Global Scientific JOURNALS

GSJ: Volume 10, Issue 2, February 2022, Online: ISSN 2320-9186
www.globalscientificjournal.com

REPRESENTATION OF THE THREE CUBE ROOTS OF UNITY AS THE VERTICES OF AN EQUILATERAL TRIANGLE

Parimal Mridha[1], Bint E Sayera Ruhu[2], Jaba Rani Sana[3]

[1]*(Lecturer in Mathematics, Military Collegiate School Khulna (MCSK), Bangladesh).*

[2]*Lecturer in Mathematics, Military Collegiate School Khulna (MCSK), Bangladesh.*

[3]*(Assistant teacher, 155 no. Mothbari Kalitola Govt. Primary School, Dumuria, Khulna, Bangladesh).*

Abstract: A complex number is a number of the form $a + ib$, where a and b are real numbers and i is a square root of -1, that is, i satisfies the quadratic equation $i^2 + 1 = 0$. Gauss thought of a complex number $z = a + ib$ geometrically as a point (a, b) in the real two-dimensional space. This represents the set **C** of complex numbers as a real two-dimensional plane, called the complex plane. The x-axis is called the real axis and the real number a is called the real part of z. The y-axis is called the imaginary axis and the real number b is called the imaginary part of z. Cube root can also be denoted in index form as numbers raised to the power 1/3. The cube root of unity meaning is the cube root of '1'. There are three values of the cube root of unity. In case of cube root of unity there are two complex values and one is real value. In this paper we will try to show that the representation of this three-cube root of unity is an equilateral triangle.

Keywords: Complex number, Imaginary number, Cube root, Equilateral triangle.

Representation of the three cube roots of unity as the vertices of an equilateral triangle

1. Introduction

A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is an indeterminate satisfying $i^2 = -1$. For example, $2 + 3i$ is a complex number. This way, a complex number is defined as a polynomial with real coefficients in the single indeterminate i, for which the relation $i^2 + 1 = 0$ is imposed. Based on this definition, complex numbers can be added and multiplied, using the addition and multiplication for polynomials. The relation $i^2 + 1 = 0$ induces the equalities $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, and $i^{4k+3} = -i$, which hold for all integers k; these allow the reduction of any polynomial that results from the addition and multiplication of complex numbers to a linear polynomial in i, again of the form $a + bi$ with real coefficients a, b. The real number a is called the *real part* of the complex number $a + bi$; the real number b is called its *imaginary part*. To emphasize, the imaginary part does not include a factor i; that is, the imaginary part is b, not bi. Formally, the complex numbers are defined as the quotient ring of the polynomial ring in the indeterminate i, by the ideal generated by the polynomial $i^2 + 1$. The polar form of a complex number is another form of representing and identifying a complex number in the argand plane. The polar form makes the use of the modulus and argument of a complex number, to represent the complex number. The complex number z = a + ib, can be represented in polar form as z = r(Cosθ + iSinθ). Here r is the modulus (r = sqrt{a^2 + n^2}\), and θ is the argument of the complex number(θ = Tan^{-1} b/a).

2. Cube Root of 1

Whenever a number (x) is multiplied three times, then the resultant number is known as the cube of that number. Thus, the cube for the number (x) becomes x^3 or x-cubed. For example, let us take the number 5. We know that 5 × 5 × 5 = 125. Hence, 125 is called the cube of 5. While on the other hand, the cube root of a number is the reverse process of the cube of a number and is denoted by ∛. Considering the same example, 5 is called the cube root of the number 125. On this page, we will learn more about the cubes and cube roots of a number. When we think about the words cube and root, the first picture that might come to our mind is a literal cube and the roots of a tree. Isn't it? Well, the idea is similar. Root means the primary source or origin. So, we just need to think "cube of which number should be taken to get the given number". In mathematics, the definition of cube root is written as Cube root is the number that needs to be multiplied three times to get the original number. Now, let us look at the cube root formula, where y is the cube root of x. $\sqrt[3]{x} = y$. The radical sign ∛ is used as a cube root symbol for any number with a small 3 written on the top left of the sign. Another way to denote cube root is to write 1/3 as the exponent of a number. Cube root is an inverse operation of the cube of a number. The cube root of 1 is the number which when multiplied by itself three times gives the product as 1. In fact, the n-th root of 1 is always 1. Therefore, the cube root of 1 = $\sqrt[3]{1} = 1$.

A sequence of steps is to be followed to find the cube root of unity.

Step 1:

The cube root of unity is equated to a variable, say 'x'. As $\sqrt[3]{x} = 1$.

Step 2:

Cube and cube root of a number are inverse operations. So, if the cube root is shifted to the other side of the equation, it becomes the cube of the number on the other side. So, $1 = x^3$

Step 3:

Shift '1' also to the other side of the equation. So, the value on LHS will be zero.

$$x^3 - 1 = 0$$

Step 4:

From the algebraic identity of $a^3 - b^3 = (a - b)(a^2 + ab + a^2)$, factorize $x^3 - 1$.

$$(x-1)(x^2 + x + 1) = 0$$

Step 5:

Simplify the factors further to evaluate the value of 'x'.

From the equation in step 4, either $x - 1 = 0$ or $x^2 + x + 1 = 0$.

If $x - 1 = 0$, $x = 1$

$x^2 + x + 1 = 0$ is simplified using the formula method of solving quadratic equations.

According to the formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In the above formula, the general form of a quadratic equation is considered as

$ax^2+bx+c=0$. Comparing the general equation and $x^2 + x + 1 = 0$, a = 1, b = 1, and c = 1.

So, the complex cube roots of unity obtained by solving $x^2 + x + 1 = 0$ are

$$x = \frac{-1 \pm \sqrt{1^2 - 41.1}}{2.1}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

The three values of the cube root of unity are:

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

3. Representation of the three cube roots of unity

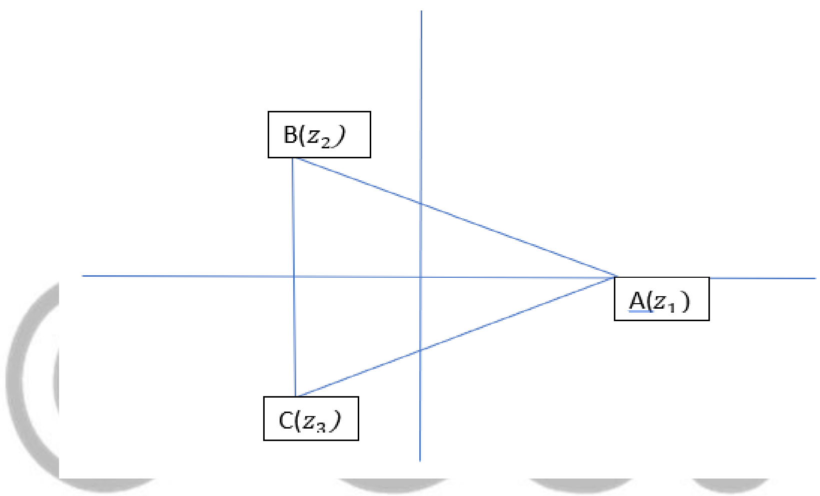
Let, $z_1 = 1$, $z_2 = \frac{-1+i\sqrt{3}}{2}$, $z_3 = \frac{-1-i\sqrt{3}}{2}$

i.e; $z_1 = 1 + 0.i = (1,0) = A(Let)$

$$z_2 = \frac{-1+i\sqrt{3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = -0.5 + \frac{\sqrt{3}}{2}i = \left(-0.5, \frac{\sqrt{3}}{2}\right) = B(Let)$$

$$z_3 = \frac{-1-i\sqrt{3}}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = -0.5 - \frac{\sqrt{3}}{2}i = (-0.5, -\frac{\sqrt{3}}{2}) = C(Let)$$

Now Plot these three points,



Now, coordinate of A is A (1,0), coordinate of B is B (-0.5, $\frac{\sqrt{3}}{2}$), coordinate of C is C (- 0.5,$-\frac{\sqrt{3}}{2}$)

So, $AB = \sqrt{(-0.5-1)^2 + (\frac{\sqrt{3}}{2} - 0)^2} = \sqrt{(-\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{12}{4}} = \sqrt{3}$

$$BC = \sqrt{(-0.5+0.5)^2 + (-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2})^2} = \sqrt{(\frac{2\sqrt{3}}{2})^2} = \sqrt{3}$$

$$CA = \sqrt{(1+0.5)^2 + (0 + \frac{\sqrt{3}}{2})^2} = \sqrt{(\frac{3}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

So, $AB = BC = CA$. So all the sides are equal to each other's. So cube roots of unity can be expressed as the vertices of an equilateral triangle.

4. References:

1. B. Blank, An Imaginary Tale Book Review, in Notices of the AMS Volume 46, Number 10, November 1999, pp. 1233-1236

2. Erwin Kreyszig, Herbert Kreyszig, Edward J. Norminton, “ADVANCED ENGINEERING MATHEMATICS”, 10edition , JOHN WILEY & SONS, INC., 2011
3. George B. Thomas, Jr., “THOMAS CALCULUS” , 13edition , Maurice D. Weir, Joel Hass. c2010
4. Howard Anton, Irl Bivens, Stephen Davis,“CALCULUS EARLY TRANSCENDENTALS”10edtion, JOHN WILEY & SONS, INC, 2012.
5. B. L. van der Waerden, A History of Algebra, Springer Verlag, NY 1985.
6. Narayan, S. and Mittal, P.K. (2005) Theory of Functions of a Complex Variable. S. Chand & Company Ltd., New Delhi.
7. Schneider, C. (2011) De Moivre’s Theorem—A Literature and Curriculum Project on Roots, Powers, and Complex Numbers. Department of Mathematics and Statistics, Portland State University.
8. www.cuemath.com/algebra/cube-root-of-1/
9. https://byjus.com/maths/cube-root-of-unity/
10. www.vedantu.com/maths/cube-root-of-unity