Global Scientific JOURNALS
GSJ: Volume 9, Issue 12, December 2021, Online: ISSN 2320-9186
www.globalscientificjournal.com

STABILITY ANALYSIS OF DISEASE PREVALENCE EQUILIBRUM STATE OF THE DYNAMICS OF DIABETES AND ITS COMPLICATIONS INCORPORATING CONTROL

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Abstract

This study give an overview of the dynamics of diabetes mellitus and its complications in a population. Five stages of the disease progression and control are incorporated into the formulated mathematical model of the dynamics of diabetes and its complications. The disease prevalence equilibrum states are obtained from the model equations. The stability of the disease prevalence equilibrum states of the system are analyzed using the constructed quadratic Lyapunov function method. The disease prevalence equilibrum states was found to be locally and globally asymptotically stable.

Keywords: Diabetes Mellitus, Complications, Lyapunov Function, Disease Prevalence Equilibrum, Stability, Mathematical Model

1.0 Introduction

The concept of stability in problems arising from theory and application of differential equations is very important and an effective approach is the method of Lyapunov (Omeike, 2008). The method of Lyapunov functions was introduced by Aleksandra M. Lyapunov, a Russian Mathematician. The fundamental of his proof was centred on the established fact that the sum of energy in a system is decreasing or constant as it approaches state of equilibrum. Lyapunov functions are useful tools in determining stability, asymptotic stability, uniform stability, global stability or out-right instability of differential system (Ogundare 2012, Ezeilo *et al.,* 2010, Omeike, 2008, Afuwape, 2010). Lyapunov theorem allow stability of linear and nonlinear system to be verified without differential equations solution being required. The presence of Lyapunov function implies asymptotic stability for linear time-invariant systems (Lars *et al.,* 2006). Lyapunov functions have been constructed for linear equa-tions on the platform that given any V that is definite positive, we have another definite positive function U such that $-U = \overset{*}{V}$

Also, if we consider linear system that is autonomous and we want to applied theory of Lyapunov to examine the stability of the system, for the sake of argument, let $V = x^T Mx$, where M is a positive definite matrix. Then, $\dot{V} = E^T M + ME$. Let $E^T M + ME = -Q$, Q is positive definite. The existence of positive definite Q guaranteed stability (global asymptotic) of the linear system. $E^T M + ME = -Q$ is called the equation of Lyapunov. Before solving for M, we established that E is stable, so that given any $Q > 0$, we have $M > 0$, the normal method is to solve for M and set $Q = 1$ (Hedrick and Girard, 2005; Aye *et al.,*2018; Ignat'ev, 2011; Griggs *et al.,* 2010).

2.0 Model Formulation

The model equations are formulated using first order differential equations. Improving on the work of Enagi *et al.(2017),* we proposed a mathematical model of diabetes and its complications incorporating positive lifestyle and effective management of diabetes condition as control. Based on their health status, the model population are classified into five classes. They are healthy class *H(t)*, susceptible class *S(t),* diabetic without complications class *D(t),* diabetic with complications class*C(t)* and diabetic with complications that undergo treatment class *T(t).* We assume that diabetes disease infections can either be acute or chronic.

In this model, we assume that the healthy individual will give birth to a healthy children that will be born into healthy compartment while parent who is diabetic or have history of diabetes will give birth to children with genetic factors that will be born into susceptible compartment. The proportion of children born into healthy compartment is denoted by θ while proportion of children that are born into susceptible compartment is denoted by $1-\theta$.

In formulating this model, we introduce a control parameters $\phi_1, \phi_2 \cdot \phi_1$, is a measure of positive lifestyle in the susceptible class, such that $0 \leq \phi_1 \leq 1 \cdot \phi_1 = 0$ indicate negative lifestyle and $\phi_1 = 1$ indicate positive lifestyle. ϕ_2, is a measure of effective management of diabetes condition in the compartment of diabetics without complications, such that $0 \leq \phi_2 \leq 1 \cdot \phi_2 = 0$ indicate ineffective management of diabetes condition and $\phi_2 = 1$ indicate effective management of diabetes condition.

Table 3.1: Definition of variables of the model

S/N	Variables	Description
1	*H(t)*	Healthy class
2	*S(t)*	Susceptible class
3	*D(t)*	Diabetics without complications class
4	*C(t)*	Diabetics with complications class
5	*T(t)*	Diabetics with complications undergoing treatment class
6	*N(t)*	Total population

Table 3.2: Definition of parameters of the model

S/N	Parameters	Description
1	α	Probability rate of incidence of diabetes
2	β	Birth rate
3	μ	Natural mortality rate
		Rate at which healthy individual become susceptible
4	τ	Rate at which susceptible individual become healthy
5	σ	Rate at which *D(t)* develop a complications
6	λ	Rate at which *C(t)* are treated
7	γ	
8	ω	Rate at which *C(t)* after treatment return to *D(t)*
9	δ	Mortality rate due to complications
10	θ	Proprotion of children born into the healthy class
11	ϕ_1	Measure of positive lifestyle in *S(t)* class
12	ϕ_2	Measure of effective management of diabetes condition in *D(t)* class
13	$1-\theta$	Proportion of children born into the susceptible class

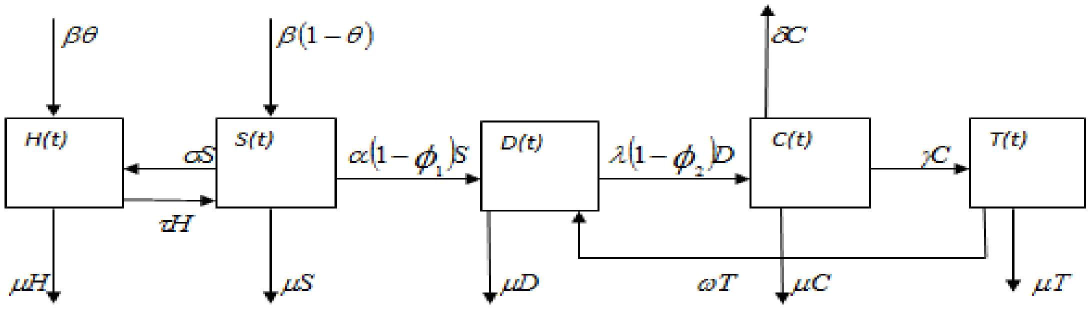

Figure 2.1 : Schematic diagram of model

2.2 The Model Equations

The model equations are stated as follows in (2.1) to (2.5)

$$\frac{dH(t)}{dt} = \sigma S(t) - \mu H(t) - \tau H(t) + \beta\theta \tag{2.1}$$

$$\frac{dS(t)}{dt} = \beta(1-\theta) - \mu S(t) + \tau H(t) - \alpha(1-\phi_1)S(t) - \sigma S(t) \tag{2.2}$$

$$\frac{dD(t)}{dt} = \alpha(1-\phi_1)S(t) + \omega T(t) - \lambda(1-\phi_2)D(t) - \mu D(t) \tag{2.3}$$

$$\frac{dC(t)}{dt} = \lambda(1-\phi_2)D(t) - \gamma C(t) - \delta C(t) - \mu C(t) \tag{2.4}$$

$$\frac{dT(t)}{dt} = \gamma C(t) - \omega T(t) - \mu T(t) \tag{2.5}$$

The initial values conditions are $H(o) = H_o$, $S(o) = S_o$, $D(o) = D_o$, $C(o) = C_o$ and $T(o) = T_o$.

3.3 Disease Prevalence Equilibrum State of Equations (2.1) to (2.5)

$$\beta\theta - \tau H(t) + \sigma S(t) - \mu H(t) = 0 \tag{2.6}$$

$$\beta(1-\theta) + \tau H(t) - \mu S(t) - \alpha(1-\phi_1)S(t) - \sigma S(t) = 0 \tag{2.7}$$

$$\alpha(1-\phi_1)S(t) + \omega T(t) - \mu D(t) - \lambda(1-\phi_2)D(t) = 0 \tag{2.8}$$

$$\lambda(1-\phi_2)D(t) - \mu C(t) - \delta C(t) - \gamma C(t) = 0 \tag{2.9}$$

$$\gamma C(t) - \omega T(t) - \mu T(t) = 0 \tag{2.10}$$

Rearranging (2.10), we obtain

$$C = \frac{1}{\gamma}(\mu + \omega)T \tag{2.11}$$

From (2.6)

$$S = \frac{1}{\sigma}(\mu + \tau)H - \frac{\beta\theta}{\sigma} \tag{2.12}$$

Substitute (2.12) into (2.7) gives

$$\beta(1-\theta) + \tau H - \left(\mu + \sigma + \alpha(1-\phi_1)\right)\left[\frac{1}{\sigma}(\mu + \tau)H - \frac{\beta\theta}{\sigma}\right] = 0$$

$$\beta(1-\theta) + \tau H - \frac{1}{\sigma}\left(\mu + \sigma + \alpha(1-\phi_1)\right)(\mu + \tau)H + \frac{1}{\sigma}\left(\mu + \sigma + \alpha(1-\phi_1)\right)\beta\theta = 0$$

$$\beta(1-\theta)+H\left[\frac{\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)}{\sigma}\right]=-\frac{1}{\sigma}(\mu+\sigma+\alpha(1-\phi_1))\beta\theta$$

Multiply throughout by σ

$$H[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]=-(\mu+\sigma+\alpha(1-\phi_1))\beta\theta-\beta\sigma(1-\theta)$$

$$H=\frac{-\beta[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]} \quad (2.13)$$

Substitite (2.13) into (2.12) gives

$$S=\frac{1}{\sigma}\left[\frac{-(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}-\beta\theta\right] \quad (2.14)$$

Substitute (2.14) into (2.8) gives

$$\frac{\alpha(1-\phi_1)}{\sigma}\left[\frac{-(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}-\beta\theta\right]+\omega T-(\mu+\lambda(1-\phi_2))D=0 \quad (2.15)$$

Substitute (2.11) into (2.9) gives

$$\lambda(1-\phi_2)D-(\mu+\delta+\gamma)\left[\frac{1}{\gamma}(\mu+\omega)T\right]=0$$

$$D=\frac{1}{\lambda\gamma(1-\phi_2)}(\mu+\delta+\gamma)(\mu+\omega)T \quad (2.16)$$

Substitute (2.16) into (2.15) gives

$$\frac{\alpha(1-\phi_1)}{\sigma}\left[\frac{-(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}-\beta\theta\right]$$
$$+\omega T-\frac{(\mu+\lambda(1-\phi_2))}{\lambda\gamma(1-\phi_2)}(\mu+\delta+\gamma)(\mu+\omega)\tau=0$$

$$T\left[\frac{\omega\lambda\gamma(1-\phi_2)-(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}{\lambda\gamma(1-\phi_2)}\right]$$
$$=\frac{-\alpha(1-\phi_1)}{\sigma}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}-\beta\theta\right]$$

$$T=\frac{\alpha\lambda\gamma(1-\phi_1)(1-\phi_2)}{\omega\lambda\gamma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right]$$
...(2.17)

Substitute (2.17) into (2.16) gives

$$D=\frac{\alpha(1-\phi_1)(\mu+\delta+\gamma)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right] \quad \text{...(2.18)}$$

Substitute (2.17) into (2.11) gives

$$C=\frac{\alpha\lambda(1-\phi_1)(1-\phi_2)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right] \quad \text{...(2.19)}$$

The disease prevalence equilibrum state $E^*(H^*, S^*, D^*, C^*, T^*)$ of the system (2.1) to (2.5) is given as follow:

$$H^*=\frac{-\beta[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]} \quad (2.20)$$

$$S^*=\frac{1}{\sigma}\left[\frac{-(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}-\beta\theta\right] \quad (2.21)$$

$$D^*=\frac{\alpha(1-\phi_1)(\mu+\delta+\gamma)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right] \quad (2.22)$$

$$C^*=\frac{\alpha\lambda(1-\phi_1)(1-\phi_2)(\mu+\omega)}{\omega\lambda\gamma\sigma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right] \quad (2.23)$$

$$T^*=\frac{\alpha\lambda\gamma(1-\phi_1)(1-\phi_2)}{\omega\lambda\gamma(1-\phi_2)-\sigma(\mu+\lambda(1-\phi_2))(\mu+\delta+\gamma)(\mu+\omega)}\left[\frac{(\mu+\tau)[(\mu+\sigma+\alpha(1-\phi_1))\theta+\sigma(1-\theta)]\beta}{[\tau\sigma-(\mu+\sigma+\alpha(1-\phi_1))(\mu+\tau)]}+\beta\theta\right] \quad (2.24)$$

3.0 Asymptotic Stability of the Disease Prevalence Equilibrum State Equations (2.1) to (2.5)

3.1 Lyapunov Stability of Linear System

Given that the dynamical system is of linear form:

$$\dot{x}=Ex \quad (3.1)$$

Let $M>0$ be a symmetric, positive definite matrix, then we define

$$V(x)=x^T Mx \quad (3.2)$$

$$\dot{V}(x)=\dot{x}^T Mx+x^T M\dot{x}$$

$$=E^T x^T Mx+x^T MEx$$

$$=x^T\left(E^T M+ME\right)x$$

$$\dot{V}(x)=E^T M+ME \quad (3.3)$$

Theorem 3.1: A linear system $\dot{x} = Ex$ is local asymptotically stable if and only if for any symmetric, positive definite *Q,* there exist a corresponding symmetric, positive definite *M* so that

$$E^T M + ME = -Q \qquad (3.4)$$

Theorem 3.2: Let $\overset{*}{x} = E(x), x \in \Re^n$.The system (origin) is globally asymptotically stable if and only if there exists a positive definite matrix $M = M^T > 0$ so that $E^T M + ME$ is negative definite or $E^T M + ME$. Equivalently if, for a given $Q = Q^T > 0$, it is possible to find a $M = M^T > 0$ so that

$$E^T M + ME = -Q$$

then the system is globally asymptotically stable (Griggs *et al.,* 2010).

Proof:

Choose $M = \int_0^\infty e^{tE^T} Q e^{tE} dt$

$$E^T M + ME = \int \left(E^T e^{tE^T} Q e^{tE} + e^{tE^T} Q e^{tE} E \right) dt$$
$$= \int_0^\infty \frac{d}{dt}\left(e^{tE^T} Q e^{tE} \right) dt$$
$$= e^{tE^T} Q e^{tE} \Big|_0^\infty$$
$$= -Q$$

Theorem 3.3: If $\Re e\, \lambda_K(E) < 0 \quad \forall k$, then for given every $Q = Q^T > 0$ there exists a unique $M = M^T > 0$ satisfying the Lyapunov equation

$$E^T M + ME = -Q$$

so that the system is globally asymptotically stable (Griggs *et al.,* 2010).

Proof:

$$Jacobian matrix = \begin{bmatrix} -(\mu+\tau) & \sigma & 0 & 0 & 0 \\ \tau & -(\mu+\sigma+\alpha(1-\phi_1)) & 0 & 0 & 0 \\ 0 & \alpha(1-\phi_1) & -(\mu+\lambda(1-\phi_2)) & 0 & \omega \\ 0 & 0 & \lambda(1-\phi_2) & -(\mu+\delta+\gamma) & 0 \\ 0 & 0 & 0 & \gamma & -(\mu+\omega) \end{bmatrix} \qquad (3.5)$$

Let

$$E = \begin{bmatrix} -\eta_1 & \sigma & 0 & 0 & 0 \\ \tau & -\eta_2 & 0 & 0 & 0 \\ 0 & \varepsilon & -\eta_3 & 0 & \omega \\ 0 & 0 & \varphi & -\eta_4 & 0 \\ 0 & 0 & 0 & \gamma & -\eta_5 \end{bmatrix} \qquad (3.6)$$

Where

$$\eta_1 = \mu + \tau \qquad (3.7)$$

$$\eta_2 = \mu + \sigma + \alpha(1-\phi_1) \tag{3.8}$$

$$\eta_3 = \mu + \lambda(1-\phi_2) \tag{3.9}$$

$$\eta_4 = \mu + \delta + \gamma \tag{3.10}$$

$$\eta_5 = \mu + \omega \tag{3.11}$$

$$\varepsilon = \alpha(1-\phi_1) \tag{3.12}$$

$$\varphi = \lambda(1-\phi_2) \tag{3.13}$$

To show that (3.6) is stable, we first determine the eigenvalue.

$$(-\eta_1-\rho)\begin{vmatrix} -\eta_2-\rho & 0 & 0 & 0 \\ \varepsilon & -\eta_3-\rho & 0 & \omega \\ 0 & \varphi & -\eta_4-\rho & 0 \\ 0 & 0 & \gamma & -\eta_5-\rho \end{vmatrix} - \sigma\begin{vmatrix} \tau & 0 & 0 & 0 \\ 0 & -\eta_3-\rho & 0 & \omega \\ 0 & \varphi & -\eta_4-\rho & 0 \\ 0 & 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$(-\eta_1-\rho)(-\eta_2-\rho)\begin{vmatrix} -\pi_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} - \tau\sigma\begin{vmatrix} -\eta_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$((-\eta_1-\rho)(-\eta_2-\rho)-\tau\sigma)\begin{vmatrix} -\eta_3-\rho & 0 & \omega \\ \varphi & -\eta_4-\rho & 0 \\ 0 & \gamma & -\eta_5-\rho \end{vmatrix} = 0$$

$$(\eta_1\eta_2+\eta_1\rho+\eta_2\rho+\rho^2-\tau\sigma)(-\eta_3-\rho)(-\eta_4-\rho)(-\eta_5-\rho) = 0 \tag{3.14}$$

From (3.14), we obtain the eigenvalue as follows:

$$\rho_{1,2} = \frac{-(\eta_1+\eta_2)\pm\sqrt{(\eta_1+\eta_2)^2-4(\eta_1\eta_2-\tau\sigma)}}{2}, \rho_3=-\eta_3, \rho_4=-\eta_4, \rho_5=-\eta_5$$

All the eigenvalue is nonpositive, (3.16) is stable.

$$E^T = \begin{bmatrix} -\eta_1 & \tau & 0 & 0 & 0 \\ \sigma & -\eta_2 & \varepsilon & 0 & 0 \\ 0 & 0 & -\eta_3 & \varphi & 0 \\ 0 & 0 & 0 & -\eta_4 & \gamma \\ 0 & 0 & \omega & 0 & -\eta_5 \end{bmatrix} \tag{3.15}$$

We choose $Q = Q^T = I$ so from equation $E^T M + ME = -I$, we obtain

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{bmatrix} \tag{3.16}$$

$$E^T M = \begin{bmatrix} -\eta_1 & \tau & 0 & 0 & 0 \\ \sigma & -\eta_2 & \varepsilon & 0 & 0 \\ 0 & 0 & -\eta_3 & \varphi & 0 \\ 0 & 0 & 0 & -\eta_4 & \gamma \\ 0 & 0 & \omega & 0 & -\eta_5 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{bmatrix} \tag{3.17}$$

$$E^T M = \begin{bmatrix} \tau M_{12} - \eta_1 M_{11} & \tau M_{22} - \eta_1 M_{12} & \tau M_{23} - \eta_1 M_{13} & \tau M_{24} - \eta_1 M_{14} & \tau M_{25} - \eta_1 M_{15} \\ \sigma M_{11} - \eta_2 M_{12} + \varepsilon M_{13} & \sigma M_{12} - \eta_2 M_{22} + \varepsilon M_{23} & \sigma M_{13} - \eta_2 M_{23} + \varepsilon M_{33} & \sigma M_{14} - \eta_2 M_{24} + \varepsilon M_{34} & \sigma M_{15} - \eta_2 M_{25} + \varepsilon M_{35} \\ \varphi M_{14} - \eta_3 M_{13} & \varphi M_{24} - \eta_3 M_{23} & \varphi M_{34} - \eta_3 M_{33} & \varphi M_{44} - \eta_3 M_{34} & \varphi M_{45} - \eta_3 M_{35} \\ \gamma M_{15} - \eta_4 M_{14} & \gamma M_{25} - \eta_4 M_{24} & \gamma M_{35} - \eta_4 M_{34} & \gamma M_{45} - \eta_4 M_{44} & \gamma M_{55} - \eta_4 M_{45} \\ \omega M_{13} - \eta_5 M_{15} & \omega M_{23} - \eta_5 M_{25} & \omega M_{33} - \eta_5 M_{35} & \omega M_{34} - \eta_5 M_{45} & \omega M_{35} - \eta_5 M_{55} \end{bmatrix}$$

$$ME = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{bmatrix} \begin{bmatrix} -\eta_1 & \sigma & 0 & 0 & 0 \\ \tau & -\eta_2 & 0 & 0 & 0 \\ 0 & \varepsilon & -\eta_3 & 0 & \omega \\ 0 & 0 & \varphi & -\eta_4 & 0 \\ 0 & 0 & 0 & \gamma & -\eta_5 \end{bmatrix}$$

$$ME = \begin{bmatrix} \tau M_{12}-\eta_1 M_{11} & \sigma M_{11}-\eta_2 M_{12}+\varepsilon M_{13} & \varphi M_{14}-\eta_3 M_{13} & \gamma M_{15}-\eta_4 M_{14} & \omega M_{13}-\eta_5 M_{15} \\ \tau M_{22}-\eta_1 M_{12} & \sigma M_{12}-\eta_2 M_{22}+\varepsilon M_{23} & \varphi M_{24}-\eta_3 M_{23} & \gamma M_{25}-\eta_4 M_{24} & \omega M_{23}-\eta_5 M_{25} \\ \tau M_{23}-\eta_1 M_{13} & \sigma M_{13}-\eta_2 M_{23}+\varepsilon M_{33} & \varphi M_{34}-\eta_3 M_{33} & \gamma M_{35}-\eta_4 M_{34} & \omega M_{33}-\eta_5 M_{35} \\ \tau M_{24}-\eta_1 M_{14} & \sigma M_{14}-\eta_2 M_{24}+\varepsilon M_{34} & \varphi M_{44}-\eta_3 M_{34} & \gamma M_{45}-\eta_4 M_{44} & \omega M_{34}-\eta_5 M_{45} \\ \tau M_{25}-\eta_1 M_{15} & \sigma M_{15}-\eta_2 M_{25}+\varepsilon M_{35} & \varphi M_{45}-\eta_3 M_{35} & \gamma M_{55}-\eta_4 M_{45} & \omega M_{35}-\eta_5 M_{55} \end{bmatrix} \quad (3.18)$$

$$-Q = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad (3.19)$$

Evaluating (3.4), we obtain equations (3.20) to (3.44)

$$2\tau M_{12} - 2\eta_1 M_{11} = -1 \quad (3.20)$$
$$\tau M_{22} - (\eta_1 + \eta_2) M_{12} + \sigma M_{11} + \varepsilon M_{13} = 0 \quad (3.21)$$
$$\tau M_{23} - (\eta_1 + \eta_3) M_{13} + \varphi M_{14} = 0 \quad (3.22)$$
$$\tau M_{24} - (\eta_1 + \eta_4) M_{14} + \gamma M_{15} = 0 \quad (3.23)$$
$$\tau M_{25} - (\eta_1 + \eta_5) M_{15} + \omega M_{13} = 0 \quad (3.24)$$
$$\sigma M_{11} - (\eta_1 + \eta_2) M_{12} + \varepsilon M_{13} + \tau M_{22} = 0 \quad (3.25)$$

$$2\sigma M_{12} - 2\eta_2 M_{22} + 2\varepsilon M_{23} = -1 \quad (3.26)$$
$$\sigma M_{13} - (\eta_2 + \eta_3) M_{23} + \varepsilon M_{33} + \varphi M_{24} = 0 \quad (3.27)$$
$$\sigma M_{14} - (\eta_2 + \eta_4) M_{24} + \varepsilon M_{34} + \gamma M_{25} = 0 \quad (3.28)$$
$$\sigma M_{15} - (\eta_2 + \eta_5) M_{25} + \varepsilon M_{35} + \omega M_{23} = 0 \quad (3.29)$$
$$\varphi M_{14} - (\eta_1 + \eta_3) M_{13} + \tau M_{23} = 0 \quad (3.30)$$
$$\varphi M_{24} - (\eta_2 + \eta_3) M_{23} + \sigma M_{13} + \varepsilon M_{33} = 0 \quad (3.31)$$
$$2\varphi M_{34} - 2\eta_3 M_{33} = -1 \quad (3.32)$$
$$\varphi M_{44} - (\eta_3 + \eta_4) M_{34} + \gamma M_{35} = 0 \quad (3.33)$$

$$\varphi M_{45} - (\eta_3 + \eta_5) M_{35} + \omega M_{33} = 0 \quad (3.34)$$

$$\gamma M_{15} - (\eta_1 + \eta_4) M_{14} + \tau M_{24} = 0 \quad (3.35)$$

$$\gamma M_{25} - (\eta_2 + \eta_4) M_{24} + \sigma M_{14} + \varepsilon M_{34} = 0 \quad (3.36)$$

$$\gamma M_{35} - (\eta_3 + \eta_4) M_{34} + \varphi M_{44} = 0 \quad (3.37)$$

$$2\gamma M_{45} - 2\eta_4 M_{44} = -1 \quad (3.38)$$

$$\gamma M_{55} - (\eta_4 + \eta_5) M_{45} + \omega M_{34} = 0 \quad (3.39)$$

$$\omega M_{13} - (\eta_1 + \eta_5) M_{15} + \tau M_{25} = 0 \quad (3.40)$$

$$\omega M_{23} - (\eta_2 + \eta_5) M_{25} + \sigma M_{15} + \varepsilon M_{35} = 0 \quad (3.41)$$

$$\omega M_{33} - (\eta_3 + \eta_5) M_{35} + \varphi M_{45} = 0 \quad (3.42)$$

$$\omega M_{34} - (\eta_4 + \eta_5) M_{45} + \gamma M_{55} = 0 \quad (3.43)$$

$$2\omega M_{35} - 2\eta_5 M_{55} = -1 \quad (3.44)$$

Solving (3.20) to (3.44), we obtain the component of symmetric matrix *M.*

3.2 Condition for local stability

From Theorem (3.2), the local asymptotic stability of the equilibrum points of system (2.1) to (2.5) holds, provided the following conditions are met.

$$\Delta_1 = |M_{11}| > 0$$

$$\Delta_2 = \begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix} > 0$$

$$\Delta_3 = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{vmatrix} > 0$$

$$\Delta_4 = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{vmatrix} > 0$$

$$\Delta_5 = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} > 0$$

$\Delta_r > 0, r = 1, \cdots, 5$, it means that M is definite positive matrix. The model is locally asymptotically stable.

3.3 Condition for global stability

From Theorem (3.3) , we have

$$\Delta_M = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} = \Delta_{M^T} = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{12} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{13} & M_{23} & M_{33} & M_{34} & M_{35} \\ M_{14} & M_{24} & M_{34} & M_{44} & M_{45} \\ M_{15} & M_{25} & M_{35} & M_{45} & M_{55} \end{vmatrix} =$$

$M = M^T > 0$ satisfying the Lyapunov equation $E^T M + ME = -Q$. Equation (2.1) to (2.5) is globally asymptotically stable.

4.0 Conclusion

The study analyzed the stability of disease prevalence equilibrum state of the model. The model equations has no disease free equilibrum state; this is consistent with the dynamics of the disease as it has no cure and hence the disease remain prevalent in the population. Quadratic Lyapunov function method was constructed and used to analyze the condition for local and global stability of Disease Prevalence Equilibrum state of the model.
With reference to the control measure incorporated into the model equations, the epidemiological implication of the Local Asymptotical Stability (LAS) analysis and Global Asymptotical Stability (GAS) analysisof the system (2.1) - (2.5) at the disease prevalence equilibrum state show that the disease can be controlled.

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