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SOME TOPOLOGICAL PROPERTIES AND STONE-ČECH COMPACTIFICATION OF LINEAR STRONGLY B-CONVERGENT TOPOLOGICAL SPACE OF MADDOX

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Abstract

Topological properties on strongly B - convergent sequence space of Maddox were studied. The strongly B - convergent sequence space of Maddox and its Stone-$\breve{C}$ech compactification, $\beta\gamma$ were characterized as O_z, Quasi - O_z and Baire space. It is shown that $\beta\gamma$ and Borel set G_δ in $\beta\gamma$ are Baire spaces. Finally, the Stone $\breve{C}$ech compactification of linear strongly B- convergent sequence space of Maddox were characterized as βA compactification by continuum hypothesis.

Key Words:

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INTRODUCTION

The classical topological spaces of all bounded, convergent, null, absolute p-summable sequence and space of finite sequences were denoted by l_∞, c, c_o, $l_p$andc_{oo} . These spaces are metric spaces and they are in sense the simplest of all metric spaces and their structures have been well understood for many years back. Some topological notions such as; separability, continuity, compactness, were studied for these classical sequences spaces. Some of these spaces were extended to strongly B-convergent to zero, strongly B-convergent and strongly B-bounded sequences spaces of Maddox. As a result of such extensions, recent works in [(Aydin and Basar, 2014), (Malkowsky and Basra, 2017), and (Eloi and Micheal, 2017)] revealed the study of some topological properties. Motivated in this line, this paper studies separability, continuity, compactness, complete regularity and Stone-Čech compactification of the extended sequence spaces of Maddox and report where some of these properties fail to hold.

Notations

The following notations would be found necessary in the next discussions:

i. $c_o(p) = \{x = x_k: |x_k|^{p_k} \to o \; as \; k \to \infty\}, and$

ii. $c(p) = \{x = x_k: |x_k|^{p_k} \to o \; as \; k \to \infty \; for \; some \; l \in C\}.$

iii. $l_\infty(p) = \{x = \{x_k\}: \sup_k |x_k|^p k < \infty\}$

iv. $l(p) = \{x = \{x_k\}: \sum_{k=1}^{\infty} |x_k|^p k < \infty\}$

See (Nakano, 1951), (Simons, 1965) and by (Maddox 1977).

Also, we list the fundamental properties of Maddox's sequence spaces $[B,p]_o$, $[B,p]$ and $[B,p]_\infty$. The definition of the spaces here can be found in (Malkowsky, 2017), special cases were studied, for instance, in (Velickovic, 2011).

Let $p = (p_k)_{k=1}^{\infty}$ be a sequence of positive reals, $x \in w \; and \; B = \{b_{nk}\}$. Then we write $|x|^p = \{|x_k|^p k\}_{K=1}^{\infty}$, $B_n(|x|^p) = \sum_{k=1}^{\infty} b_{nk} |x_k|^{p_k}$ for $n \in N$, and $B(|x|^p) = (B_n|x|^p)_{n=1}^{\infty}$, provided that all the series converges, and $[B,p]_o = \{x \in w: B(|x|^p) \in c_o\}$, $[B, p] = \{x \in w: B(|x - te|^p) \in c_o \; for some \; t \in \mathbb{C}\} \; and \; [B, p]_\infty = \{x \in w: B(|x|^p) \in l_\infty\}$, are the set of sequences that are strongly B-convergent to zero, strongly B-convergent, and strongly B-bonded. If $x \in [B, p], then \; t \in C \; with \; B(|x - te|^p) \in c_o$ is referred to as a strong B-limit, or [B,p]-limit, of the sequence x. We write $x_k \to t[B, p] \; if \; \lim_{n\to\infty} B_n\{|x - te|^p\} = 0$, condition for the uniqueness of the [B, p] – limits of the sequences in [B, p] and of convergent sequences were given in (Cafer, 2014) for a certain class of matrices B. Let A denotes the class of all infinite matrix $A = \{a_{nk}\}_{n,k=1}^{\infty}$ for which there exist a positive integer M such that

i. $a_{nk} \geq o \; for \; each \; n \geq 1 \; and \; for \; each \; k > M$

ii. $|a_{nk}| - a_{nk} \to o(n \to \infty; 1 \leq k \leq M).$

It is clear from the definition of the sets

$[B,p]_o, [B, p] \; and \; [B,p]_\infty, that \; [B, p]_o \subset [B, p] \; and \; [B, p]_o \subset [B, p]_\infty, but [B, p]$ is not included in $[B, p]_\infty$. In general, Maddox's sets are obtained as a special case of the above sets as follows;

$If\ B = I, [I, p]_o = c_o(p) = \{x \in w\colon \lim_{k\to\infty} |x_k|^{p_k} = 0\},$

$[I,p] = c(p) = \left\{x \in w\colon \lim_{k\to\infty} |x_k - t|^{p_k} = o\ for\ some\ t \in \mathbb{C}\right\},$

$[I, p]_\infty = l_\infty(p) = \left\{x \in w\colon \sup_k |x_k|^{p_k} < \infty\right\},$

$B = C_1\colon \left[C_1,p\right]_o = w_o(p) = \left\{x \in w\colon \lim_{n\to\infty} (\frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}\right) = 0\Big\},$

$[C_1, p] = w(p) = \{x \in w\colon \lim_{n\to\infty} (\frac{1}{n}\sum_{k=1}^{n} |x_k - t|^{p_k} = o\ for\ some\ t \in \mathbb{C}\},$

$[C_1, p]_\infty = w_\infty(p) = \{x \in w\colon \sup_n(\frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k}) < \infty\},$

$B = [E, p] : [E,p]_\infty = l(p) = \{x \in w\colon \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\}.$

where w denotes the vector spaces of all complex valued sequences.

RESULTS

The following results clearly show the existence and uniqueness of Stone $\breve{C}$ ech Compactification of linear strongly B- convergence sequence space of Maddox. The result is divided into three parts , part A compare some topological properties on strongly B- convergent sequence space of Maddox, part B characterize the space and its Stone $\breve{C}$ech compactification as O_z , Quasi$-$ O_z and Baire space. It is shown that the Borel set G_δ in $\beta\gamma$ is a Baire space. Lastly part C characterizes the *Stone*-Čech compactification $\beta\gamma$ of the Maddox space γ as GA compactification.

A. *Comparative study of some topological properties on Maddox space* **Y** *of strongly B – convergent and its Stone*-**Čech Compactification βY**

Theorem 4.1: suppose X and Y are Maddox spaces of Natural and Rational components repectively. Then, βY is continuous image of βX.

Proof: Let X be Maddox space of Natural components and Y be Maddox space of Rational components. Then, Y is completely regular. Let denotes the Stone-**Č**ech compactification of $Y by \beta Y$. Since component of Y is not connected, implies that βY is connected. Also, since Y is not locally compact, then Y is not subspace of βY. Hence X and Y have the same cardinality, i.e

$|X| = |Y|$, therefore, ∃ a bijection from X to $Y \subset \beta Y$. Therefore X is a discrete space, certainly, g is continuous. Therefore, ∃ a continuous extension $\widehat{g}$ of g from βX to βY. Therefore, $\widehat{g}(\beta X)$ is compact in βY , and also closed in βY .We note that $Y \subset \widehat{g}(\beta X)$, therefore, it follows that$\widehat{g}(\beta X) = \beta Y$,which shows that βY is a continuous image of βX.

Theorem 4.2: Suppose βY is a Stone-Čech compactification of a space Y with the discrete topology. Then Y is extremely disconnected.

Proof: Suppose $A = O \cap Y$, where O is an open subset of Y and, if $\exists\, U_A \ni F \subset U_A$, then, every U_B of F must have an intersection point with an open set O, since A and B are not disjoint, then $F \subset Cl(0)$.

Conversely, if $\exists\, M \; and \; U_{Y \setminus A} \cap O = \emptyset$, then $Cl(0) = U_A$. Therefore, βY is extremely disconnected provided that $(0) = U_A$, then we have $Cl(U_A) = U_A$, so βY is zero dimensional. In fact, if $Y = A \cup B,\ A \cap B = \emptyset$, then $U_A \cup U_B = \beta Y$ and $U_A \cap U_B = \emptyset$.

Theorem 4.3: Suppose $A \subset Y$ is a subspace of topological space Y, then every bounded continuous real-valued function g on A has a bounded continuous extension toβY.

Proof: If A is a subspace of Y and g is a bounded continuous real-valued function on A and suppose ĝ is a bonded continuous extension of g on Y, then by the construction of βY, there exist a bounded continuous extension $\hat{g}_1$ of ĝ defined on βY. It is not difficult to see that $\hat{g}_1$ is a bounded continuous extension of g defined on βY.

Conversely, suppose that g has a bounded continuous extension ɵ defined on βY, then the restriction of ɵ on Y is represented by g_1.Therefore it is observed that g_1 is a bounded continuous extension of g on Y.

Theorem 4.4: Every bounded continuous real-valued function on the subspace $A \subset Y$ can be extended to a bounded continuous real-valued function defined on Y iff $Cl_{\beta Y}A = \beta A$.

Proof: Suppose g is every bounded continuous real-valued function on A and has a bounded continuous extension ĝ on Y. Hence, g must have a bounded continuous extension $\hat{g}_1$on βY. This is observed that g has a bounded continuous extension defined on $Cl_{\beta Y}A$, which is unique. Since

$Cl\,A = Cl_{\beta Y}A$ and $Cl_{\beta Y}A$ is compact Hausdorff space, then $Cl_{\beta Y}A = \beta A$. Conversely, suppose $Cl_{\beta Y}A = \beta A$. Then, every bounded continuous real-valued function g on A has a unique bounded continuous extension ĝ defined on βA, and also on $Cl_{\beta Y}A$. By Tietze's extension theorem ĝ has a bounded continuous real-valued extension $\hat{g}_1$ defined on βY. By **theorem 4.3** above, it follows that every bounded continuous real-valued function A has a bounded continuous extension on Y.

Corollary 4.1: Suppose $A \subset Y$ is a compact subset of Y, then every bounded continuous real-valued function on A can be extended to a bounded continuous real-valued function on Y.

Proof: Let A be a compact subset of Y, then $Cl_{\beta Y}A = A$ and $\beta A = A$. Therefore, $Cl_{\beta Y}A = \beta A$ and from **theorem 4.4** above, every bounded continuous real-valued function on A has a bounded continuous extension on Y.

Theorem 4.5: If $A \subset Y$ is clopen subset of Y, then $Cl_{\beta Y}A$ and $Cl_{\beta Y}Y \setminus A$ are disjoint complementary open subset of βY.

Proof: Suppose A is clopen subset of Y, then $Y \setminus A$ is clopen in Y. So, there exists a bounded continuous real-valued function g on Y such that $g(A) = 0$ and $g(Y \setminus A) = 1$. Indicating that every bounded continuous real-valued function on Y has a unique bounded continuous extension on βY, and if g is the unique bounded continuous extension of g,

then $g(A) = 0$ and $g\big(Cl_{\beta Y}(Y \setminus A)\big) = 1$. Provided that $Cl\,Y = \beta Y$, implies that $Cl_{\beta Y}A \cup Cl_{\beta Y}Y \setminus A = \beta Y$ **(4.1)**

Therefore from (4.1) above, we see that $Cl_{\beta Y}A$ and $Cl_{\beta Y}Y \setminus A$ are disjoint complementary open subsets of βY.

Theorem 4.6: Suppose $x \in Y$ is an isolated point of Maddox topological space Y. Then x is an isolated point of βY.

Proof: Let x be an isolated point of Y and since Y is T_1, $\{\ x\ \}$ is clopen in Y. Then by **Theorem (4.5) above**, $Cl_{\beta Y}\{x\}$ is clopen in βY. However, since βY is Hausdorff, then $Cl_{\beta Y}\{x\} = \{x\}$. Therefore, x is an isolated point of βY.

Theorem 4.7: If Stone-Čech compactification βY of topological space Y is connected, then Y is connected.

Proof: Suppose $\beta Y$is connected. Let M be a proper subset of $Y \ni M$ is clopen in Y. By **Theorem (4.5) above**, $Cl_{\beta Y}M$ is clopen in $\beta Y$and $Cl_{\beta Y}(Y \setminus M) \cap Cl_{\beta Y}M = \emptyset$. It then follows that $Cl_{\beta Y}M \subset \beta Y$, which contradict the fact that βY is connected. Therefore Y is certainly connected.

Conversely, suppose Y is connected. Let $A \subset \beta Y \; and \; A$ be clopen, and hence $\beta Y \setminus A$ is clopen. Since $Cl\, Y = \beta Y$ and it follows that both $\beta Y \setminus A$ and A contains $x \in Y$. Thus $A \cap Y \subset Y$ which is clopen in Y. This leads to contradiction of the hypothesis that Y is connected. Hence βY is connected.

Theorem 4.8:$Y \subset \beta Y$ is open subset of βY iff Y is locally compact.

Proof: Suppose Y is locally compact. If $x \in Y$ and A is a compact neighborhood of $x \in Y$. Hence,$\exists$ an open set $B \in Y \ni x \in B \subset A$. So, A is a compact subset of βY and hence is closed, provided βY is Hausdorff. Sice $B \subset Y$ and open in Y, $\exists$ an open set $M \in \beta Y \ni M \cap Y = B \subset A$. It is clear that $Cl_{\beta Y}M \cap Y \subset Cl_{\beta Y}M \cap Cl_{\beta Y}Y = Cl_{\beta Y}M$. If $\exists x \in Cl_{\beta Y}M$, then every open set $U \in \beta Y$ which contains x, also contains a point of M. Hence, $U \cap (M \cap Y) = (U \cap M) \cap Y$ where $U \cap M \neq \emptyset$ and open. But since $Cl\, Y = \beta Y$, every nonempty open set has a non-empty intersection with Y. Hence, $U \cap (M \cap Y) \neq \emptyset$ and it follows that $x \in Cl_{\beta Y}(M \cap Y)$ and $Cl_{\beta Y}M \cap Y = Cl_{\beta Y}M$. Therefore,$M \subset Cl_{\beta Y}M = Cl_{\beta Y}M \cap Y \subset A \subset Y$ and Y has to be open.

Conversely, if Y is open in $\beta Y$and βY is a compact Hausdorff space, then βY is regular. If $t \in \beta Y$, then every neighborhood of t contains a closed neighborhood of x, Y, being open, is a neighbourhood for each $t \in Y$. Therefore, each $t \in Y$ has a compact neighborhood in Y, since closed sets are compact in βY. So, Y is locally compact.

Theorem 4.9: If Y is a discrete Maddox metric space and βY the Stone-Čech compactification of Y, then βY is not a metric space.

Proof: Suppose βY*is a metric space. Let* $\Theta(Y)$ denote the image of $Y \subset \beta Y$. Since Y is discrete, then is not compact, so $\exists$ a sequence $M = \{x_n\}_{n \in N}$ in Y which has no convergence subsequence $\{y_{n_k}\} \subset \{y_n\}$. Let $y_{n_k} \to y \in \beta Y$ and consider the points $x_n^1 = \Theta^{-1}(y_{n_k})$. From $M_1 = \{x_n^1\} \subset M$, the sequence M_1 contains no convergent subsequence. For this $M_1 \subset Y$ is a closed subset of Y. Now, we define the real function $g{:}M_1 \to J = [0,1]$ by $g(x_n^1) = \{^{0,\ for\, n=2l}_{1,\ for\, n=2l-1}\ for\ l \in N.$

Since M_1 contains no convergent subsequence, the function $g{:}M_1 \to J$ is continuous and since $M_1 \subset Y$ is a closed subset of discrete metric space Y, by using Tietze's extension theorem, we can extend the function, to a continuous function $\hat{g}{:}M_1 \to J$. Now, the function ge^{-1} has a continuous extension ĝ on the space βY. Since $\hat{g}(y_{n_k}) = ge^{-1}(y_{n_k}) = g(y_{n_k}) = \{^{0,\ for\, n=2l}_{1,\ for\, n=2l-1}$ and $y_{n_k} \to y$, for this, the function ĝ cannot be continuous at the point y, This is a contradiction, which shows that βY is not a metric space.

B. ***The characterizations of the Stone-Čech compactification of the linear strongly B-converrgent sequence space of Maddox as Oz space, Qausi-Oz space and Baire Space.***

LEMMA 4.1: (Blair R. L. 1976). A space Y is an Oz-space iff every regular closed subset of Y is a zero-set in Y.

LEMMA 4.2: (Rudd D. 1975). For a zero-set M of a space Y, the following properties are satisfied.

i. $Cl_{\beta Y}M$ is a zero-set of βY
ii. ∃ a real-valued continuous function f on Y which satisfies the following properties
 a. $M = f^{-1}(0)$
 b. if a subset K of Y is completely separated from M, then $\inf\{f(x){:}x \in k\} > 0$

Theorem 4.10: For an Oz-space Y, the following are satisfied

a. βY is Oz
b. For each regular closed subset K of Y, ∃ a sequence $\{U_i{:}i < w\}$ of regular open subsets of Y which satisfies the following properties
 i. $K \subset U_i\ for\ each\ i < w$
 ii. For any regular open subset U of Y contains in K, ∃ some $U_i \ni U_i \subset U$

Proof: $(a) \Longrightarrow (b)$. Suppose K is a regular closed subset of Y. $Cl_{\beta Y}K$ is a regular closed subset of βY. Hence $Cl_{\beta Y}K$ has a countable neighborhood basis $\{V_i : i < w\}$ consisting of regular open subsets of βY, since βY is a compact Oz-space, for each $i < w, let\ U_i = V_i \cap Y$. Then, we shall show that the sequence $\{U_i : i < w\}$ has the properties (i) and (ii) above. (i) is obviously satisfied. (ii) Let U be a regular open subset of Ycontaining K, then K and $Y \setminus U$ are completely separated provided that K and $Y \setminus U$ are regular closed subsets of an Oz-space Y. Hence $Cl_{\beta Y}A \subset U^{\beta}$. Therefore, for some i, $Cl_{\beta Y}A \subset V_i \subset U^{\beta}$. Thus $U_i \subset U$ for some$i$. Hence **(ii)** is satisfied.

$(b) \Longrightarrow (a)$. Suppose N is a regular closed subset of βY. Then $K = N \cap Y$ is a regular closed subset of Y. Hence $\exists$ a sequence $\{U_i : i < w\}$ of regular open subsets of Y which satisfies the properties (i) and (ii) above. Then it is obvious that $Cl_{\beta Y}K = N = \bigcap\{U_i^{\beta} : i < w\}$. Hence N is a zero-set of βY, since βY is normal.

Corrollary 4.2: For a normal space Y, the following are equivalent;

i. βY is Oz

ii. Every regular closed subset of Y has a countable neighborhood basis.

Theorem 4.11: Suppose βY is Oz, then for any regular closed subset $X$of Y, $\partial_Y X$ is relatively pseudocompact.

Proof: Let X be a regular closed subset of Y. Suppose that $\partial_Y X$ is not relatively pseudocompact. Then, we shall show that the condition (b) in the **theorem 4.10** above is not satisfied. To do this, let $\{U_i : i < w\}$ be a sequence of regular open subsets of Ycontaining$X$. Since $\partial_Y X$ is not relatively pseudocompact, then $Cl_{\beta Y}(\partial_Y X) \cap (\beta Y \setminus \upsilon Y)$ is nonempty. Suppose x is a point of $Cl_{\beta Y}(\partial_Y X) \cap (\beta Y \setminus \upsilon Y)$, it is obvious that $x \in Cl_{\beta Y}(U_i \setminus X)$ for each $i < w$. Since $x \notin \upsilon Y$, there exist a discrete sequence $f_i \subset U_i \setminus X$ for each $i < w$. Now, let $U = Y \setminus \cup \{F_i : i < w\}$Y\∪{F_i:i<w}. Then, U is a regular open subsets of Ycontaining$X$. But U does not contain any family of $\{U_i : i < w\}$ by the construction. Therefore $\partial_Y X$ is relatively pseudocompact.

Theorem 4.12: Suppose υY is of countable type, then the following are satisfies:

i. βY is Oz

ii. For any regular closed subset $X$of Y, $\partial_Y X$ is a relatively pseudocompact zero-set.

Proof: $(i) \Longrightarrow (ii)$, this imply that Y must be Oz, $\partial_Y X$ is a zero-set for any regular closed subset $X$of Y. Then by **theorem 4.11** above, this implication is obvious.

$(ii) \Longrightarrow (i)$. Let A be a regular closed subset of βY. Then $A \cap Y$ is a regular closed subset of Y. So, $\partial_Y(A \cap Y)$ is a relatively pseudocompact zero-set of Y. Provided that $\partial_{\beta Y} A = Cl_{\beta Y}(\partial_Y(A \cap Y))$, hence $\partial_{\beta Y} A$ is a compact zero-set of vY. Therefore, by the assumption that vY is of countable type, implying that $\partial_{\beta Y} A$ is a $G_\delta -$ set in βY. Hence A is G_δ in βY.

Corollary 4.3: Suppose Y is a real compact space and every closed subset of Y has a countable neighborhood basis, then Y is perfectly normal.

Theorem 4.13: Suppose Y is a real compact space. Then, the following are equivalent;

i. βY is Oz
ii. Any regular closed subset X of Y has a countable neighborhood basis in Y.
iii. For any regular closed subset X of Y,then $\partial_Y X$ is a compact subset which has a countable neighbourhood basis in Y

Proof: Before we provide the proof for this, we need to state the following Lemma below;

LEMMA 4.3: (Skljarenko E.G 1953) For any open subset X of a space Y, the equality $\partial_{\beta Y}(X^\beta) = Cl_{\beta Y}(\partial_Y X)$ holds.

Now, from this Lemma, for any regular closed subset X of Y, implies that $Cl_{\beta Y} X = Cl_{\beta Y}(\partial_Y X) \cup (Int_Y X)^\beta$ and $Cl_{\beta Y} Y \setminus X = Cl_{\beta Y}(Y \setminus X) \cup (Y \setminus X)^\beta = Cl_{\beta Y}(\partial_Y X) \cup (Y \setminus X)^\beta$. Thus, $Cl_{\beta Y} X \cap Cl_{\beta Y}(Y \setminus X) = Cl_{\beta Y}(\partial_Y X)$. Thus, $Cl_{\beta Y} X \cap Cl_{\beta Y}(Y \setminus X) = Cl_{\beta Y}(\partial_Y X)$. Therefore $Cl_{\beta Y}(\partial_Y X)$ is G_δ in βY, since βY is Oz. by **Theorem 4.12 above**, $\partial_Y X$ is relatively pseudocompact in Y. Since Y is realcompact, $\partial_Y X$ must be compact. Hence $\partial_Y X$ has a countable neighbourhood basis in Y.

$(iii) \Longrightarrow (ii)$ from **lemma 4.3** above,

$(ii) \Longrightarrow (i)$ we observe from the **lemma 4.4** specified below

LEMMA 4.4: Suppose Y is a real compact space and X is a closed subset of Y. If X has a countable neighbourhood basis in Y, then $Cl_{\beta Y}X$ is a zero-set of βY. We note if Y is extremely disconnected or pseudocompact Oz, then βY is Oz. Otherwise, we have the following theorem:

Theorem 4.14: If βY is Oz, then for each discrete sequence $\{U_i : i < w\}$ of open subsets of Y, $\exists\ M_o \ni U_i$ extremely disconnected for each $i \geq M_o$.

Proof: We prove this theorem from the contradictory point of view. Suppose βY is not Oz. then $\exists$ a sequence $\{U_i : k < w\}\ of\ \{U_i : i < w\} \ni U_{ik}$ is not extremely disconnected for each k. For each k, let T_k be an open subset of U_{ik} such that $Cl_{U_{ik}}T_k$ is not open. Let $A = \cup\{Cl_Y T_k : k < w\}$, obviously A is regular and closed. Then we shall show that the condition **(b) of theorem 4.10 above** is not satisfied. Therefore let $\{W_i : i < w\}$ be a sequence of regular open subsets of Y containing A. then, for each k, $\exists$ a regular closed subset B_k of Y such that $B_k \subset (W_k \cap U_{ik}) \setminus A$. Let us designate $U = X \setminus \cup\{B_k : k < w\}$. Then U is a regular open subset of Y which has no member of sequence $\{W_i : i < w$, which ends the proof.

Corollary 4.4: If every open subset of a space Y is not extremely disconnected, then the following condition are satisfied

i. βYYis Oz
ii. Y is pseudocompact and Oz

Theorem 4.15: Let Y be an Oz-space whose Hewitt realcompactification vY is of countable type. Then, the following conditions are equivalent;

i. βY is Oz
ii. Y can be expressed as the union of an extremely disconnected open subset and a closed relatively pseudocompact subset.

Proof: To prove **theorem 4.15** we will need the support of **corollary 4.4** above.

$(i) \Longrightarrow (ii)$ Suppose M is a family of all extremely disconnected open subsets of Y, then M is partially ordered by the inclusion relation $\subset$. Suppose M_i is a linearly ordered subset of M. Then, it is easy to set that $\cup\{U : U \in M_i\} \in M$, hence, using Zorn's lemma, there exist a maximal member E of M. So that $A = Y \setminus E$. Let us assume that A is not relatively pseudocompact. Then

$\exists$ a discrete sequence $\{U_i : i < w\}$ of open subsets of Y such that $U_i \cap A \neq \emptyset$ for each i. If U_i is extremely disconnected, then $U_i \cup E$ is also extremely disconnected. But this contradicts the maximality of E. Hence, each U_i is not extremely disconnected, therefore by **theorem 4.14** above shows a contradiction. Thus A is relatively pseudocompact.

$(ii) \Longrightarrow (i)$. If $Y = E \cup A$, where E is an extremely disconnected open subset and p is a closed relatively pseudocompact subset, we shall show that for each regular closed subset X of Y, $\partial_Y X$ is Oz. Now, we can show that $\partial_Y X \subset A$. This follows from the following that:

$$\partial_Y X = Cl_Y(Int_Y X) \setminus Int_Y X = Cl_Y(((Int_Y X) \cap E) \cup ((Int_Y X) \cap A)) \setminus Int_Y X$$
$$= (Cl_Y(Cl_E((Int_Y X) \cap E)) \setminus Int_Y X) \cup (Cl_Y((Int_Y X) \cap A) \setminus Int_Y X) \subset A \cup A = A$$

Theorem 4.16: A topological space Y is said to be a quasi Oz-space iff $R(Y) = Z(Y)^*$

To prove this theorem we need the following definition

Proof: Suppose that $G(Y) = \{M \in Z(Y)^* : Cl_Y(Y \setminus M) \in Z(Y)^*\}$. Thus $G(Y) \subseteq Z(Y)^* \subseteq R(Y)$. Suppose $R(Y) = G(Y)$, then Y is clearly a quasi Oz-space.

Conversely, suppose $A \subset Y$ is an open subset of Y, then $Cl_Y(A), Cl_Y(Y \setminus Cl_Y(A)) \in R(Y) = Z(Y)^*$. So, $Cl_Y(A) \in G(Y)$. Thus, $R(Y) = G(Y)$. Therefore, Y is quasi Oz-space.

Theorem 4.17: For a Oz topological space Y, the following must be satisfied:

i. Y is quasi Oz-space
ii. Every open subset $M \subset Y$ is Z^*-embedded in Y
iii. Every dense open subset $M \subset Y$ of Y is Z^*-embedded in Y

Proof: $(i) \Longrightarrow (ii)$, suppose $Z \in Z(M)$, $\exists$ a closed subset $K \in Y \, and \, Z = K \cap M$. Since M is open, then $Cl_M(int_M(Z)) = Cl_Y(int_Y(K)) \cap M$ and $Cl_Y(int_Y(K)) \in \Re(Y) = Z(Y)^*$. Hence, M is Z^*-embedded in Y

$(ii) \Longrightarrow (iii)$ is trivial

$(iii) \Longrightarrow (i)$, suppose $M \subset Y$ is any open subset of Y. Let $A = M \bigcup (Y \setminus Cl_Y(M))$ then, A is open and dense in Y.We define a map $g{:}A \longrightarrow R$ by $g(y) = 0$ if $y \in M$ and $g(y) = 1$ if $y \in Y \setminus Cl_Y(M)$, then g is continuous and $g^{-1}(0) = M,$ such that M is a zero-set in A. Since $A \subset Y$ is open and dense in Y, $\exists$ a cozero-set $Z \in Y \ni Cl_A(Int_A(M)) = Cl_Y(M) \bigcap A = Cl_Y(Int_Y(Z)) \bigcap A$. since A is dense in Y and $Cl_Y(M), Cl_Y(Int_Y(Z))$ are regular closed sets in Y, so $Cl_Y(M) = Cl_Y(Int_Y(Z))$. Hence, $R(Y) \subseteq Z(Y)^*$, therefore, Y is a quasi Oz-space.

Definition 4.2: A topological space Y is said to be basically disconnected if every cozero-set $X \in Y$ is C*-embedded in Y.

Definition 4.3: Let Y be a topological space and $X \subset YYX$ is said to be Z*- embedded in Y if for any $A \in Z(X)^* \exists\, U \in Z(Y)^* \ni A = U \bigcap X$.

Definition 4.4: Suppose $U(Y)$ is the set of clopen sets in a space Y. Then Y is said to be basically disconnected iff $B(Y) = Z(Y)^*$.

Definition 4.5: A completely regular space Y is said to be 3 a quasi Oz-space if for any regular closed set $K \in Y$. $\exists$ a zero-set $X \in Y \ni K = Cl_Y(Int_Y(X))$.

LEMMA 4.5: $A \subseteq Y$ is said to be dense in Y iff $A \bigcap K \neq \emptyset$ whenever $\emptyset \neq K \in Y$

Theorem 4.18: Suppose βY is a Stone-Čech compactification of a topological space Y. Then βY is a Baire space.

Proof: Let $A_i \subseteq \beta Y$ be dense and open for each $i \in N$. Suppose $\emptyset \neq K \in \tau$. By **Lemma 4.5** above, $K \bigcap A_1 \neq \emptyset$∩A_1≠∅, so, we can choose $x_i \in K \bigcap A_i$. Due to the regularity of the space Y, we can find an open set $U_1 \ni x_1 \in U_1 \subseteq Cl\,U_1 \subseteq K \bigcap A_1$. Since A_2 is dense, $U_1 \bigcap A_2 \neq \emptyset$. Also, we find an open set $U_2 \neq \emptyset \ni U_2 \subseteq Cl\,U_2 \subseteq U_1 \bigcap A_2$. By iteration, we find non-empty open set $U_i \ni U_n \subseteq Cl\,U_n \subseteq U_{n-1} \bigcap U_n, \forall\, n$. Now, all $Cl\,U_n$ are non-empty and nested interval $Cl\,U_1 \supseteq Cl\,U_2 \supseteq \ldots$ holds. Then, the family $\{Cl\,U_i\}$ has the finite intersection property, so that compactness gives $\bigcap_i Cl\,U_i \neq \emptyset$. By construction, $\bigcap_i Cl\,U_i \subseteq K \bigcap \bigcap_i A_i$. Therefore, $K \bigcap \bigcap_i A_i \neq \emptyset$ which proved that βY is a Baire space.

The **theorem 4.19** below provides a result that the Borel set $G_\delta\ in\ \beta Y$ as shown in **theorem 4.12 and 4.13** above is a Baire space

Definition 4.6: suppose $\beta\gamma$ is a Stone-$\breve{C}$ech compactification of a topological space .Then $\gamma \subset \beta\gamma$ is said to nowhere dense if the interior of its closure is empty.

Definition 4.7: The space $\gamma \subset \beta\gamma$ is of first category, if it is a countable union of nowhere dense subsets of $\beta\gamma$.

Definition 4.8: The space $\gamma \subset \beta\gamma$ is of second category, if it is not first category; that is if it cannot be express as a countable union of nowhere dense subsets of $\beta\gamma$.

Theorem 4.19: Every $G_\delta - set$ in $O_z - space\, \beta\gamma$ is a Baire space.

Proof : Let $\beta\gamma$ be a compact Housdorff space and $\gamma \subseteq \beta\gamma\ is\ G_\delta, thus, \gamma = \cap_i K_i\ with\ K_i \subseteq \beta\gamma$ open for every $i \in N$ and let $A_i \subseteq \gamma$ be dense open for every $i \in N$. Clearly, $(\beta\gamma)^I = Cl\gamma$ is Compact Housdorff and $\gamma \subseteq \beta\gamma$ is dense. Then ${K^I}_i = K_i \cap (\beta\gamma) \subseteq (\beta\gamma)^I$ are open and dense in $\beta\gamma$, since each containsY. Further, there are open set $B_i \subseteq (\beta\gamma)^I$ such that $A_i = \gamma \cap B_i$. Since γ is dense in $(\beta\gamma)^I$ and A_i is dense in γ, each A_i is dense in $(\beta\gamma)^I, thus\ the\ open\ sets A_i \subseteq B_i$ are dense in $\beta\gamma$. Therefore $N = \cap_i A_i = \cap_i(\gamma \cap B_i) = \cap_i K_i \cap \ \cap_i B_i$ is countable intersection of dense open sets in $(\beta\gamma)^I$ and dense in $(\beta\gamma)^I$, By theorem above. Hence, $Cl_\gamma N = Cl(N) \cap \gamma = (\beta\gamma)^I \cap \gamma = \gamma$. Therefore N is dense in γ.

C. ***The characterizations of the Stone-Čech compactification of the linear strongly B-convergent sequence space of Maddox as GA compactification, using continuum hypothesis.***

From **theorem 4.8**, since Y is locally compact, then we have the **theorem 4.20** below:

Theorem 4.20: The Stone-Cech compactification of locally compact space Y such that $|\beta Y| \leq 2^{\chi_0}$ is a GA compactification.

Proof: if $\mathcal{B}$ is an open basis for Stone-Cech compactification of locally compact space Y such that $|\ \beta Y\ | \leq 2^{\chi_0}$. Without loss of generality, if we assume that $\mathcal{B}$ is closed under finite intersections and finite unions. Then, we can define the set

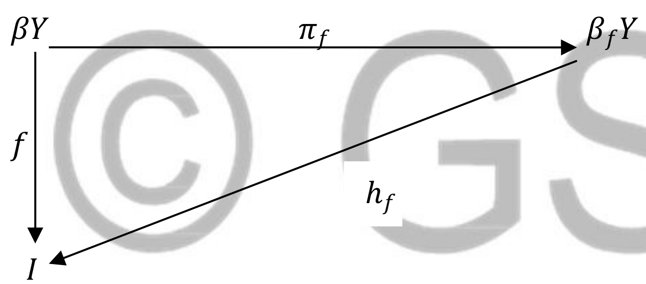
$$M = \{(Cl_{\beta Y}K_0),(Cl_{\beta Y}K_1)\backslash K_i \in \mathcal{B}, \text{for } (i = 0,1,2,...) \text{ and } Cl_{\beta Y}K_0 \cap Cl_{\beta Y}K_1 = \emptyset\} \quad \textbf{(4.2)}$$

From **(4.2)** above, for each pair $(Cl_{\beta Y}(K_0)),(Cl_{\beta Y}(K_1) \in M$, we can choose a function $f \in C(\beta Y, I) \ni f(Cl_{\beta Y}(K_0)) = 0$ and $f(Cl_{\beta Y}(K_1)) = 1$, by Uryshon function

Suppose $\mathcal{F}$ denotes the set of mappings by Uryshon function above such that $|\mathcal{F}| \leq 2^{\chi_0}$. Then, by transfinite induction, we construct for each $f \in F$ a

$$\zeta_f \in (0,1) \ni Cl_{\beta Y}(f^{-1}[0,\zeta_f) \cap Cl_{\beta Y}(g^{-1}[0,\zeta_f)) \neq 0 \Rightarrow Cl_{\beta Y}(f^{-1}[0,\zeta_f) \cap Cl_{\beta Y}(g^{-1}[0,\zeta_g)) \cap Y \neq 0 \quad \textbf{(4.3)}$$

Suppose $f \in F$ and $N = \{f^{-1}(t): Y\backslash t \in f(\beta Y\backslash Y)\} \cup \{\{x\}: x \in Y\}$ is an upper semi-continuous decomposition of βY, locally compact space of Y such that the decomposition space $\beta_f Y$ is a Stone-Cech compactification of locally compact space Y for which $f(\beta Y\backslash Y)$ is the remainder. Next, if π_f respresent the identity projection map on the space Y, and $h_f: \beta_f Y \to I$ defined as $h_f: f o \pi_{f^{-1}}$. Then, $h_f$is continuous and the diagram below is commutes;



As we see that $h_f\backslash_{\beta_f Y\backslash Y}$ is a homeomorphism. Then, we shall identify $\beta_f Y\backslash Y$ and $h_f(\beta_f Y\backslash Y)$ as follows:

Let $f_0 \in \mathcal{F}$ be the first point of $\mathcal{F}$and $\zeta_f = \frac{1}{2}$. Also, if every ζ_g have been constructed $\forall f > g$, where $g \in F$ such that **(4.3)** above is satisfied. Then, we follow the following step for our identifications:

STEP I: Let $X \in \beta_f Y$ be an open subset of $\beta_f Y$ and

$$A = \{\zeta \in (0,1): Cl(h_{f^{-1}}[0,\zeta)) \cap ClX \cap (\beta_f Y\backslash Y) \neq Cl(h_{f^{-1}}[0,\zeta) \cap X) \cap (\beta_f Y\backslash Y)\} \quad \textbf{(4.4)}$$

If we let $B \subset \beta_f Y$, then $Cl(B)$ represent closure of $B \subset \beta_f Y$. Then (A) in **(4.4)** above will be a subset of $f(\beta_f Y\backslash Y)$, i.e $A \subset \beta Y\backslash Y$, while A is countable. Also, if we choose $\zeta \in I\backslash$ f$(\beta Y\backslash Y)$, then,

$Cl(h_{f^{-1}}[0,\zeta)) \cap ClX \cap (\beta_f Y \backslash Y) \subset h_{f^{-1}}[0,\zeta) \cap ClX \cap (\beta_f Y \backslash Y) = h_{f^{-1}}[0,\zeta) \cap ClX \cap (\beta_f Y \backslash Y) \subset Cl(h_{f^{-1}}[0,\zeta)) \cap ClX \cap (\beta_f Y \backslash Y)$, provided that $f^{-1}(\zeta) \cap\subset \beta Y \backslash Y = \emptyset$

Now, we shall show that A is countable, suppose A is not countable. Then as $A \subset R$ is uncountable subset of real numbers, then it must contain one condensation point. Also, it is obvious that there exist a condensation point ζ_0 which is a limit point from the left. Now, if U is an open Neighbourhood of $\zeta_0 \in \beta_f Y$, $\exists\, \zeta_1 \in U \cap A \ni \zeta_1 < \zeta_0$ and consequently, $\zeta_1 \in h_{f^{-1}}[0,\zeta_0) \cap U \cap ClX \subset U \cap Cl\big(h_{f^{-1}}[0,\zeta_0)\big) \cap X)$.

Therefore, it follows that, $\zeta_0 \in Cl(h_{f^{-1}}[0,\zeta_0)) \cap X \cap (\beta_f Y \backslash Y)$, and is a contradiction.

STEP II:$\exists$ a $\zeta_0 \in (0,1) \ni Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \cap Cl\big(h_{f^{-1}}[0,\zeta_0)\big) \cap (\beta_f Y \backslash Y) =$

$Cl\big(g^{-1}[0,\zeta_g)\big) \cap h_{f^{-1}}[0,\zeta_0) \cap Y \cap ((\beta_f Y \backslash Y) \forall f > g$ and $g \in F$. Also, as $|g \in F{:}f > g| < 2^{\mathcal{X}_0}$, since Y is locally compact and also open in $\beta_f Y$, we can conclude from **step I** above that;

$\big|\cup_{f>g}\{\zeta \in (0,1){:}Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \cap Cl\big(h_{f^{-1}}[0,\zeta)\big) \cap (\beta_f Y \backslash Y) \neq Cl(g^{-1}[0,\zeta)) \cap h_{f^{-1}}([0,\zeta) \cap Y) \cap \beta_f Y \backslash Y\}\big| < \mathcal{X}_0.2^{\mathcal{X}_0} = 2^{\mathcal{X}_0}$, with this, the choice for ζ_0 is possible.

STEP III: Let $\zeta_f = \zeta_0$. Then, we need to claim that **(4.3)** above is satisfied. Let $g \in F \ni f > g$ and we suppose that; $Cl_{\beta Y}(f^{-1}[0,\zeta_f) \cap Cl_{\beta Y}\big(g^{-1}[0,\zeta_g)\big) \neq \emptyset$. Then $\pi_f(Cl_{\beta Y}(f^{-1}[0,\zeta_f)) \cap \pi_f(Cl_{\beta Y}\big(g^{-1}[0,\zeta_g)\big) \neq \emptyset$ and also, $Cl(\pi_f f^{-1}[0,\zeta_f)) \cap Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \neq \emptyset$, since it is easily observed that $\pi_f(Cl_{\beta Y} X) = Cl(X \cap Y)$ for each open subset $X \subset \beta Y$. Therefore, $Cl(h_{f^{-1}}[0,\zeta_f) \cap Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \neq \emptyset$ **(4.5)**

Now, if we assume that **(4.5)** $\cap\, Y$ is disjoint. We have $Cl(h_{f^{-1}}[0,\zeta_f) \cap Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \cap \big(\beta_f Y \backslash Y\big) = Cl(h_{f^{-1}}[0,\zeta_f) \cap \big(g^{-1}[0,\zeta_g) \cap Y\big) \cap \big(\beta_f Y \backslash Y\big) \neq \emptyset$, **STEP II** above $\Longrightarrow h_{f^{-1}}[0,\zeta_f) \cap g^{-1}[0,\zeta_g) \cap Y \neq \emptyset$ is a contradiction. Therefore, **(4.5)** $\cap\, Y \neq \emptyset$. Now, if we let $x \in Cl(h_{f^{-1}}[0,\zeta_f) \cap Cl\big(g^{-1}[0,\zeta_g) \cap Y\big) \cap Y$; this show that **(4.3)** above holds for ζ_f.

STEP IV: Let $\phi = \{Cl_{\beta Y}(f^{-1}[0,\zeta_f){:}f \in F\}$. Then, ϕ is a closed base for βY. Also we see that for k_0 , $k_1 \in \phi$, where $k_0 \cap k_1 \neq \emptyset \Longrightarrow k_0 \cap k_1 \cap Y \neq \emptyset$. Next, we need to show that ϕ is weakly normal an τ_1. To do these, we proceed as follows;

Let k_0 , $k_1 \in ϕ \ni k_0 \cap k_1 = \emptyset$. By the fact that βY is compact Hausdorff space, there exist closed set $B_0, B_1 \in ϕ \ni k_0 \cap B_1 = B_0 \cap k_1 = \emptyset$ and $\beta Y = B_0 \cup B_1$. From above, since $\mathcal{B}$ is closed under finite intersections and unions, $\exists\ L_0, L_2 \in \mathcal{B} \ni k_0 \subset L_0 \subset Cl_{\beta Y}(L_0)$ and $B_1 \subset L_1 \subset Cl_{\beta Y}(B_1)$ and $Cl_{\beta Y}(L_1) \cap Cl_{\beta Y}(L_1) = \emptyset$. Now, if we let $k_1{}^* \in ϕ \ni Cl_{\beta Y}(L_1) \subset k_1{}^*$ and $k_1{}^* \cap Cl_{\beta Y}(L_0) = \emptyset$. Similarly, we shall find $k_0{}^* \in ϕ \ni B_0 \subset k_0{}^*$ and $k_0{}^* \cap k_1{}^* = \emptyset$. Therefore, $k_0 \cap k_1{}^* = k_0{}^* \cap k_1 = \emptyset$ and $k_0{}^* \cup k_1{}^* = \beta Y$, which shows that, ϕ is a normal closed base.

Next, to show that ϕ is τ_1 –space, we suppose ϕ is τ_1 and $K_o \in ϕ$. We shall Show that cl $\{ K_o \} = \{ K_o \}$, to prove this, let $K_1 \in Cl\{ K_o \}$ such that $k_1 \notin \{ K_o \}$. As $K_o \neq K_1$ and ϕ is τ_1 –space, Ǝ open set B_o, $B_1 \in \tau \ni K_o \in B_o$ but $K_1 \notin B_1$. $K_1 \in Cl\{ K_o \}$ imply that K_1 is a limit point of $\{ k_o \}$ imply that $[B_o \sqcap \{ K_o \}] \setminus \{ K_o \} \neq ø$ (where $K_1 \in B_o$ and $B_o \in \tau$) $\Longrightarrow \{ K_o \} \setminus \{K_1 \} \neq ø$ (provided $K_o \in B_o$) $\Longrightarrow ϕ \neq ø$; which is a contradiction. Hence $Cl\{ K_o \} \subset \{ K_o \}$. As usual $\{ K_o \} \subset cl \{ K_o \}$, then we have $Cl\{ K_o \} = \{ K_o \}$.

Conversely, let ϕ be a topological space such that $\{ K_o \}$ is closed set for each $K_o \in ϕ$. To prove that ϕ is a τ_1 –space, we let $K_1 \neq K_o$ in ϕ. As $Cl\{ K_o \}=\{ K_o \}$, we have $K_1 \notin \{ K_o \} = cl\{ K_o \}$. Hence, $K_1 \in ϕ \setminus \{ K_o \} = ϕ \setminus Cl\{ K_o\}$. Similarly, $K_o \notin \{ K_1 \} = Cl\{ K_1 \} \Longrightarrow K_o \in ϕ \setminus \{ K_1 \} = ϕ \setminus Cl\{K_1\}$. We define $B_o = ϕ \setminus \{ K_o \}$ and $B_1 = ϕ \setminus Cl\{ K_1 \}= ϕ \setminus \{K_1\}$. Then $B_o, B_1 \in \tau$ such that $K_1 \in B_O$ but $K_o \notin B_o$ and $K_o \in B_1$ but $K_1 \notin B_1$.Hence, ϕ is a τ_1 –space.

Conclusion

In this research work, some topological properties on Maddox topological space Y and its Stone-Čech compactification were compared. Then, the Stone-Čech compactification of linear strongly B – convergent topological space of Maddox were characterized as Oz space, Qausi-Oz space and Baire spaces. The study also characterizes this Stone-Čech compactification as a GA compactification by using continuum hypothesis.

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