcal{R}_e - 1)[\psi[\rho\sigma C + \xi BD] + \theta\xi BG]}{\mu[CG(\mathcal{R}_e - 1)(AB - \rho\pi) + b_1]}$$

This leads to the following theorem.

Theorem 3: The model system (2) has a unique endemic equilibrium (EE) state whenever $\mathcal{R}_e > 1$.

3.5 Sensitivity Analysis of the Model Parameters

We conducted the sensitivity analysis of the model parameters to know the relative contribution of different model parameter that is responsible for the transmission and control of the disease. This helps to the parameters that have the highest impact in reducing the effective reproduction number.

The normalized forward sensitivity index method of $\mathcal{R}_e$, is used to investigate the relative change in $\mathcal{R}_e$, to relative change in the model parameter g. This is also defined using partial derivatives if $\mathcal{R}_e$ is a differentiable function of the model parameter g,as is defined in [30], [30].

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial g} \cdot \frac{g}{\mathcal{R}_e} \qquad (17)$$

Where $Z_g^{\mathcal{R}_e}$ is the sensitivity index of $Z_g^{\mathcal{R}_e}$ from a parameter, g.

Table 3: The Parameter values for COVID- 19 infection used to calculate $\mathcal{R}_e$.

Parameters	Description	Value	Reference
φ	Recruitment rate into Susceptible class	11.0811	Estimated
μ	Natural death rate	$1E-6$	[32]
β	Transmission rate	0.2	[19]
θ	Recovery rate of infected individuals due to immune response	0.33029	[19]
ψ	Recovery rate of treated individuals	0.11624	[19]
d	Death induced rate of infected individuals	0.0079	[19]
d_1	Rate of disease induced death on treated individuals	0.0068	[33]
π	Progress from Quarantine to Susceptible class	1/14	[34], [35]
ξ	Progress from Exposed to Infected class	1/7	[14]

ρ	Progress from Exposed to Quarantined class	1.8887E-7	[19]
σ	Progress from Quarantine to Treated class	0.1259	[19]
γ	Progress from Infected to Treated class (contact Tracing)	0.13266	[19]
γ_1	Progress from Infected to Treated (self submission)	0.001	Assumed
η	Modification factor for the Exposed	0.3	[21]
η_1	Modification factor for the Quarantined	0	[21]
η_2	Modification factor for the Treated (Isolated)	0.1	[21]

The sensitivity index of $\mathcal{R}_e$ with respect to each parameter of $\mathcal{R}_e$ using the parameter values in table 3 is given in table 4. For example the sensitivity index for β is given by

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \beta}.\frac{\beta}{\mathcal{R}_e} = +1.$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \mu}.\frac{\mu}{\mathcal{R}_e}$$

$$= -\left[\frac{\eta}{A^2} + \frac{\xi(A+C)}{(AC)^2} + \frac{\rho\eta_1(A+B)}{(AB)^2} + \frac{\xi\eta_2 D(AC+AG+CG)}{(ACG)^2} + \frac{\rho\sigma\eta_2(AB+AG+BG)}{(ABG)^2}\right] \times \frac{\mu}{\mathcal{R}_e}$$

$$= -9.22744 \times E^{-5}$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \eta}.\frac{\eta}{\mathcal{R}_e} = \frac{1}{A} \times \frac{\eta}{\mathcal{R}_e} = 2.36216$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \eta_1}.\frac{\eta_1}{\mathcal{R}_e} = \frac{\rho}{AB} \times \frac{\eta_1}{\mathcal{R}_e} = 0$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \eta_2}.\frac{\eta_2}{\mathcal{R}_e} = \frac{\rho\sigma C + \xi BD}{ABCG} \times \frac{\eta_2}{\mathcal{R}_e} = 0.25896$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \xi}.\frac{\xi}{\mathcal{R}_e} = \left[-\frac{\eta}{A^2} + \frac{\mu+\rho}{A^2C} - \frac{\rho\eta_1}{A^2B} + \frac{\eta_2 D(\mu+\rho)}{A^2CG} - \frac{\eta_2\rho\sigma}{A^2BG}\right] \times \frac{\xi}{\mathcal{R}_e} = -2.36214$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \sigma}.\frac{\sigma}{\mathcal{R}_e} = \left[-\frac{\rho\eta_1}{AB^2} + \frac{\eta_2\rho(\mu+\pi)}{AB^2G}\right] \times \frac{\sigma}{\mathcal{R}_e} = 2.79137E^{-7}$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \pi}.\frac{\pi}{\mathcal{R}_e} = \left[-\frac{\rho\eta_1}{AB^2} - \frac{\eta_2\rho\sigma}{AB^2G}\right] \times \frac{\pi}{\mathcal{R}_e} = -2.79134E^{-7}$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \rho}.\frac{\rho}{\mathcal{R}_e} = \left[-\frac{\eta}{A^2} - \frac{\xi}{A^2C} + \frac{\eta_1(\mu+\xi)}{A^2B} - \frac{\eta_2\xi D}{A^2CG} - \frac{\eta_2\sigma(\mu+\xi)}{A^2BG}\right] \times \frac{\rho}{\mathcal{R}_e} = -4.547E^{-6}$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \gamma}.\frac{\gamma}{\mathcal{R}_e} = \left[-\frac{\xi}{AC^2} + \frac{\eta_2\xi(d+\mu)}{AC^2G}\right] \times \frac{\gamma}{\mathcal{R}_e} = -0.66587$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \gamma_1}.\frac{\gamma_1}{\mathcal{R}_e} = \left[-\frac{\xi}{AC^2} + \frac{\eta_2 \xi (d+\mu)}{AC^2 G}\right] \times \frac{\gamma_1}{\mathcal{R}_e} = -5.01935E^{-3}$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \psi}.\frac{\psi}{\mathcal{R}_e} = \left[-\frac{\eta_2 \xi D}{ACG^2} - \frac{\eta_2 \rho \sigma}{ABG^2}\right] \times \frac{\psi}{\mathcal{R}_e} = -0.24466$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial \theta}.\frac{\theta}{\mathcal{R}_e} = \left[-\frac{\xi}{AC^2} - \frac{\eta_2 \xi D}{AC^2 G}\right] \times \frac{\theta}{\mathcal{R}_e} = -1.66856$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial d}.\frac{d}{\mathcal{R}_e} = \left[-\frac{\xi}{AC^2} + \frac{\eta_2 \xi D}{AC^2 G}\right] \times \frac{d}{\mathcal{R}_e} = -0.03991$$

$$Z_g^{\mathcal{R}_e} = \frac{\partial \mathcal{R}_e}{\partial d_1}.\frac{d_1}{\mathcal{R}_e} = \left[-\frac{\eta_2 \xi D}{ACG^2} - \frac{\eta_2 \rho \sigma}{ABG^2}\right] \times \frac{d_1}{\mathcal{R}_e} = -0.014312$$

Table 4: The effect of the parameters on $\mathcal{R}_e$.

Parameters	Value	Effect on $\mathcal{R}_e$
μ	$1E-6$	$-9.22744E^{-5}$
β	0.2	1
θ	0.33029	-1.66856
ψ	0.11624	-0.24466
d	0.0079	-0.03991
d_1	0.0068	-0.014312
π	1/14	$-2.79134E^{-7}$
ξ	1/7	-2.36214
ρ	1.8887E-7	$-4.547E^{-6}$
σ	0.1259	$2.79137E^{-7}$
γ	0.13266	-0.66587
γ_1	0.001	$-5.01935E^{-3}$
η	0.3	2.36216
η_1	0	0
η_2	0.1	0.25896

The sensitivity indices $Z(\eta), Z(\eta_2), Z(\beta), Z(\sigma)$ are positive and this shows that the value of $\mathcal{R}_e$ increases as the value of $\eta, \eta_2, \beta \; and \; \sigma$ increases. The remaining indices $Z(\theta), Z(d_1), Z(\gamma_1), Z(\xi), Z(\gamma), Z(d)\; Z(\pi), Z(\psi), Z(\mu) and\; Z(\rho)$ are negative, indicating that the value $\mathcal{R}_e$ decreases as $\theta, d_1, \gamma_1,\; \xi, \gamma, d, \pi, \psi, \mu \; and \; \rho$ increases. Actually, the effectiveness of control may be measured by its effect on $\mathcal{R}_e$. Once the parameters $\theta, d_1, \gamma_1,\; \xi, \gamma, d, \pi, \psi, \mu \; and \; \rho$ will reduce $\mathcal{R}_e < 1$, then it is curative if the reduction can be maintained. This implies that these parameters can help in reducing the rate of COVID – 19 infection over time and if it is maintained, the transmission of the disease may decrease, causing the cases in the population to go below an endemicity threshold.

IV. Optimal Control Analysis

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamic system over a period of time such that an objective, function is optimized [36]. It is an extension of calculus of variations, and is used for deriving cotrol policy [37]. The optimal control can be derived using Pontryagin's maximum principle (a necessary condition known as Pontryagin's minimum principle or simply Pontryagin's principle) or by solving the Hamilton – Jacobi – Bellman equation (a sufficient condition).

We are interested in minimizing the cost function

$$M(v_1, v_2) = \int_0^{t_f} \left[A_1 E(t) + A_2 Q(t) + A_3 I(t) + A_4 T(t) + \frac{1}{2} C_1 v_1^2 + \frac{1}{2} C_2 v_2^2 \right] dt \qquad (18)$$

subject to the system of differential equations (2), where t_f is the final time. This performance specification involves the number of individuals of exposed, quarantined, infected or treated (isolated), respectively as well as the cost for applying quarantine controlv_1 and treatment (isolation) control v_2. The total cost includes not only the consumption for every individual but also the cost of organization, management, and co-operation etc. Base on the literature for optimal control of epidemics, the cost of controls is assumed to be non linear and quadratic [38],[39],[40],[41][42]. The coefficients, A_3, A_4, C_1, and C_2 are balancing cost factor due to scales and importance of the six parts of the objective function.

Our aim is to find an optimal control pair, $v_1^* \; and \; v_2^*$ such that

$$M(v_1^*, v_2^*) = \min_{\phi} J(v_1, v_2) \qquad (19)$$

where the control effects $v_1 \; and \; v_2$ are assumed to be bounded and Lebesgue measurable time-dependent functions on the interval $[0, t_f]$.

Therefore,

$$\phi = \{(v_1, v_2) \mid 0 \le v_1 \le 1, 0 \le v_2 \le 1, 0 \le t \le t_f\}$$

Applying Pontryagin's maximum principle [43] which provides the necessary conditions for an optimal control problem. This converts the COVID-19 model system (2) with equations (18) and (19) intoa problem of minimizing a Hamiltonian, H, pointwisely with respect to $v_1 \; and \; v_2$:

$$H = A_1 E(t) + A_2 Q(t) + A_3 I(t) + A_4 T(t) + \frac{1}{2} C_1 v_1^2(t) + \frac{1}{2} C_2 v_2^2(t) + \sum_{i=1}^{6} \lambda_i f_i \qquad (20)$$

Where f_i is the right hand side of the differential equation of ith state variables by applying

$$H = A_1E(t) + A_2Q(t) + A_3I(t) + A_4T(t) + \frac{1}{2}C_1v_1^2(t) + \frac{1}{2}C_2v_2^2(t) + \lambda_1(\varphi + \pi Q - \mu S - \lambda S) + \lambda_2(\lambda S - (\mu + \xi + \rho)E) + \lambda_3(\rho E - (\mu + \sigma + \pi)Q) + \lambda_4(\xi E - (d + \mu + \gamma + \gamma_1)I) + \lambda_5(\sigma Q + (\gamma + \gamma_1)I - (\psi + \mu + d_1)T) + \lambda_6(\psi T + \theta I - \mu R) \quad (21)$$

Applying pontryagin's maximum principle together with the existence result for control pairs from (19), we have the following proposition.

Proposition1: Given an optimal control pair (v_1^*, v_2^*) and corresponding solution $S^*, E^*, Q^*, I^*, T^*, R^*$ that maximizes $M(v_1, v_2)$ over ϕ. Then there exist adjoint variables $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t), \lambda_6(t)$ satisfying.

$$\frac{d\lambda_1}{dt} = \frac{\beta(I^* + \eta E^* + \eta_1 Q^* + \eta_2 T^*)}{N}(\lambda_1 - \lambda_2) + \mu\lambda_1$$

$$\frac{d\lambda_2}{dt} = -A_1 + \frac{\eta\beta S^*}{N}(\lambda_2 - \lambda_1) + \lambda_2\left(\mu + \xi + \rho v_1^*(t)\right) - \lambda_3\rho v_1^*(t) - \lambda_4\xi$$

$$\frac{d\lambda_3}{dt} = -A_2 + \frac{\eta_1\beta S^*}{N}(\lambda_2 - \lambda_1) + \mu\lambda_3 + \sigma(\lambda_3 - \lambda_5) + \pi(\lambda_3 - \lambda_1) \quad (21)$$

$$\frac{d\lambda_4}{dt} = -A_3 + \frac{\beta S^*}{N}(\lambda_1 - \lambda_2) + \lambda_4(\mu + d) + (\gamma v_2^*(t) + \gamma_1)(\lambda_4 - \lambda_5) + \theta(\lambda_4 - \lambda_6)$$

$$\frac{d\lambda_5}{dt} = -A_4 + \frac{\eta_2\beta S^*}{N}(\lambda_2 - \lambda_1) + \lambda_5(\mu + d_1) + \psi(\lambda_5 - \lambda_6)$$

$$\frac{d\lambda_5}{dt} = \frac{\beta S^*}{N}(\lambda_2 - \lambda_1) + \mu\lambda_6$$

With transversality condition

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \lambda_4(t) = \lambda_5(t) = \lambda_6(t) = 0 \quad (22)$$

and optimality condition is given by

$$v_1^*(t) = \frac{\rho E^*(\lambda_2 - \lambda_3)}{C_1};\ v_2^*(t) = \frac{\gamma I^*(\lambda_4 - \lambda_5)}{C_2} \quad (23)$$

Proof:

The system of differential equations in (21) is obtained by differentiation of Hamiltonian function, H, evaluated at the optimal control. This is written as

$$-\frac{d\lambda_1}{dt} = \frac{\partial H}{\partial S^*}, \qquad \lambda_1(t_f) = 0;$$

$$-\frac{d\lambda_2}{dt} = \frac{\partial H}{\partial E^*}, \qquad \lambda_2(t_f) = 0;$$

$$-\frac{d\lambda_3}{dt} = \frac{\partial H}{\partial Q^*}, \qquad \lambda_3(t_f) = 0;$$

$$-\frac{d\lambda_4}{dt} = \frac{\partial H}{\partial I^*}, \qquad \lambda_4(t_f) = 0;$$

$$-\frac{d\lambda_5}{dt} = \frac{\partial H}{\partial T^*}, \qquad \lambda_5(t_f) = 0;$$

$$-\frac{d\lambda_6}{dt} = \frac{\partial H}{\partial R^*}, \qquad \lambda_6(t_f) = 0;$$

This gives the costate system in equation (21). The optimality conditions are given in the interior of the control set as

$$V = \{v_1^*, v_2^* \mid 0 \leq v_1(t), v_2(t) \leq 1\}$$

In adding, equating to zero the derivative of Hamiltonian with respect to the control variables in the control set, V, that is

$\frac{\partial H}{\partial v_1^*} = 0 \; and \; \frac{\partial H}{\partial v_2^*} = 0$, we solve $v_1 \; as \; v_1^*$ and $v_2 \; as \; v_2^*$, this gives

$$v_1^* = \frac{\rho E^*(\lambda_2 - \lambda_3)}{C_1}; \; v_2^* = \frac{\gamma I^*(\lambda_4 - \lambda_5)}{C_2}$$

Using the bounds on the controls we divide the optimality conditions as follows

$$v_1^* = \max\left\{0, \min\left\{1, \frac{\rho E^*(\lambda_2 - \lambda_3)}{C_1}\right\}\right\}$$

$$v_2^* = \max\left\{0, \min\left\{1, \frac{\gamma I^*(\lambda_2 - \lambda_3)}{C_2}\right\}\right\}$$

Owing to the pair boundedness of the state and adjoint functions and the resulting Lipschitz structure of ODE's, we obtain the uniqueness of the optimal control pair which follows from the optimality system as in [39],[41],[42],[44],[45].

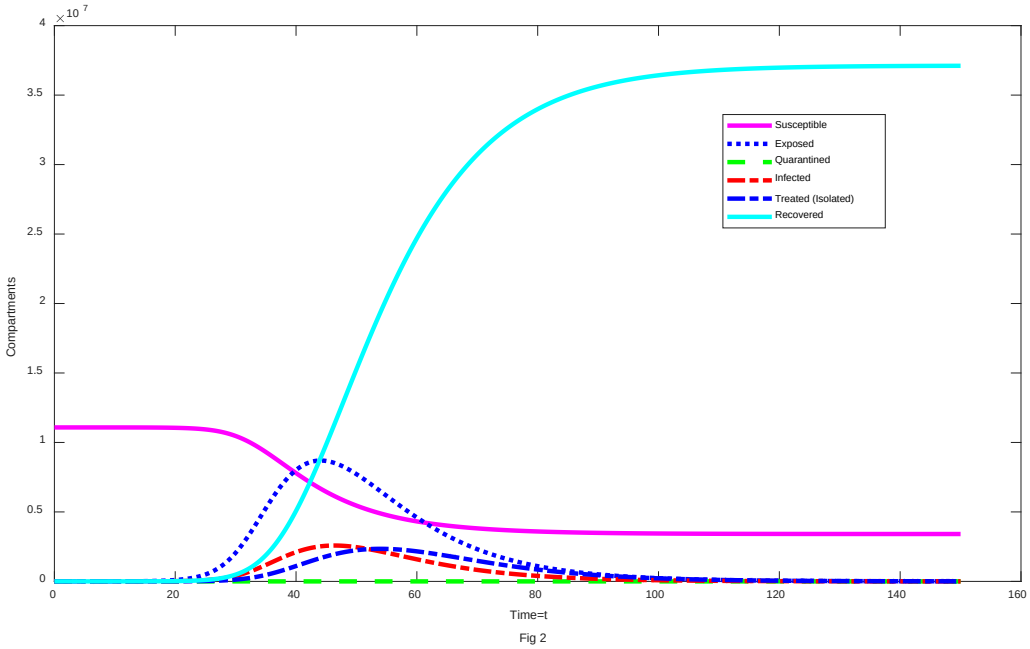

Fig 2

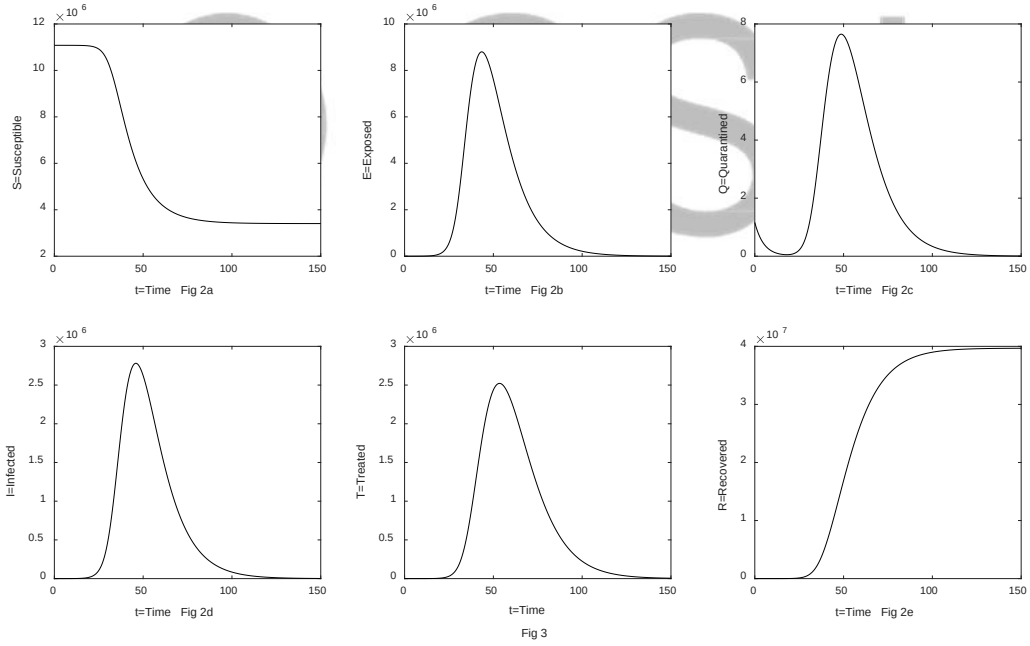

Fig 3

Fig. 2 and 3 are dynamics of COVID – 19 infected individuals when both Quarantine Control (v_1) and Treatment (Isolation) (v_2) are employed as optimal control.

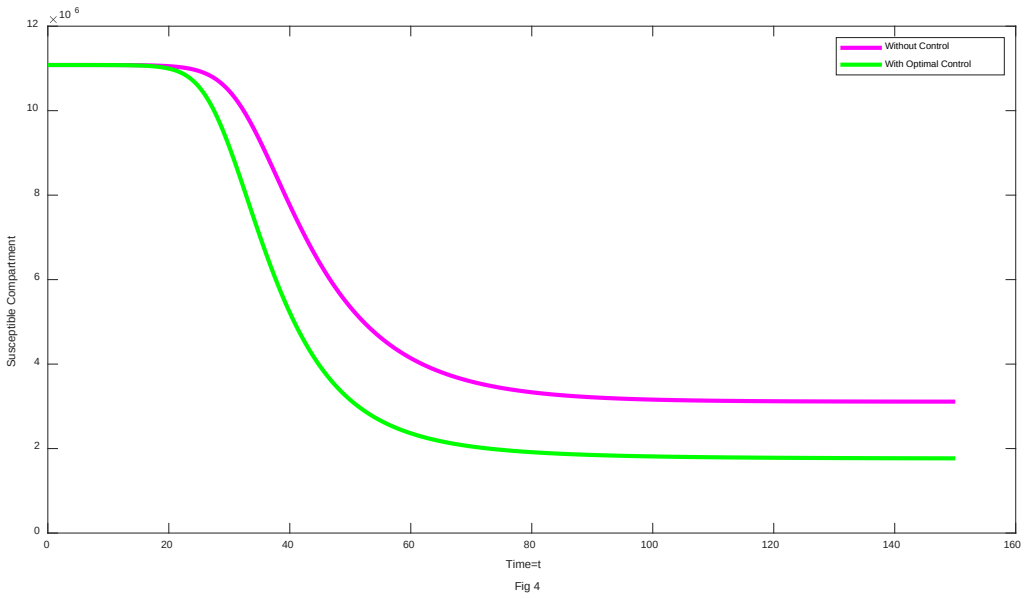

Fig. 4: Dynamics of Susceptible individuals when both Quarantine Control (v_1) and Treatment (Isolation) (v_2) are employed as optimal control.

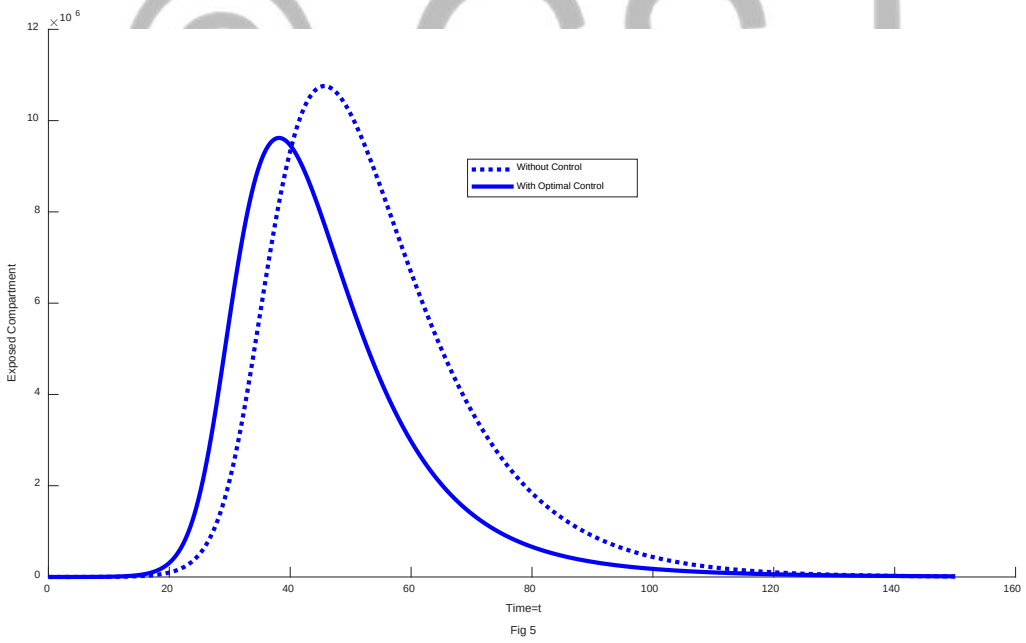

Fig. 5: Dynamics of Exposed individuals when both Quarantine Control (v_1) and Treatment (Isolation) (v_2) are employed as optimal control.

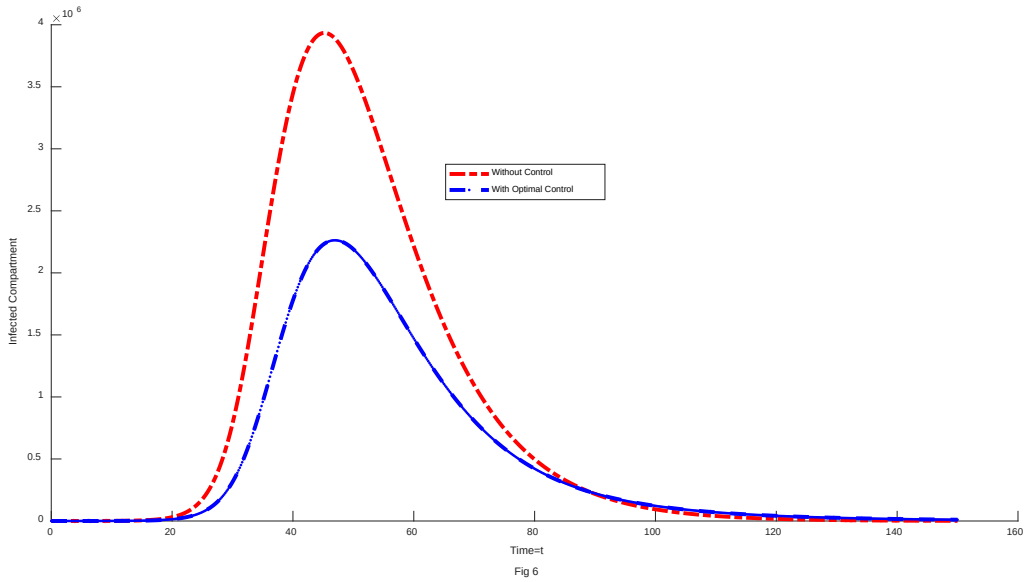

Fig. 6: Dynamics of Infected individuals when both Quarantine Control (v_1) and Treatment (Isolation) (v_2) are employed as optimal control.

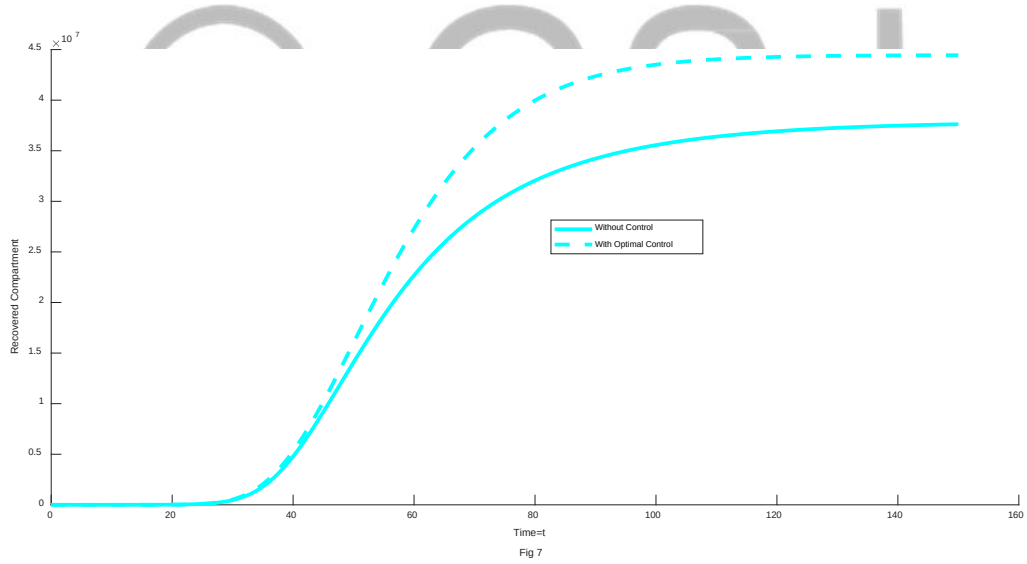

Fig. 5: Dynamics of Recovered individuals when both Quarantine Control (v_1) and Treatment (Isolation) (v_2) are employed as optimal control.

V. Numerical Results and Discussion

In this section, numerical simulations of the optimal control model (2) have been perfomed using Runge Kutta method written in MATLAB programming. We use a set of logical parameter values. Graphical results are displayed using the initial values: $S = 11{,}081{,}000, E = 105.1, Q = 1.1642, I = 27.679, T = 1, R = 2$ andall the parameters showed in table 3. The

simulations are performed with timeline of 150 days. Firstly, the optimal control model is simulated. For the simulation of optimal control system (2), we solved the optimality sytems when there is no control and when there is control. Here, the numerical simulations of the optimal system (2) are performed considering both controls, that is, quarantine (v_1) and treatment (Isolation) (v_2) and the results are shown in figures 2 and 3, 4 – 7. We observe the effect of quarantine and treatment (Isolation) on the susceptible, exposed, quarantine, infected, treated (isolated) and recovered individuals for five (6) months timeline. It has been seen that the control measures signicantly influences the susceptible, exposed, quarantine, infected, treated (isolated) and recovered individuals. As expected , both the exposed and infected individuals have increased in the absence of control (quarantineand treatment), than the individuals with the optimal control measure. On the contrary, the number of recovered individuals increases when quarantine and treatment control are applied compared to the individuals without optimal control. Furthermore, the number of recovered individuals increases when quarantine and treatment (isolation) controls are applied compared to the individuals without optimal control.

VI. Conclusions

In this paper an optimal control model has been formulated considering two control variables, i.e., quarantine and treatment (isolation) by using the most well-known pontryagin's maximum principle. Numerical simulation was performed toillustrate the analytic results. From investigation, it was observed that the optimal quarantine andtreatment are much more effective for reducing the number of exposed and infected individuals to maximize the recovered individuals and also to minimize the cost of control measures. Since there are quarantine and treatment strategies available for COVID – 19 infection, so from the simulations, ithas been stabliushed that the optimal combination of quarantine and treatment are effective to control the disease progression. So to reduce the infection (COVID-19), quarantine exposed and subsequent treatment of infected individuals should be stated on time.

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