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On the Complete Stability Set of the First Kind for Parametric Multi-objective Linear Programming Problems

(Parameters in the Objective Functions)

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ABSTRACT

This paper uses parametric study for providing essential information about the problem's behavior to the decision maker. Two novel algorithms are presented in this work. The first algorithm is obtained to find the complete stability set of the first kind for parametric multi- objective linear programming problems. It is based on the weighting method for scalarizing the multi- objective linear programming problems and Kuhn- Tucker conditions for mathematical programming in general.

The second algorithm presents a technique to decompose of the parametric space in parametric linear programming problem according to the complete stability set of the first kind.

Two numerical examples are introduced to clarify the obtained results.

KeyWords

Multi-objective Linear Programming; Efficient Solution; Solvability Set; Stability Set of the First Kind; Multicriteria Simplex Method.

1. INTRODUCTION

Multi-objective optimization is concerned with mathematical optimization problems involving two or more objective functions and constraints, which need to be optimized (minimized or maximized) simultaneously. A multi-objective problem is linear program, if all objective functions and constrains are linear [5, 9].

These objectives are usually in incommensurate and conflicting with one another, there normally exist:

1- Infinite number of efficient (non- dominated, Pareto- optimal, or non-inferior) solutions in the MOOPs. The non-dominated set of the entire feasible decision space is called the Pareto-optimal and the boundary defined by the set of all solutions mapped from the Pareto optimal set is called the Pareto-optimal front.

2- Not all objectives can simultaneously arrive at their optimal levels. So, an assumed value function is used to choose appropriate solutions **[**6**,** 7**,** 8**,** 15**]**.

In early work, the notions of the solvability set and the stability of the first kind are analyzed for parametric convex programming problems [12].

The relation to the importance of the results in multi- objective convex programming problems can be seen by the fundamental role which parametric techniques play in multi- objective programming.

In this paper, the basic concepts are defined and analyzed quantitatively for parametric multi- objective linear programming problems.

Also, the connectedness nature of the efficient extreme solutions of such problems is utilized together with the fruitful relation that the Kuhn- Tucker conditions play in solving such problems.

This paper is structured as follows: the parametric multi- objective linear programming problem is introduced in the second section. The third section covers the definitions with characterization of the basic notions of solvability, stability set of the first kind, and the complete stability set of the first kind for such problems. Also, an algorithm for determining the complete stability set of the first kind for a given parameter is presented. The fourth section is devoted to present an algorithm for decomposing parametric space according to the complete stability set of the first kind. Two examples are given which clarify the developed theory and algorithms. Finally, conclusion is reported in the fifth section.

2. Parametric multi- objective linear programming problem

Consider the following parametric multi-objective linear programming (MOLP) problem with parameters in the objective functions:

(MOLP):

Min $F(\lambda, X) = \left(\sum_{j=1}^{n} C_{1j}(\lambda_1)\, x_j, \sum_{j=1}^{n} C_{2j}(\lambda_2)\, x_j, \ldots\ldots, \sum_{j=1}^{n} C_{kj}(\lambda_k)\, x_j \right)$,

Subject to

$G = \{X \in R^n \mid \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ i=1, 2, \ldots, m\}$

Where:

$C_{ij}(\lambda_i)$, i=1, 2, …,k, j=1, 2, …., n are continuous functions in λ_i,

$\lambda_i \in R^{\ell_k}$, i= 1, 2, …., k are real valued parameters,

$$\lambda = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_k) \in R^{\ell_1} \otimes R^{\ell_2} \otimes \ldots, \otimes R^{\ell_k}. \qquad \textbf{(1)}$$

It must be noted that the nonnegativity constraints $x_j \ge 0$, j=1, 2, …., n are included in the feasible region "G".

Using the weighting method to solve (MOLP) problem, we get the following single objective parametric linear programming problem "P(W)" as follows:

P(W): **Min** $\sum_{i=1}^{k} \sum_{j=1}^{n} w_i c_{ij}(\lambda_i)\, x_j$,

Subject to

$G = \{ X \in R^n \mid \sum_{j=1}^{n} a_{ij} x_j \le b_i,\ x_j \ge 0,\ i=1, 2, \ldots, m,\ j=1, 2, \ldots., n \}$ [13],

$\sum_{i=1}^{k} w_i = 1,\ w_i \ge 0,\ i = 1, 2, \ldots.., k,$

$W = (w_1, w_2, \ldots\ldots, w_k) \in R^k$.

It is well known [4, 11] that; if $\overline{X}$ is an efficient solution of (MOLP) for $\lambda = \overline{\lambda}$, then there exists $\overline{W} \ge 0$, such that $\overline{X}$ solves P(W) for $\lambda = \overline{\lambda}$, and if $\overline{X}$ solves P(W) uniquely for $\lambda = \overline{\lambda}$, $\overline{W} \ge 0$, or if $\overline{X}$ solves P(W) for $\lambda = \overline{\lambda}$, $\overline{W} > 0$ then $\overline{X}$ is an efficient solution of (MOLP) for $\lambda = \overline{\lambda}$. Moreover, if we take $\lambda = \overline{\lambda}$, then all the efficient solutions of (MOLP) could be generated from P(W) by taking only positive values of W.

3- The complete stability set of the first kind

For the parametric multi- objective linear programming problem, the following basic notions can be defined:

Definition (The solvability set)

The solvability set of problem (MOLP) which is denoted by B, is defined by:

$B = \{ \lambda \in R^{\ell_1} \otimes R^{\ell_2} \otimes \ldots, \otimes R^{\ell_k}$ | there exists an efficient solution of problem (MOLP) for the given λ}. **(2)**

It is clear that if G is bounded then $B = R^{\ell_1} \otimes R^{\ell_2} \otimes \ldots, \otimes R^{\ell_k}$. **(3)**

Definition (The stability set of the first kind)

Assume that for $\bar{\lambda} \in B$, $\bar{X}$ is an efficient solution of problem (MOLP) corresponding to $\bar{\lambda}$, then the stability set of the first kind of problem (MOLP) corresponding to $\bar{X}$ which is denoted by $S(\bar{X})$ is defined by:

$S(\bar{X}) = \{ \lambda \in B \mid \bar{X}$ is an efficient solution of the problem (MOLP) corresponding to $\lambda\}$. **(4)**

Definition (The complete stability set of the first kind)

Assume that for $\bar{\lambda} \in B$, all the corresponding efficient solutions of the problem (MOLP) are the points X_i, $i \in L_1$ then the complete stability set of the first kind of problem (MOLP) denoted by $S_C(\bar{\lambda})$ is defined by:

$$S_C(\bar{\lambda}) = \bigcap_{i \in L_1} S(X_i). \quad \textbf{(5)}$$

It must be noted that the efficient solutions of problem (MOLP) for any $\bar{\lambda} \in B$ is connected [1, 9], then $S_C(\bar{\lambda})$ can be obtained by:

$$S_C(\bar{\lambda}) = \bigcap_{i \in L} S(X_i), \quad \textbf{(6)}$$

Where L is a finite set, X_i, i= 1, 2, ..., L are the efficient extreme solutions of (MOLP) corresponding to $\lambda = \bar{\lambda}$.

Assume that $\bar{X}$ is an efficient solution of problem (MOLP) for $\bar{\lambda} \in B$, and $I = \{ i \mid \sum_{j=1}^{n} a_{ij} x_j = b_i, i=1, 2, \ldots, m\}$. **(7)**

i.e., i denote the set of all indices corresponding to the active constraints of G at $\bar{X}$, with cardinality **r**.

Then the corresponding Kuhn- Tucker (K.T.) conditions [3, 10] of the P(W) for $\lambda \in B$ takes the form:

$$\left.\begin{array}{l} \sum_{i=1}^{k} w_i c_{ij}(\lambda_i) + \sum_{i \in I} a_{ij} u_i = 0, \quad j=1, 2, \ldots, n, \\ u_i \geq 0, i \in I, \end{array}\right\} \text{K.T. problem} \quad \textbf{(8)}$$

where

u_i, $i \in I$ are the corresponding multipliers of the K.T. problem.

The equations in K.T. problem is a linear system of equations in the multipliers u_i, $i \in I$ which could be solved by any suitable technique from which either one of the following three cases can arise assuming the linear independence of the equations:

1) r = n

In this case u_i, $i \in I$ could be obtained from the equations in **K.T. problem**, substituting in $u_i \geq 0$, we get **r** conditions on w_i, λ_i, i =1, 2, …., k.

2) r < n

In this case, we use **r** of the **n** equations to get u_i, $i \in I$ and substituting in the rest of the equations and in the inequalities $u_i \geq 0$, we get **n** conditions on w_i, λ_i, i=1, 2, ….. , k.

3) r > n

In this case, **n** of multipliers u_i, $i \in I$ are obtained from the equations, and substituting in the inequalities $u_i \geq 0$, $i \in I$ we get **r** conditions on w_i, λ_i, i= 1, 2, …., k.

Let $A(\bar{X}) = \{(W, \lambda) \in R^k \times B \mid (W, \lambda)$ solves **K.T. problem**}, then the stability set of the first kind $S(\bar{X})$ can be obtained by:

$S(\bar{X}) = \{ \lambda \in B \mid (W, \lambda) \in A(\bar{X})\}$. **(9)**

In the following, the algorithm ALg.(I) is presented to obtain the complete stability set of the first kind of problem (MOLP) for a given $\lambda \in B$.

ALg. (I)

1. Start with any $\bar{\lambda} \in B$.

2. Solve problem (MOLP) for $\lambda = \bar{\lambda}$ using the multicriteria simplex method [16] to get all the efficient extreme points of the problem "X_i", $i \in L$.
3. Put i = 1, get I_1 such that
$I_1 = \{ i \mid \sum_{j=1}^{n} a_{ij} x_{1j} = b_i, i=1, 2, \ldots, m\}$. **(10)**
4. Solve K.T. problem to get $A(X_1)$ from which we obtain $S(X_1)$.
5. Go to step (**3**) and put i= 2, repeat step (**4**) to get the set $S(X_2)$.
6. The process is repeated till all the efficient extreme points X_i, $i \in L$ are exhausted.
7. The complete stability set of the first kind is obtained from the relation:

$$S_C(\bar{\lambda}) = \bigcap_{i \in I} S(X_i). \quad \textbf{(11)}$$

Example 1:

Let us consider the following bicriterion parametric linear programming problem with two parameters λ_1, λ_2 in the first objective function f_1 and one parameter λ_3 in the second objective function f_2. $\lambda_1, \lambda_2, \lambda_3 \in R$.

Min F = ($f_1 = \lambda_1 x_1 + \lambda_2 x_2, \quad f_2 = -3 x_1 + \lambda_3 x_2$ **),**

Subject to

$$G = \{ X \in R^2 \mid x_1 + x_2 \le 4,$$
$$x_1 + 2 x_2 \le 6,$$
$$x_1 \ge 0,$$
$$x_2 \ge 0\}.$$

Since G is bounded then $B = R^3$. If we take $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) = (1, 1, 1)$, the problem will have two efficient extreme solutions $X_1 = (0, 0)$, $X_2 = (4, 0)$, (see Figure (1)).

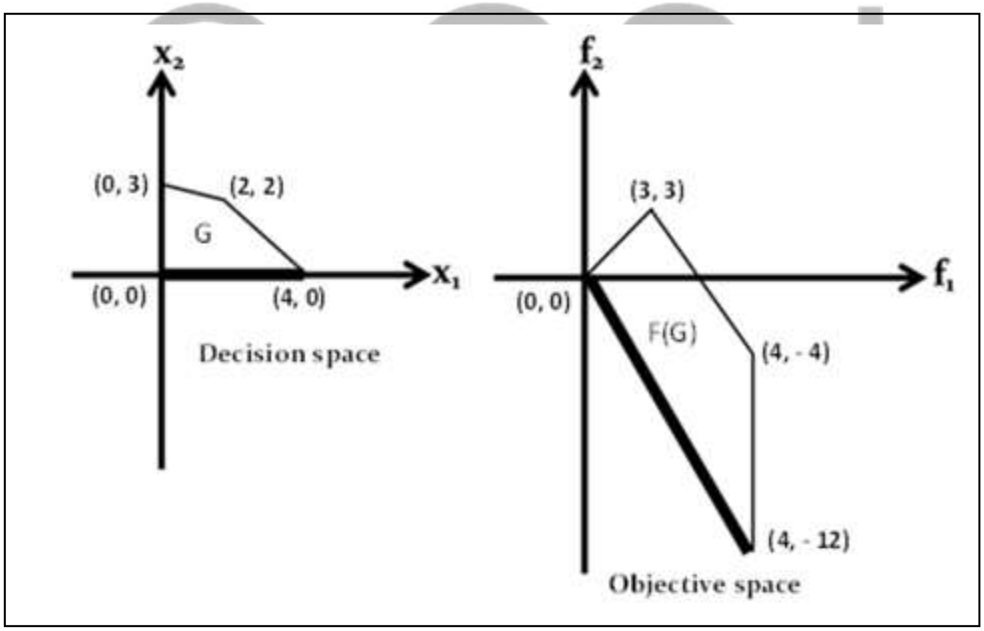

Figure (1): G and F(G) When ($\lambda_1, \lambda_2, \lambda_3$) = (1, 1, 1).

(1) For X_1 = (0, 0)

It is clear that $I(X_1) = \{3, 4\}$.

The Kuhn- Tucker conditions at X_1 for the corresponding weighting problem take the form:

$$w \lambda_1 - 3(1 - w) - u_3 = 0,$$
$$w \lambda_2 + (1 - w) \lambda_3 - u_4 = 0,$$
$$u_3 \ge 0,$$
$$u_4 \ge 0.$$

From which, we get:

$A(X_1) = \{(w, \boldsymbol{\lambda}) \in R^4 \mid w(\lambda_1 + 3) \geq 3,\ w(\lambda_3 - \lambda_2) \leq \lambda_3,\ 0 < w < 1\}$. **(12)**

Therefore,

$S(X_1) = \{\boldsymbol{\lambda} \in R^3 \mid (w, \boldsymbol{\lambda}) \in A(X_1)\}$. **(13)**

(2) For $X_2 = (4, 0)$

It is clear that $I(X_2) = \{1, 4\}$.

The Kuhn- Tucker conditions at X_2 for the corresponding weighting problem take the form:

$$\begin{aligned} w\,\lambda_1 - 3(1-w) + u_1 &= 0,\\ w\,\lambda_2 + (1-w)\,\lambda_3 + u_1 - u_4 &= 0,\\ u_1 &\geq 0,\\ u_4 &\geq 0. \end{aligned}$$

From which, we get:

$A(X_2) = \{(w, \boldsymbol{\lambda}) \in R^4 \mid w(\lambda_1 + 3) \leq 3,\ w(3 + \lambda_1 - \lambda_2 + \lambda_3) \leq \lambda_1 + 3,\ 0 < w < 1\}$. **(14)**

Therefore,

$S(X_2) = \{\boldsymbol{\lambda} \in R^3 \mid (w, \boldsymbol{\lambda}) \in A(X_2)\}$. **(15)**

The complete stability set of the first kind for the given problem for $\boldsymbol{\lambda} = (1, 1, 1)$ is given as

$S_C(1, 1, 1) = S(X_1) \cap S(X_2)$. **(16)**

4- Decomposition of the parametric space

It is clear that the parametric space is of dimension $\sum_{i=1}^{k} \lambda_i$, the solvability set B which is a subset of this space could be decomposed according to the complete stability sets of first kind as well be clear from the following algorithm which is denoted to ALg. (II):

ALg. (II)

1. Start with any $\lambda_1 \in B$.
2. Use ALg. (I) to get $S_C(\lambda_1)$.
3. Choose $\lambda_2 \in B - S_C(\lambda_1)$, and use ALg. (1) to get $S_C(\lambda_2)$.
4. The process is repeated by taking $\lambda_{i+1} \in B - S_C(\lambda_i)$, i=1, 2, …., τ.
5. The algorithm is terminated when $B - S_C(\lambda_\tau) = \phi$.

It must be noted that, the complexity of determining the complete stability set of the first kind depends mainly on the structure of $C_{ij}(\lambda_i)$, i=1, 2, …, k and j=1, 2, ….., n.

Remark 1:

If for any $\bar{\lambda} \in B$, $\bar{X}$ is the unique efficient extreme point of problem (MOLP), then it must be noted that there is a difference between the sets $S(\bar{X})$ and $S_C(\bar{X})$ since $S(\bar{X})$ gives the set of $\lambda\underline{s}$ that maintains $\bar{X}$ as an efficient extreme point while $S_C(\bar{\lambda})$ gives the set of $\lambda\underline{s}$ that maintains $\bar{X}$ as the unique efficient extreme point. In this case $S_C(\bar{\lambda}) = \{\bar{\lambda}\}$.

Remark 2:

If at step**1** of ALg. (II) for $\lambda_1 \in B$, we get unique efficient extreme point, then choose $\lambda_2 \in B$, $\lambda_2 \neq \lambda_1$, and go to step**1**.

Example 2:

Let us consider the following bicriterion parametric linear programming problem with the same parameter $\lambda \in R$ in the two objective functions f_1, and f_2.

Min F = ($f_1 = \lambda x_1 + 3x_2$, $f_2 = 5x_1 - \lambda x_2$ **)**,

Subject to

$$\begin{aligned} G = \{X \in R^2 \mid\ & x_1 + x_2 \leq 4,\\ & x_1 + 2x_2 \leq 6,\\ & x_1 \geq 0,\\ & x_2 \geq 0\}. \end{aligned}$$

Since G is bounded, then $B \in R$.

(1) If we take $\lambda = 0$, the problem will have one efficient extreme solution $X_1 = (0, 0)$, (see Figure (2)).

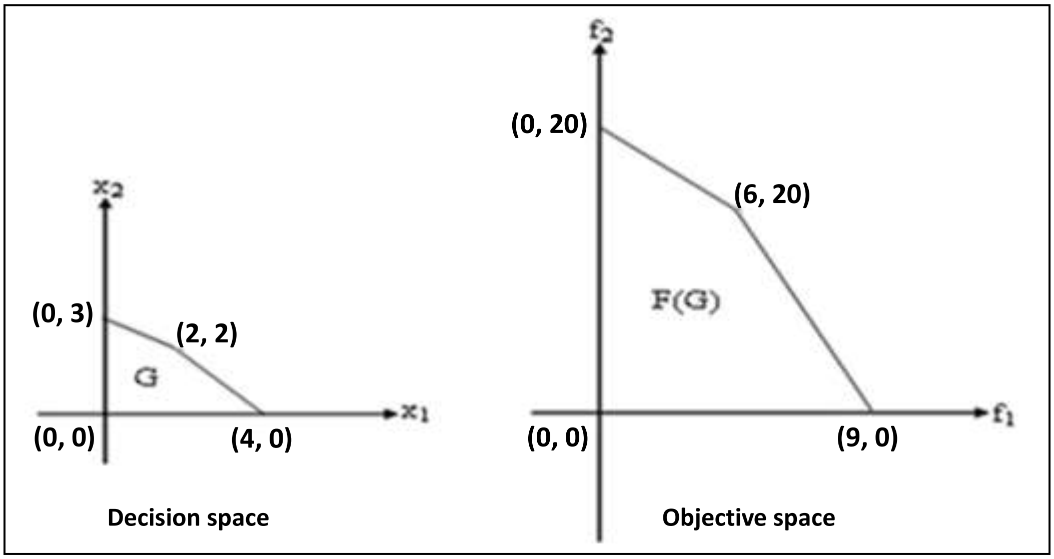

Figure (2): G and F(G) When λ = 0.

It is clear that $I(X_1) = \{3, 4\}$. The Kuhn- Tucker conditions at X_1 for the corresponding weighting problem take the form:

$$w\lambda + 5(1-w) - u_3 = 0,$$
$$3w - \lambda(1-w) - u_4 = 0,$$
$$u_3 \geq 0,$$
$$u_4 \geq 0.$$

From which, we get:

$$A(X_1) = \{(w, \lambda) \in R^2 \mid \lambda \geq , \frac{-5(1-w)}{(w)}, \lambda \leq \frac{(3w)}{(1-w)}, \ 0 < w < 1\}. \quad \textbf{(17)}$$

Therefore,

$$S(X_1) = \{\lambda \in R \mid (w, \lambda) \in A(X_1)\}. \quad \textbf{(18)}$$

From Figure (3), it is clear that $S(X_1) \equiv R$.

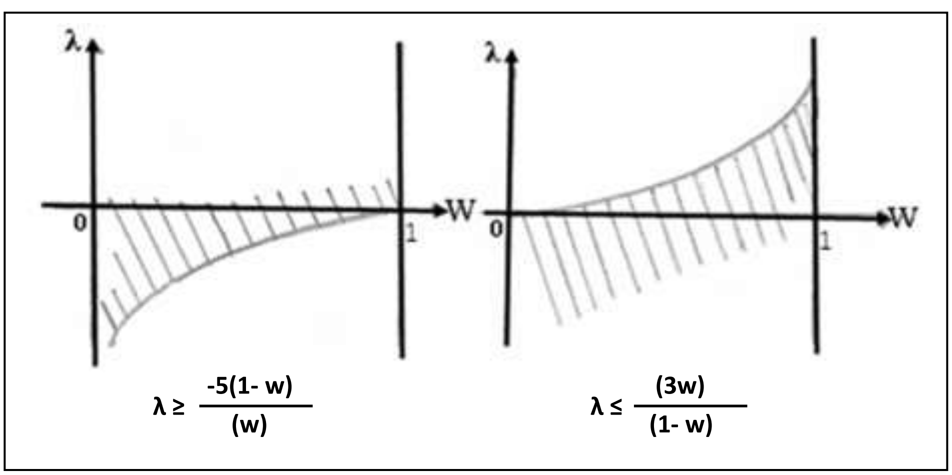

Figure (3): The Relation Between λ and W When $X_1 = (0, 0)$.

Then from **remark 1**, it follows that $S_C(0) = \{0\}$.

(2) Take $\lambda = 1$, the problem will have two efficient extreme solutions $X_1 = (0, 0)$, $X_2 = (0, 3)$, (see Figure (4)).

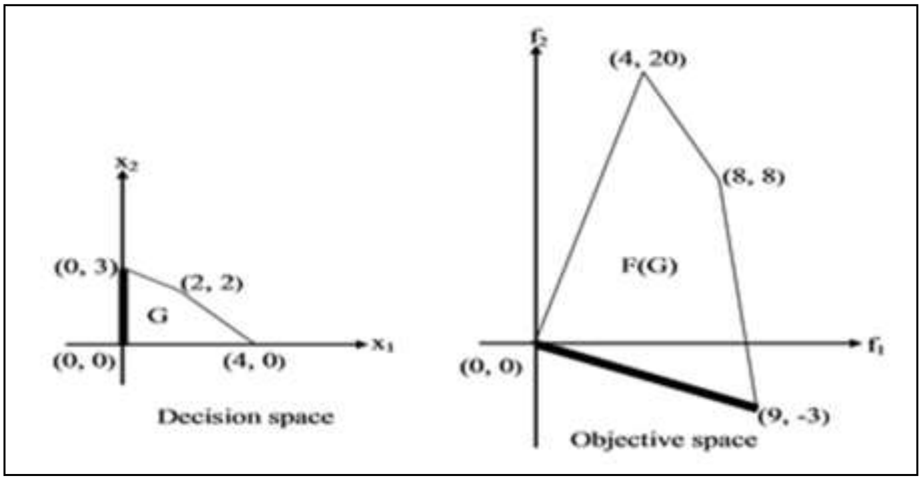

Figure (4): G and F(G) When $\lambda = 1$.

It is clear that $I(X_2) = \{2, 3\}$.
The Kuhn- Tucker conditions at X_2 for the corresponding weighting problem take the form:

$$w\lambda + 5(1-w) + u_2 - u_3 = 0,$$
$$3w - \lambda(1-w) + u_2 = 0,$$
$$u_2 \geq 0,$$
$$u_3 \geq 0.$$

From which, we get:

$$A(X_2) = \{(w, \lambda) \in R^2 \mid \lambda \geq \frac{(3w)}{(1-w)},\ \lambda \geq 8w-5,\ 0 < w < 1\}. \quad (19)$$

Therefore,

$$S(X_2) = \{\lambda \in R \mid (w, \lambda) \in A(X_2)\}. \quad (20)$$

From Figure (5), it is clear that

$$S(X_2) = \{\lambda \in R \mid \lambda > 0\}. \quad (21)$$

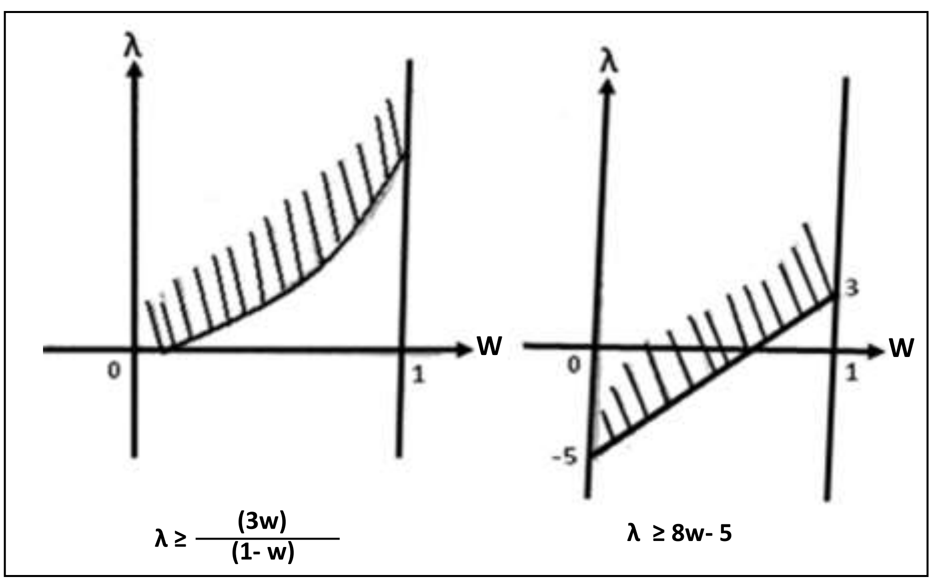

Figure (5): The Relation Between λ and W When $X_1 = (0, 0)$, and $X_2 = (0, 3)$.

Then $S_C(1) = S(X_1) \cap S(X_2) = \{ \lambda \in R \mid \lambda > 0\}$. **(22)**

3) Take $\lambda = -1$, the problem will have two efficient extreme points $X_1 = (0, 0)$, $X_3 = (4, 0)$, (see Figure (6)).

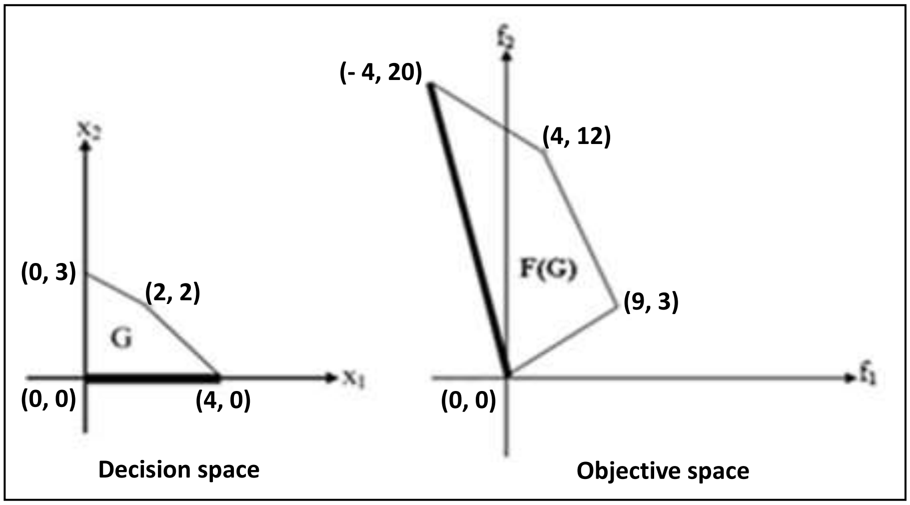

Figure (6): G and F(G) When λ = - 1.

It is clear that $I(X_3) = \{1, 4\}$. The Kuhn- Tucker conditions at X_3 for the corresponding weighting problem take the form:

$w\lambda + 5(1-w) + u_1 = 0,$
$3w - \lambda(1-w) + u_1 - u_4 = 0,$
$u_1 \geq 0,$
$u_4 \geq 0.$

From which, we get:

$$A(X_3) = \{(w, \lambda) \in R^2 \mid \lambda \leq \frac{-5(1-w)}{(w)},\ \lambda \leq 8w-5,\ 0 < w < 1\}. \quad (23)$$

Therefore,

$$S(X_3) = \{ \lambda \in R \mid (w, \lambda) \in A(X_3) \}. \quad (24)$$

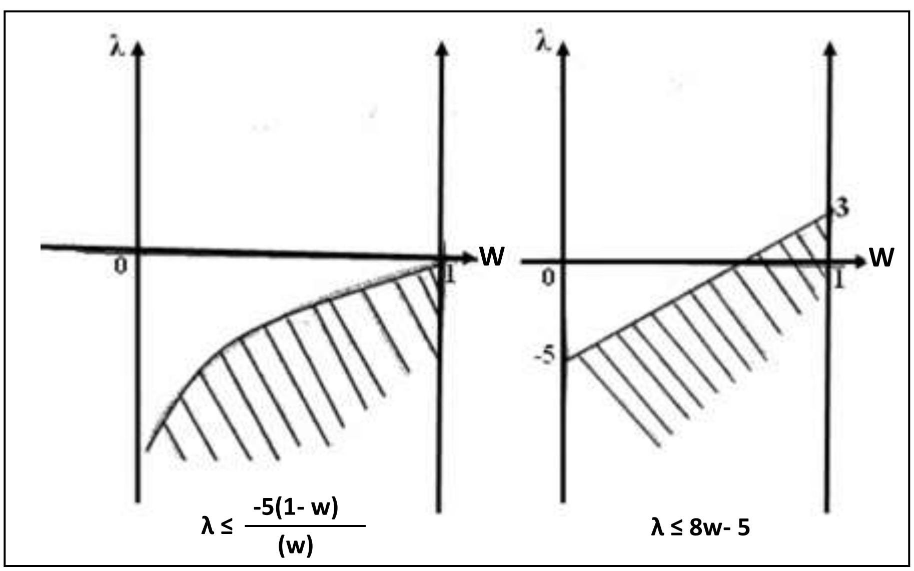

Figure (7): The Relation Between λ and W When $X_1 = (0, 0)$, and $X_2 = (4, 0)$.

From Figure (7) , it is clear that

$S(X_3) =\{ \lambda \in R \mid \lambda < 0\}$. **(25)**

Then $S_C(-1) = S(X_1) \cap S(X_3) = \{ \lambda \in R , \lambda < 0\}$. **(26)**

A summary of these results are shown in the following figure.

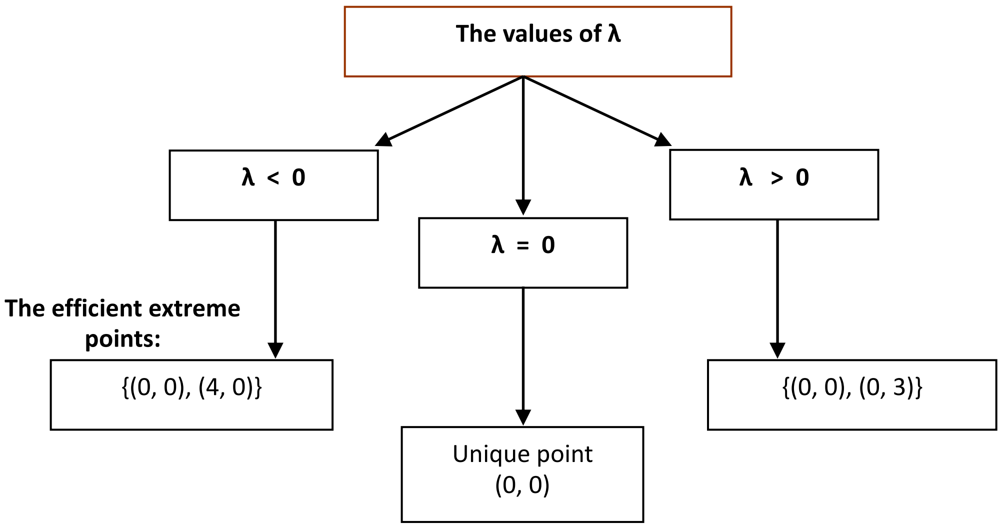

Figure (8): The Decomposition of The Parametric Space of The Example.

5- Conclusion

In this paper, the efficient set is studied for the parametric multi- objective linear programming problems. The aim was not only to measure and evaluate the solutions but also to determine the complete stability efficient sets of the first kind for such problems based on two new algorithms.

The problem was treated for general parameters and many special cases can be derived from this general structure. The complexity of determining the complete stability efficient set of the first kind of this problem is depended on the structure nature of the parameters.

This work could be generalized for parametric multi- objective linear programming problems with parameters in constraints, and quadratic programming problems.

Finally, this study is essentially a new trend towards a good decision for multi- objective linear programming problems.

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