

On construction of magic rectangles

S. M. Elzadidi
Department of Mathematics
Faculty of Science
University of Benghazi
P.O. Box 9480
Benghazi, Libya

Abstract

In this paper we give a method of constructing magic rectangles of sizes $2m \times 2n$ ($n, m > 1$). We have illustrated the $2 \times 2n$ case for n odd and n even.

Introduction

A magic rectangle is an array of, say r rows and c columns containing all integers from 1 to rc , such that the row sums are all the same and the column sums are all the same.

To construct the $2m \times 2n$ magic rectangles we follow the following steps:

1. Write out a 2 row matrix with $a_{1c} = C$ and $a_{2c} = 2n + 1 - C$ as following

$$\left(\begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 \end{array} \right), \left(\begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 20 & 19 & 18 & 17 & 16 & 15 & 14 & 13 \end{array} \right)$$

2. Quarter the matrix by partitioning down the middle and then down the 'Quarter marks' (as shown).

3. Swap top & bottom rows in the central sections

$$\left(\begin{array}{cccccc|cc} 1 & 2 & 14 & 13 & 12 & 11 & 7 & 8 \\ 16 & 15 & 3 & 4 & 5 & 6 & 10 & 9 \end{array} \right), \left(\begin{array}{cccccc|cc} 1 & 2 & 18 & 17 & 16 & 15 & 14 & 13 \\ 20 & 19 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \right)$$

4. If n is odd swap top & bottom in columns n and $2n$

$$\left(\begin{array}{cccccc|cc} 1 & 2 & 18 & 17 & 5 & 15 & 14 & 13 \\ 20 & 19 & 3 & 4 & 16 & 6 & 7 & 8 \end{array} \right)$$

You now have magic rectangles (This can be proved for general n).

To make $2m \times 2n$ rectangles add matrices of the form $A = (a_{rc})$ where

$$a_{rc} = \begin{cases} 4n \left(\frac{r-1}{2} \right) & \text{if } r \text{ odd, } 1 \leq c \leq n \\ 4n \left(m - \frac{r+1}{2} \right) & \text{if } r \text{ odd, } n+1 \leq c \leq 2n \\ 4n \left(m - \frac{r}{2} \right) & \text{if } r \text{ even, } 1 \leq c \leq n \\ 4n \left(\frac{r}{2} - 1 \right) & \text{if } r \text{ even, } n+1 \leq c \leq 2n \end{cases}$$

to the $2m \times 2n$ matrix M formed by repeating the $2 \times 2n$ matrix m times as illustrated by the following examples

To construct 6×8 we have the following

$$\begin{pmatrix} 1 & 2 & 14 & 13 & 12 & 11 & 7 & 8 \\ 16 & 15 & 3 & 4 & 5 & 6 & 10 & 9 \\ 1 & 2 & 14 & 13 & 12 & 11 & 7 & 8 \\ 16 & 15 & 3 & 4 & 5 & 6 & 10 & 9 \\ 1 & 2 & 14 & 13 & 12 & 11 & 7 & 8 \\ 16 & 15 & 3 & 4 & 5 & 6 & 10 & 9 \end{pmatrix} + 16 \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 14 & 13 & 44 & 43 & 39 & 40 \\ 48 & 47 & 35 & 36 & 5 & 6 & 10 & 9 \\ 17 & 18 & 30 & 29 & 28 & 27 & 23 & 24 \\ 32 & 31 & 19 & 20 & 21 & 22 & 26 & 25 \\ 33 & 34 & 46 & 45 & 12 & 11 & 7 & 8 \\ 16 & 15 & 3 & 4 & 37 & 38 & 42 & 41 \end{pmatrix}$$

which is a magic rectangle of constant sums 196 & 147.

To construct a magic rectangle of 6×10 we have the following

$$\begin{pmatrix} 1 & 2 & 18 & 17 & 5 & 15 & 14 & 13 & 9 & 11 \\ 20 & 19 & 3 & 4 & 16 & 6 & 7 & 8 & 12 & 10 \\ 1 & 2 & 18 & 17 & 5 & 15 & 14 & 13 & 9 & 11 \\ 20 & 19 & 3 & 4 & 16 & 6 & 7 & 8 & 12 & 10 \\ 1 & 2 & 18 & 17 & 5 & 15 & 14 & 13 & 9 & 11 \\ 20 & 19 & 3 & 4 & 16 & 6 & 7 & 8 & 12 & 10 \end{pmatrix} + 20 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 18 & 17 & 5 & 55 & 54 & 53 & 49 & 51 \\ 60 & 59 & 43 & 44 & 56 & 6 & 7 & 8 & 12 & 10 \\ 21 & 22 & 38 & 37 & 25 & 35 & 34 & 33 & 29 & 31 \\ 40 & 39 & 23 & 24 & 36 & 26 & 27 & 28 & 32 & 30 \\ 41 & 42 & 58 & 57 & 45 & 15 & 14 & 13 & 9 & 11 \\ 20 & 19 & 3 & 4 & 16 & 46 & 47 & 48 & 52 & 50 \end{pmatrix}$$

which is a magic rectangle of constant sums 305 & 183.

To prove the process always generates distinct entries, writing $B = M + A$. If $b_{ij} = b_{kl}$ then $M_{ij} \equiv m_{kl} \pmod{4n}$ since $a_{rc} \equiv 0 \pmod{4n}$. But then $M_{ij} = M_{kl}$ since all entries are less than or equal to $4n$ & non-zero. By the construction of M , $M_{ij} = M_{kl} \Rightarrow j = l$ and $i \equiv k \pmod{2}$. By construction of A , $a_{ij} = a_{kj}$ only if $i = k$ thus $j = l$ and so $b_{ij} = b_{kl}$ only if $j = l$ and $i = k$.

References

- Bier, T. and Kleinschmidt, A. (1997). Centrally symmetric and magic rectangles. *Discrete Math.*
- Hagedorn, T. (1999). Magic rectangles revisited. *Discrete Math.*



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