Bohr Radius Extensions for Bounded Linear Operators on Infinite-Dimensional Space

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Abstract

This manuscript investigates the concept of the operator Bohr radius within the framework of functional analysis, particularly in the context of bounded linear operators on Banach and Hilbert spaces. We extend classical Bohr-type inequalities to operator-valued analytic functions and derive bounds for various operator classes including nuclear and Hilbert–Schmidt operators. The study culminates in a sharp characterization using the unilateral shift operator on $\ell^2(\mathbb{N})$.

1 Introduction

The classical Bohr radius problem, originating from Harald Bohr's work in complex analysis, seeks the largest radius r_0 such that the sum of the moduli of coefficients of a bounded analytic function remains bounded by 1 within the unit disk. This concept has been extended to operator-valued functions, where the coefficients are bounded linear operators on a Banach or Hilbert space.

2 Background

This work started the journey, Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic in the unit disk with $|f(z)| < 1$.[5]

Bohr's inequality states that

$$\sum_{n=0}^{\infty} |a_n| \, |z|^n \leq 1 \quad \text{for } |z| \leq \tfrac{1}{3}.$$

The constant $\frac{1}{3}$ is known as the *Bohr radius*[7][6]

3 Definition

Linear Operator
A **linear operator** $T : V \to W$ between vector spaces V and W over the same field satisfies:

$$T(u+v) = T(u) + T(v), \quad T(\alpha v) = \alpha T(v)$$

for all $u, v \in V$ and scalars α[1][8]

Bounded Operator
A **bounded operator** is a linear operator $T : X \to Y$ between normed spaces such that there exists $C > 0$ with

$$\|T(x)\| \leq C\|x\| \quad \forall x \in X.$$

The smallest such C is called the operator norm of T. [9]

Dual Space
The **dual space** X^* of a normed space X is the set of all bounded linear functionals $f : X \to \mathbb{R}$ (or $\mathbb{C}$). [2].

Continuity
An operator $T : X \to Y$ is **continuous** if for every sequence $x_n \to x$ in X, we have $T(x_n) \to T(x)$ in Y. For linear operators between normed spaces, continuity is equivalent to boundedness. [10]

Banach Space
A **Banach space** is a complete normed vector space, meaning every Cauchy sequence in the space converges to a limit within the space [1][3].

4 Main results

Let X be a Banach (or Hilbert) [5] space and $L(X)$ the space of bounded linear operators on X. [4] For an operator-valued analytic function

$$F(z) = \sum_{n=0}^{\infty} A_n z^n, \quad A_n \in L(X), \quad |F(z)| \leq 1 \text{ for } |z| < 1,$$

the operator Bohr radius R is the largest $r_0 > 0$ such that

$$\sum_{n=0}^{\infty} |A_n| r^n \leq 1 \quad \text{for all } 0 \leq r < r_0.$$

5 Radius of Convergence

Theorem 5.1. *Let R_{conv} be the radius of convergence of $\sum_{n=0}^{\infty} A_n z^n$ in operator norm. Then*

$$R_{\text{conv}} = \frac{1}{\limsup_{n\to\infty} |A_n|^{1/n}},$$

and any Bohr radius r_0 must satisfy $r_0 \leq R_{\text{conv}}$.

6 Special Operator Classes

6.1 Nuclear Operators

Proposition 6.1. *If A_n are nuclear (trace-class) operators with nuclear norm $|\cdot|_1$, then*

$$r_0 \leq \frac{1}{3\sup_{n\geq 0} |A_n|_1}.$$

6.2 Hilbert–Schmidt Operators

Proposition 6.2. *If A_n are Hilbert–Schmidt operators with norm $|A_n|_2$, then*

$$r_0 \leq \frac{1}{\sup_{n\geq 0} |A_n|}, \quad \textit{and} \quad R < \frac{1}{3\sup_{n\geq 0} |A_n|_2}.$$

7 Sharp Application: Unilateral Shift

Theorem 7.1. *Let M be the unilateral shift on $\ell^2(\mathbb{N})$, defined by $M(e_n) = e_{n+1}$. Consider*

$$F(z) = \sum_{n=0}^{\infty} M^n z^n, \quad |M^n| = 1.$$

Then

$$\sum_{n=0}^{\infty} |M^n| r^n = \frac{1}{1-r} \leq 1 \quad \Longleftrightarrow \quad r \leq \tfrac{1}{3}.$$

Hence, the operator Bohr radius coincides with the classical scalar value $1/3$.

8 Conclusion

This study establishes foundational bounds for the operator Bohr radius across various operator classes and highlights the sharpness of the classical bound in the operator setting. Future work may explore analogues and connections with operator-valued Hardy spaces.

9 References

References

1. Conway, J. B. (2000). Conway 2000 *A Course in Functional Analysis* (2nd ed.). Springer.

2. Rudin, W. (1991). *Functional Analysis* (2nd ed.). McGraw–Hill.

3. Halder, H. (2025). Local Banach space theoretic approach to Bohr's theorem for vector valued holomorphic and pluriharmonic functions. arXiv preprint arXiv:2512.09091. https://doi.org/10.48550/arXiv.2512.09091

4. Arora, V., Kumar, S. and Ponnusamy, S. (2025). Asymptotic value of the multidimensional Bohr radius. Monatshefte für Mathematik, 207(2), 365–384. https://doi.org/10.1007/s00605-025-01876-2

5. Boas H. P. and Khavinson D. (1997). Bohr's power series theorem in several variables, Proc. Amer. Math. Soc. 125 (1997), no. 10, 2975ˆa2979.

6. Defant A. , Frerick L. , J. ‘a, Ortega-Cerd M. Ouna¨ıes, and Seip K. , (2012) Bohr's inequality for complex Banach spaces, Journal of Functional Analysis 263 (2012), no. 5, 1199ˆa1233.

7. Allu V. and Pal S. , Arithmetic Bohr Radius of Bounded Linear Operators, arXiv:2410.18620, 2024.

8. Dineen S.(2012) , Complex Analysis on Infinite Dimensional Spaces, Springer-Verlag

9. Curgus, B.(2024) The Definition and Vector Space of Linear Operators. Western Washington University, 2024.

10. Kreyszig, E.(1989), *Introductory Functional Analysis with Applications*, Wiley, 1989.